Groups of tropical matrices

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Tropical matrices

Let $\mathbb{FT}$ denote the set of real numbers with operations $\oplus$ and $\otimes$ defined by

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$

Let $M_n(\mathbb{FT})$ denote the set of all $n \times n$ matrices over $\mathbb{FT}$, with multiplication $\otimes$ defined as you would expect:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

Then $(M_n(\mathbb{FT}), \otimes)$ is a semigroup.

Groups lurking inside semigroups

Let S be a semigroup. Around every idempotent element ($E \in S$, $E^2 = E$) there is a unique maximal subgroup H_E.

$$H_E = \{A \in S : E = PA = AQ \text{ and } A = XE = EY\}$$

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Now let $S = M_n(\mathbb{FT})$.
What are the maximal subgroups (up to isomorphism)?

Tropical matrices give tropical polytopes...

Let $\mathbb{FT}^n$ denote the set of all real n-tuples $v = (v_1, \ldots, v_n)$ with obvious operations of addition and scalar multiplication:

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Given a finite subset $X = \{x_1, \ldots, x_r\} \subset \mathbb{FT}^n$, the tropical polytope generated by X is the $\mathbb{FT}$-linear span of X:

$$\{\lambda_1 \otimes x_1 \oplus \cdots \oplus \lambda_r \otimes x_r : \lambda_i \in \mathbb{FT}\}.$$

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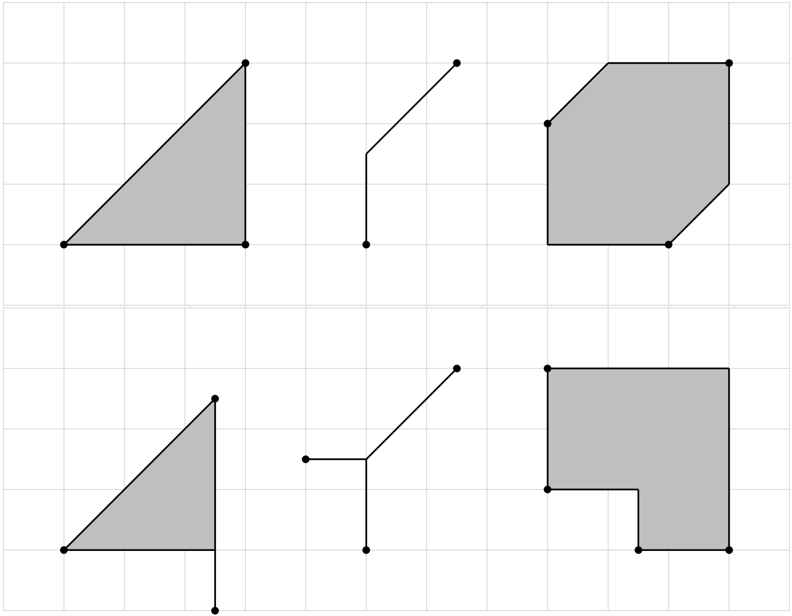
Let $A \in M_n(\mathbb{FT})$. We define the row space $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of A.

Similarly, we define the column space $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of A.

...and tropical polytopes look weird!



Maximal subgroups of $M_n(\mathbb{FT})$

- Let E be an idempotent in $M_n(\mathbb{FT})$.
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Theorem

Let E be an idempotent in $M_n(\mathbb{FT})$. Then

- H_E is isomorphic to the group of $\mathbb{FT}$-linear automorphisms of the column space $C(E)$
- H_E is isomorphic to the group of $\mathbb{FT}$-linear automorphisms of the row space $R(E)$.

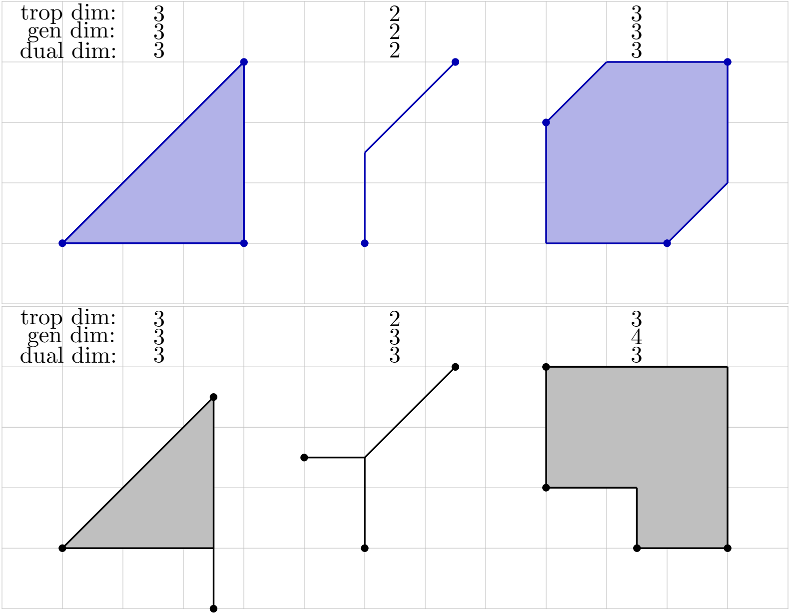
Three notions of dimension

Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

- The tropical dimension of V is the maximum topological dimension of V regarded as a subset of $\mathbb{R}^n$.
  We say that V has pure dimension if the open (within V) subsets of V all have the same topological dimension.
- The generator dimension of V is the minimum cardinality of a generating set for V.
- The dual dimension of V is the minimum k such that V embeds linearly into $\mathbb{FT}^k$.

In general, these dimensions can differ.

Dimensions of tropical polytopes



Idempotents have pretty polytopes

Theorem Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

There is a positive integer k such that V has pure tropical dimension k, generator dimension k and dual dimension k

if and only if

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- If E is an idempotent with $V = C(E)$, we say that E has **rank** k.

Maximal subgroups for idempotents of full rank

Let $\mathbb{T} = \mathbb{FT} \cup \{-\infty\}$ and consider the monoid $M_n(\mathbb{T})$.
The **units** in $M_n(\mathbb{T})$ are the tropical monomial matrices.

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Let E be an idempotent of rank n in $M_n(\mathbb{FT})$ and define

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Theorem

Let E be an idempotent of rank n in $M_n(\mathbb{FT})$.
Then $H_E \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leqslant S_n$.

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So, for an idempotent E of full rank n, the corresponding maximal subgroup is isomorphic to a direct product of $\mathbb{R}$ with a finite group $\Sigma \leqslant S_n$. What about when E has rank $< n$?

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Corollary

Let H be a maximal subgroup of $M_n(\mathbb{FT})$ containing a rank k idempotent. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leqslant S_k$.

Idempotents, groups and finite metrics

Let $[n] = \{1, \ldots, n\}$ and let $d : [n] \times [n] \to \mathbb{R}$ be a metric.
Consider the $n \times n$ matrix E with $E_{i,j} = -d(i, j)$.
Then

- $E \otimes E = E$;
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Corollary. [JK] Let G be a finite group. Then $\mathbb{R} \times G$ is a maximal subgroup of $M_n(\mathbb{FT})$, for n sufficiently large.