Projectivity of Tropical Polytopes

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The tropical semifield

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- $\otimes$ distributes over $\oplus$;
- For all $a \in \mathbb{FT}$ we have $a \oplus a = a$.

We say that $\mathbb{FT}$ is an idempotent semifield.
It is often referred to as the max-plus or **tropical semifield**.

Motivation

The tropical semifield has applications in areas such as...

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- statistical inference
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Problems in application areas typically involve finding solutions to a system of linear equations over the tropical semifield.

We are therefore interested in properties of matrices with entries in the tropical semifield and their action upon vectors.

Tropical matrices

Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$.
We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

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Example.

$$\begin{pmatrix} 0 & 1 & 2 \\ 7 & 19 & 3 \\ -5 & 2 & 6 \end{pmatrix} \otimes \begin{pmatrix} -1 & -1 & -2 \\ -20 & 4 & 5 \\ 1 & 2 & 9 \end{pmatrix}$$

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It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ forms a **semigroup**.

Tropical vectors

We write $\mathbb{FT}^n$ to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend the **addition** $\oplus$ to $\mathbb{FT}^n$ componentwise:

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Thus $\mathbb{FT}^n$ has the structure of an **$\mathbb{FT}$-module**.

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Example.
Let $A \in M_n(\mathbb{FT})$. We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of A.

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Caution: the row space need **not** be linearly isomorphic to the column space.

Projectivity and regularity

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  Tropical matrices are "$\mathcal{D}$-related" $\Leftrightarrow$ their row spaces (dually, column spaces) are isomorphic.
- A tropical polytope $X \subseteq \mathbb{FT}^n$ is projective if and only if it is isomorphic to the image of an idempotent.

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Can we give a **geometric** characterisation of the projective tropical polytopes (and hence, the regular matrices in $M_n(\mathbb{FT})$)?

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Lemma. The dual dimension of X is equal to the generator dimension of $C(A)$, where A is any matrix satisfying $X = R(A)$.

Geometric characterisation of projectivity

Theorem 2. Let $X \subseteq \mathbb{FT}^n$ be a tropical polytope. Then

$$X \text{ is projective} \quad \Leftrightarrow \quad \begin{array}{l} X \text{ has pure tropical dimension and} \\ \text{trop. dim} = \text{gen. dim} = \text{dual dim.} \end{array}$$

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$\Leftarrow$: If X has dual dimension is k then, by definition, $X \cong Y$ for some $Y \subseteq \mathbb{FT}^k$ with all dimensions equal to k.
Enough to show that every such maximal-dimension tropical polytope in $\mathbb{FT}^k$ is isomorphic to the image of an idempotent.

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Corollary. Let $A \in M_n(\mathbb{FT})$. Then
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- Theorem 2: $R(A)$ projective $\Leftrightarrow R(A)$ has pure tropical dim. and tropical dim. = generator dim. = dual dim.
- Lemma: The dual dimension of $R(A)$ is equal to the generator dimension of $C(A)$.

The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix:

tropical rank(A)	=	tropical dimension of its row or col. space.
row rank(A)	=	generator dimension of row space of A.
column rank(A)	=	generator dimension of col. space of A.
factor rank(A)	=	the minimum k such that A can be factored as $A = CR$ where C is $n \times k$ and R is $k \times n$
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Corollary. Let $A \in M_n(\mathbb{FT})$ be a regular matrix. Then all these notions of rank coincide.