Green's $\mathcal{J}$-order and the rank of tropical matrices

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Tropical semirings and tropical matrices

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and define two binary operations on $\mathbb{T}$ by

$$a \oplus b := \max(a, b), \qquad a \otimes b := a + b,$$
$$(a \oplus -\infty = -\infty \oplus a = a, \quad a \otimes -\infty = -\infty \otimes a = -\infty).$$

We say that $\mathbb{T}$ is a (commutative) idempotent semiring.
It is often referred to as the max-plus or **tropical semiring**.

We also define the **finitary tropical semiring**,
$\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$.

Throughout let $S = \mathbb{T}$ or $\mathbb{FT}$.
We study the multiplicative semigroup $(M_n(S), \otimes)$ of all $n \times n$ matrices with entries in S.

Tropical convex sets

We write S^n to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in S$ and extend $\oplus$ to S^n componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$

We also define a scaling action of S on S^n:

$$(\lambda \otimes x)_i = \lambda \otimes x_i, \text{ for all } \lambda \in S.$$

A **tropical convex set** X in S^n is a subset that is closed under $\oplus$ and scaling.

Green's relations

Let $S = \mathbb{T}$ or $\mathbb{FT}$ and let $A, B \in M_n(S)$.

(1) $A \mathcal{L} B \quad \Leftrightarrow \quad$ row space of $A =$ row space of B.

(2) $A \mathcal{R} B \quad \Leftrightarrow \quad$ col. space of $A =$ col. space of B.

(3) $A \mathcal{D} B \quad \Leftrightarrow \quad$ row space of $A \cong$ row space of B
$\Leftrightarrow \quad$ col. space of $A \cong$ col. space of B
(Hollings and Kambites, 2010.)

We shall describe **Green's $\mathcal{J}$-order** and hence the corresponding $\mathcal{J}$-relation.

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Theorem. $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$ for all $n \geqslant 1$.

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Note that we may identify $\mathcal{P}\mathbb{FT}^n$ with $\mathbb{R}^{n-1}$ via

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Recall that we have a "distance function" on $\mathbb{FT}^n$ defined by $d_H(x, y) = 0$ if x is a finite tropical multiple of y and

$$\mathrm{d_H}(x, y) = \max(y_i - x_i) - \min(y_i - x_i).$$

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It can be shown that d_H is a **metric** on $\mathcal{P}\mathbb{FT}^n$.

The key results used in the proof

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3. Metric Duality Theorem:
Let $A \in M_n(\mathbb{FT})$. There exist mutually inverse isometric embeddings between $\mathcal{P}R(A)$ and $\mathcal{P}C(A)$.

Comparing $\mathcal{D}$ and $\mathcal{J}$

Theorem. $\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$.

Sketch proof Clearly $A\mathcal{D}B \Rightarrow A\mathcal{J}B$.
Suppose for contradiction that $A\mathcal{J}B$, but $A \not\mathcal{D} B$.

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Since f is not surjective and has closed image we may choose $x_0 \in \mathcal{P}R(A)$ and $\varepsilon > 0$ such that $x_0 \notin f(\mathcal{P}R(A))$ and $\mathrm{d_H}(x_0, z) \geqslant \varepsilon$ for all $z \in f(X_0)$.

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Now set $X_i = f^i(\mathcal{P}R(A))$ and let $x_i = f^i(x_0) \in X_i$.
Since f is an isometric embedding we have $\mathrm{d_H}(x_i, y) \geqslant \varepsilon$ for all $y \in X_{i+1}$.

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In particular $\mathrm{d_h}(x_i, x_j) \geqslant \varepsilon$ for all $j > i$.
This contradicts the compactness of $\mathcal{P}R(A) \subseteq \mathcal{P}\mathbb{FT}^n = \mathbb{R}^{n-1}$.

Green's $\mathcal{J}$-order on $M_n(\mathbb{T})$

Theorem. Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.

(i) $A \leqslant_{\mathcal{J}} B$;

(ii) There is a $\mathbb{T}$-linear convex set X such that the row space of A embeds linearly into X and the row space of B surjects linearly onto X;

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(iii) There is a $\mathbb{T}$-linear convex set Y such that the col. space of A embeds linearly into Y and the col. space of B surjects linearly onto Y.

The rank of a tropical matrix

There are several (non-equivalent) notions of the **rank** of a tropical matrix. We define three such here...

factor rank(A)	$=$	the minimum k such that A can be factored as $A = CR$ where C is $n \times k$ and R is $k \times n$
det rank(A)	$=$	the maximum k such that A has a $k \times k$ minor M with $\lvert M\rvert^+ \neq \lvert M\rvert^-$
tropical rank(A)	$=$	the maximum k such that A has a $k \times k$ minor M where the max. is achieved uniquely in the permanent of M.

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Observation. The factor rank, det rank, tropical rank (and others) are all $\mathcal{J}$-class invariants in $M_n(\mathbb{T})$.

Comparing $\mathcal{D}$ and $\mathcal{J}$

Example.

$$A = \begin{pmatrix} -\infty & 0 & 1 \\ -\infty & -\infty & 1 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -\infty & 0 & 2 \\ -\infty & -\infty & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

- It is easy to see that $C(A) \subseteq C(B)$. Hence $A \leqslant_{\mathcal{R}} B$.
- It is also easy to see that $R(B) \subseteq R(A)$. Hence $B \leqslant_{\mathcal{L}} A$.
- Thus we have shown that $A \mathcal{J} B$.

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- It is also easy to see that $R(B) \subseteq R(A)$. Hence $B \leqslant_{\mathcal{L}} A$.
- Thus we have shown that $A \mathcal{J} B$.
- Claim that $\mathcal{P}C(A)$ is not isometric to $\mathcal{P}C(B)$. Thus, $A \not\mathcal{D} B$ since any isomorphism between the column spaces would induce an isometry between the projective column spaces.
- It follows that $\mathcal{D} \neq \mathcal{J}$ in $M_n(\mathbb{T})$ for all $n \geqslant 3$.