Green's $\mathcal{J}$-order and the rank of max-plus matrices

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Notation:

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and define two binary operations on $\mathbb{T}$:

$$a \oplus b := \max(a, b), \qquad a \otimes b := a + b,$$
$$(a \oplus -\infty = -\infty \oplus a = a, \quad a \otimes -\infty = -\infty \otimes a = -\infty).$$

We also define $\overline{\mathbb{T}} = \mathbb{R} \cup \{-\infty, +\infty\}$ where, as usual, we set $+\infty \otimes -\infty = -\infty \otimes +\infty = -\infty$.

Max-plus matrix semigroups

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Throughout let $S = \mathbb{T}$ or $\overline{\mathbb{T}}$.
We study the multiplicative semigroup $(M_n(S), \otimes)$ of all $n \times n$ matrices with entries in S.

Green's $\mathcal{J}$-order and the $\mathcal{J}$-relation

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This gives rise to the equivalence relation $\mathcal{J}$:

$$A \mathcal{J} B \text{ if and only if } A \leqslant_{\mathcal{J}} B \text{ and } B \leqslant_{\mathcal{J}} A.$$

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$$\begin{aligned} A \leqslant_{\mathcal{J}} B \quad &\Leftrightarrow \quad \text{row space of } A \text{ embeds in row space of } B. \\ &\Leftrightarrow \quad \text{col. space of } A \text{ embeds in col. space of } B. \\ &\Leftrightarrow \quad \operatorname{rank}(A) \leqslant \operatorname{rank}(B). \end{aligned}$$

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What is Green's $\mathcal{J}$-order on $M_n(S)$?

Let $S = \mathbb{T}$ or $\overline{\mathbb{T}}$ and let $A, B \in M_n(S)$. Then

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Theorem 1. Let $A, B \in M_n(\mathbb{T})$.
Then $A \leqslant_{\mathcal{J}} B$ in $M_n(\mathbb{T})$ if and only if $A \leqslant_{\mathcal{J}} B$ in $M_n(\overline{\mathbb{T}})$.

Thus it is enough to describe the $\mathcal{J}$-order and corresponding $\mathcal{J}$-relation on $M_n(\overline{\mathbb{T}})$.

Green's $\mathcal{J}$-order on $M_n(\overline{\mathbb{T}})$

Recall that for $A, B \in M_n(\overline{\mathbb{T}})$

$A \leqslant_{\mathcal{J}} B$ if and only if $A = PBQ$ for some $P, Q \in M_n(\overline{\mathbb{T}})$.

Theorem 2. Let $A, B \in M_n(\overline{\mathbb{T}})$. Then the following are equivalent.

(i) $A \leqslant_{\mathcal{J}} B$;

(ii) There is a $\overline{\mathbb{T}}$-linear convex set X such that the row space of A embeds linearly into X and the row space of B surjects linearly onto X;

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- **(iii)** There is a $\overline{\mathbb{T}}$-linear convex set Y such that the col. space of A embeds linearly into Y and the col. space of B surjects linearly onto Y.

Green's $\mathcal{J}$-order on $M_n(\overline{\mathbb{T}})$ and embeddings

Q: Can the $\mathcal{J}$-order on $M_n(\overline{\mathbb{T}})$ be characterised as linear embedding of row/col. spaces?

Let $A, B \in M_n(\overline{\mathbb{T}})$. It follows easily from Theorem 2 that

- If $R(A)$ embeds linearly in $R(B)$ then $A \leqslant_{\mathcal{J}} B$.
- If $C(A)$ embeds linearly in $C(B)$ then $A \leqslant_{\mathcal{J}} B$.

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- If $C(A)$ embeds linearly in $C(B)$ then $A \leqslant_{\mathcal{J}} B$.

For $n \geqslant 4$ it can be shown that the converse to the above statements is false.
i.e. there exist matrices $A, B \in M_n(\overline{\mathbb{T}})$ such that $A \leqslant_{\mathcal{J}} B$, but $R(A)$ does not embed linearly into $R(B)$.

A: No!

Green's $\mathcal{J}$-order on $M_n(R)$ and rank

Theorem 3. Let R be a commutative semiring and let $f : M_n(R) \to \mathbb{N}_0$. Then f respects the $\mathcal{J}$-order if and only if

$$f(AB) \leqslant \min(f(A), f(B)).$$

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Example. Let K be a field. It is straight-forward to verify that the rank-product inequality

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Observation. Any notion of "rank" on $M_n(R)$ satisfying the rank-product inequality will be a $\mathcal{J}$-class invariant.

The rank of a max-plus matrix

There are several (non-equivalent) notions of the rank of a max-plus matrix:

factor rank(A) = the minimum k such that A can be factored as $A = CR$ where C is $n \times k$ and R is $k \times n$

det rank(A) = the maximum k such that A has a $k \times k$ minor M with $|M|^+ \neq |M|^-$

tropical rank(A) = the maximum k such that A has a $k \times k$ minor M where the max. is achieved uniquely in the permanent of M.

GM row rank(A) = maximal number of Gondoran-Minoux linearly independent rows of A.

Max-plus rank-product inequalities

Theorem. [Akian, Gaubert, Guterman, 2009]
Let $A, B \in M_n(\mathbb{T})$. Then

$$\begin{aligned} \text{factor rank}(AB) &\leqslant \min(\text{factor rank}(A), \text{factor rank}(B)) \\ \text{det rank}(AB) &\leqslant \min(\text{det rank}(A), \text{det rank}(B)) \\ \text{tropical rank}(AB) &\leqslant \min(\text{tropical rank}(A), \text{tropical rank}(B)). \end{aligned}$$

Theorem. [Shitov, 2010]
Let $A, B \in M_n(\mathbb{T})$. Then

$$\begin{aligned} \text{GM row rank}(AB) &\leqslant \min(\text{GM row rank}(A), \text{GM row rank}(B)) \\ \text{GM col rank}(AB) &\leqslant \min(\text{GM col rank}(A), \text{GM col rank}(B)) \end{aligned}$$

Corollary. The factor rank, det rank and tropical rank are all $\mathcal{J}$-class invariants in $M_n(\mathbb{T})$.