Idempotent tropical matrices: graphs, groups and metric spaces

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The semigroup of tropical matrices

Let $\mathbb{FT}$ denote the tropical semifield $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$, where

$$a \oplus b := \max(a, b), \quad a \otimes b := a + b.$$

and let $M_n(\mathbb{FT})$ denote the set of all $n \times n$ matrices over $\mathbb{FT}$, with multiplication $\otimes$ defined in the obvious way.

It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ is a **semigroup**.

We are interested in the **algebraic** structure of this semigroup, much of which can be neatly described using some **geometric** ideas.

Tropical matrices and tropical polytopes

Let $\mathbb{FT}^n$ denote the set of all real n-tuples $v = (v_1, \ldots, v_n)$ with obvious operations of addition and scalar multiplication:

$$(v \oplus w)_i = v_i \oplus w_i, \quad (\lambda \otimes v)_i = \lambda \otimes v_i.$$

Given a finite subset $X = \{x_1, \ldots, x_r\} \subset \mathbb{FT}^n$, the tropical polytope generated by X is the $\mathbb{FT}$-linear span of X:

$$\{\lambda_1 \otimes x_1 \oplus \cdots \oplus \lambda_r \otimes x_r : \lambda_i \in \mathbb{FT}\}.$$

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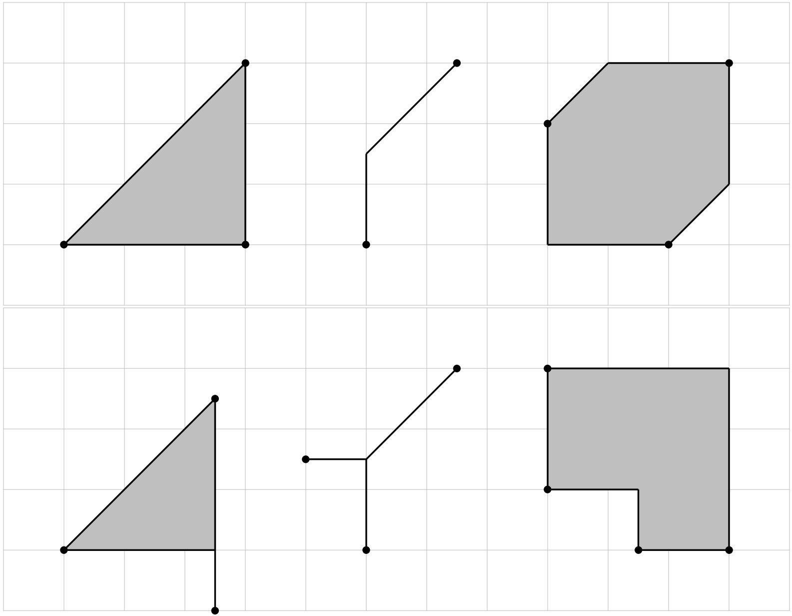
Let $A \in M_n(\mathbb{FT})$. We define the row space $R(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the rows of A.

Similarly, we define the column space $C(A) \subseteq \mathbb{FT}^n$ to be the tropical polytope generated by the columns of A.

Some tropical polytopes in $\mathbb{FT}^3$



On the structure of semigroups

Green's relations: Equivalence relations that can be defined upon any semigroup S and encapsulate the **ideal and subgroup structure** of S.

For $A, B \in S$...

- $A\mathcal{L}B$ if $\exists X, Y \in S^1$ such that $A = XB$ and $B = YA$.
- $A\mathcal{R}B$ if $\exists X, Y \in S^1$ such that $A = BX$ and $B = AY$.
- $A\mathcal{H}B$ if $A\mathcal{L}B$ and $A\mathcal{R}B$.

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- $A\mathcal{R}B$ if $\exists X, Y \in S^1$ such that $A = BX$ and $B = AY$.
- $A\mathcal{H}B$ if $A\mathcal{L}B$ and $A\mathcal{R}B$.

In $M_n(\mathbb{FT})$:
$A\mathcal{L}B$ if and only if $R(A) = R(B)$.
$A\mathcal{R}B$ if and only if $C(A) = C(B)$.
$A\mathcal{H}B$ if and only if $R(A) = R(B)$ AND $C(A) = C(B)$.

Idempotents and maximal subgroups

Let S be a semigroup.
The **idempotent elements** ($E \in S$, $E^2 = E$) play a special role in the study of the subgroup structure of S.

Around every idempotent element there is a unique **maximal subgroup** H_E. This is the $\mathcal{H}$-equivalence class of E.

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- What are the maximal subgroups of $M_n(\mathbb{FT})$?
  (i.e. What are the $\mathcal{H}$-equivalence classes of idempotents?)
- What *kinds* of group arise?
  (i.e. What are these groups up to isomorphism?)

Maximal subgroups of $M_n(\mathbb{FT})$

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Theorem Let E be an idempotent in $M_n(\mathbb{FT})$. Then

- H_E is isomorphic to the group of $\mathbb{FT}$-linear automorphisms of the column space $C(E)$
- H_E is isomorphic to the group of $\mathbb{FT}$-linear automorphisms of the row space $R(E)$.

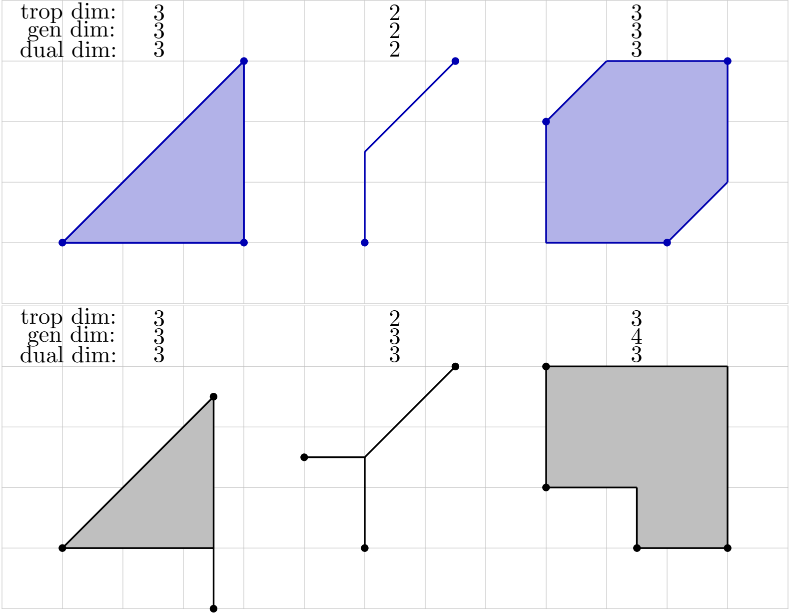
Three notions of dimension

Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

- The tropical dimension of V is the maximum topological dimension of V regarded as a subset of $\mathbb{R}^n$.
  We say that the tropical dimension is pure if the open (within V) subsets of V all have the same topological dimension.
- The generator dimension of V is the minimum cardinality of a generating set for V.
- The dual dimension of V is the minimum k such that V embeds linearly into $\mathbb{FT}^k$.

In general, these dimensions can differ.

Dimensions of tropical polytopes



Idempotents, projectivity and dimensions

Theorem Let $V \subseteq \mathbb{FT}^n$ be a tropical polytope.

There is a positive integer k such that V has pure tropical dimension k, generator dimension k and dual dimension k

if and only if

V is the column space of an idempotent

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- If E is an idempotent in $M_n(\mathbb{FT})$, we say that E has **rank k** if the dimension (in any sense) of $C(E)$ is k.
  (Note: $1 \leqslant \operatorname{rank}(E) \leqslant n$)

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- If E is an idempotent in $M_n(\mathbb{FT})$, we say that E has **rank k** if the dimension (in any sense) of $C(E)$ is k. (Note: $1 \leqslant \text{rank}(E) \leqslant n$)
- Idempotents of **full rank n** have a particularly nice structure; their **row and column spaces are convex in the ordinary sense**.

Maximal subgroups for idempotents of full rank

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Corollary
Every $\mathbb{FT}$-module automorphism of $C(E)$

(i) extends to an automorphism of $\mathbb{FT}^n$ and
(ii) is a (classical) affine linear map.

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Let E be an idempotent of rank n in $M_n(\mathbb{FT})$, so that $H_E \cong G_E$.

Theorem

Let $R = \{\lambda \otimes I_n\}$ and $\Sigma = \{G \in G_E : G \text{ has eigenvalue } 0\}$. Then $G_E = R \times \Sigma$.

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It is clear that $R \cong \mathbb{R}$ and not hard to show that the map $\Sigma \to S_n$ sending each unit G to its associated permutation is injective, giving:

Theorem
Let E be an idempotent of rank n in $M_n(\mathbb{FT})$.
Then $H_E \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leqslant S_n$.

Maximal subgroups of $M_n(\mathbb{FT})$

So, for an idempotent E of full rank n, the corresponding maximal subgroup is isomorphic to a direct product of $\mathbb{R}$ with a finite group $\Sigma \leqslant S_n$. What about when E has rank $< n$?

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Corollary

Let H be a maximal subgroup of $M_n(\mathbb{FT})$ containing a rank k idempotent. Then $H \cong \mathbb{R} \times \Sigma$, for some $\Sigma \leqslant S_k$.

Idempotents, groups and finite metrics

Let $[n] = \{1, \ldots, n\}$ and let $d : [n] \times [n] \to \mathbb{R}$ be a metric.
Consider the $n \times n$ matrix E with $E_{i,j} = -d(i, j)$.
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Corollary [JK]
Let G be a finite group. Then $\mathbb{R} \times G$ is a maximal subgroup of $M_n(\mathbb{FT})$, for n sufficiently large.