

Lie powers of modules for cyclic p -groups

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The Green ring of a cyclic p -group

Let K be a field of prime characteristic p ,

$C = \langle g \rangle$ a cyclic p -group of order q , where $q > 1$.

Consider finite-dimensional right KC -modules.

The **Green ring** R_{KC} has \mathbb{Z} -basis consisting of the indecomposable KC -modules, with multiplication coming from tensor product. We write:

$$\begin{array}{lll} KC\text{-modules:} & U \oplus V & U \otimes_K V & V^{\otimes n} \\ \text{Elements of } R_{KC} : & U + V & UV & V^n \end{array}$$

The Green ring of a cyclic p -group

There are q indecomposable KC -modules up to isomorphism.

For $r = 1, \dots, q$, let $V_r = KC/KC(g - 1)^r$.

Then V_r is indecomposable of dimension r .

Thus R_{KC} has \mathbb{Z} -basis $\{V_1, \dots, V_q\}$.

Each indecomposable V_r has basis $\{y_1, \dots, y_r\}$ and the action of g on V_r with respect to this basis is given by the Jordan block

$$\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

(Notice that V_1 is the one-dimensional trivial module and V_q is the regular KC -module.)

Lie powers

Let V be a KC -module with K -basis $\{x_1, \dots, x_r\}$.

Write $L(V)$ for the free Lie algebra on x_1, \dots, x_r over K .

Take decompositions into homogeneous components:

$$L(V) = L^1(V) \oplus L^2(V) \oplus \dots \oplus L^n(V) \oplus \dots$$

- ▶ Each $L^n(V)$ is a finite-dimensional KC -module, called the n th **Lie power** of V , where g acts by linear substitutions.
- ▶ The dimension of $L^n(V)$ is given by

$$\dim L^n(V) = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d},$$

where μ denotes the Möbius function.

The decomposition problem. For each positive integer n determine $L^n(V)$ up to isomorphism, i.e. as elements of R_{KC} .

Examples

Let $K = \mathbb{F}_3$, $C = \langle g : g^3 = 1 \rangle$, and let $V = V_3$.

Then V_3 has basis $\{y_1, y_2, y_3\}$, where

$$y_1 g = y_1 + y_2,$$

$$y_2 g = y_2 + y_3,$$

$$y_3 g = y_3.$$

We want to compute Lie powers $L^n(V_3)$ as elements of R_{KC} .

$L^2(V_3)$ has dimension 3 and basis:

$$[y_1, y_2], [y_1, y_3], [y_2, y_3].$$

The action of g on $L^2(V_3)$ with respect to this basis is given by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, it is easy to see that $L^2(V_3) = V_3$.

Examples

$L^3(V_3)$ has dimension 8 and basis:

$$\begin{aligned} & [y_1, y_2, y_1], [y_1, y_2, y_2], [y_1, y_3, y_1], [y_1, y_3, y_2], \\ & [y_1, y_3, y_3], [y_2, y_3, y_1], [y_2, y_3, y_2], [y_2, y_3, y_3]. \end{aligned}$$

The action of g on $L^3(V_3)$ with respect to this basis is given by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be shown that $L^3(V_3) = V_2 + 2V_3$
(e.g. compute the Jordan canonical form of the above matrix).

Lie resolvents

There exist \mathbb{Z} -linear functions $\Phi^n : R_{KC} \rightarrow R_{KC}$ called the **Lie resolvents** on R_{KG} such that for every KC -module V ,

$$L^n(V) = \frac{1}{n} \sum_{d|n} \Phi^d(V^{n/d}).$$

(Note: $\sum_{d|n} \Phi^d(V^{n/d}) \in nR_{KG}$, so division by n makes sense.)
Möbius inversion gives

$$\Phi^n(V) = \sum_{d|n} \mu(n/d) d L^d(V^{n/d}),$$

for all KC -modules V .

Thus we see that Φ^1 is the identity map on R_{KG} .

A factorisation theorem

By a theorem of Bryant and Schöcker we have

$$\Phi^{p^m k} = \Phi^{p^m} \circ \Phi^k,$$

for all $m \geq 0$ with $p \nmid k$, where \circ denotes composition.

In fact, Φ^k for $p \nmid k$ is well understood and the real interest is in $\Phi^p, \Phi^{p^2}, \Phi^{p^3}, \dots$

Aside:

For $p \nmid k$, $\Phi^k = \mu(k)\psi^k$, where ψ^k denotes the k th Adams operation on R_{KC} .

In recent joint work with Roger Bryant, we gave a description of the maps ψ^k for $p \nmid k$.

The cyclic group of order p

When C is cyclic of order p , the decomposition problem has been solved by Bryant, Kovács and Stöhr.

By our previous remarks, it is enough to describe the Lie resolvents $\Phi^p, \Phi^{p^2}, \Phi^{p^3}, \dots$

It turns out that

$$\Phi^p(V_r) = \begin{cases} V_1 & \text{if } r = 1, \\ p(V_p - V_{p-1}) - (p-r)V_1 & \text{if } r \text{ is odd, } 1 < r \leq p, \\ r(V_p - V_{p-1}) & \text{if } r \text{ is even, } 1 < r \leq p, \end{cases}$$

whilst $\Phi^{p^2} = \Phi^{p^3} = \dots = 0$.

(Thus we see that $\Phi^n = 0$ if n is not square-free.)

A conjecture of R. M. Bryant

Let C be a cyclic p -group, K a field of characteristic p . Then

$$\Phi^{p^2} = \Phi^{p^3} = \dots = 0.$$

Since the Lie resolvents are \mathbb{Z} -linear maps, it suffices to show that $\Phi^{p^m}(V_r) = 0$ for $r = 1 \dots, q$ and all $m > 1$.

Thus, the conjecture is equivalent to the following recursions:

$$p^m L^{p^{m+1}}(V_r) = L^p(V_r^{p^m}), \text{ for } r = 1 \dots, q, m > 0.$$

We have seen that the conjecture holds for C cyclic of order p .

It can also be shown that the conjecture holds for $p = 2$ and C cyclic of order 4. Very little is known for other cyclic p -groups.

Some properties of indecomposable KC -modules

- ▶ V_q is the unique projective indecomposable KC -module, up to isomorphism.
- ▶ $V_r \otimes V_q \cong rV_q$ for $r = 1, \dots, q$.
- ▶ For $r = 1, \dots, q$ there is a short exact sequence

$$0 \rightarrow V_{q-r} \rightarrow V_q \rightarrow V_r \rightarrow 0$$

of KC -modules, where $V_{q-r} = \Omega(V_r)$.

- ▶ The modules V_{p^i} are **permutation modules** for $1 \leq p^i \leq q$ and every permutation module can be written as a sum of the V_{p^i} .

A consequence of Bryant's conjecture for $p = 2$

Suppose that we have

$$2^m L^{2^{m+1}}(V_r) = L^2(V_r^{2^m}),$$

for $r = 1 \dots, q$, $m > 0$.

It can be shown that:

- ▶ $V_r^{\otimes 2^m}$ is a permutation module for $r = 1 \dots, q$, $m > 0$.
- ▶ If W is a permutation module then $L^2(W)$ is also a permutation module.

Hence we see that if the conjecture holds then $L^{2^{m+1}}(V_r)$ is a permutation module, for $r = 1 \dots, q$, $m > 0$.

(Note: This argument is specific to $p = 2$.)

Example

| | | | | | | | | | | Dimension | | |
|-----------------|-------|---|--------|---|---------|---|----------|---|--------------|-----------|--------------|-------|
| $L^4(V_2) =$ | V_1 | + | V_2 | | | | | | | 3 | | |
| $L^4(V_3) =$ | | | V_2 | + | $4V_4$ | | | | | 18 | | |
| $L^4(V_4) =$ | | | $2V_2$ | + | $14V_4$ | | | | | 60 | | |
| $L^4(V_5) =$ | | | V_2 | + | $5V_4$ | + | $16V_8$ | | | 150 | | |
| $L^4(V_6) =$ | V_1 | + | V_2 | + | $2V_4$ | + | $38V_8$ | | | 315 | | |
| $L^4(V_7) =$ | | | | | $3V_4$ | + | $72V_8$ | | | 588 | | |
| $L^4(V_8) =$ | | | | | $4V_4$ | + | $124V_8$ | | | 1008 | | |
| $L^4(V_9) =$ | | | | | $3V_4$ | + | $73V_8$ | + | $64V_{16}$ | 1620 | | |
| $L^4(V_{10}) =$ | V_1 | + | V_2 | + | $2V_4$ | + | $40V_8$ | + | $134V_{16}$ | 2475 | | |
| $L^4(V_{11}) =$ | | | V_2 | + | $5V_4$ | + | $19V_8$ | + | $216V_{16}$ | 3630 | | |
| $L^4(V_{12}) =$ | | | $2V_2$ | + | $14V_4$ | + | $4V_8$ | + | $316V_{16}$ | 5148 | | |
| $L^4(V_{13}) =$ | | | V_2 | + | $4V_4$ | + | $5V_8$ | + | $440V_{16}$ | 7098 | | |
| $L^4(V_{14}) =$ | V_1 | + | V_2 | | | + | $6V_8$ | + | $594V_{16}$ | 9555 | | |
| $L^4(V_{15}) =$ | | | | | | | $7V_8$ | + | $784V_{16}$ | 12600 | | |
| $L^4(V_{16}) =$ | | | | | | | $8V_8$ | + | $1016V_{16}$ | 16320 | | |
| $L^4(V_{17}) =$ | | | | | | | $7V_8$ | + | $785V_{16}$ | + | $256V_{32}$ | 20808 |
| $L^4(V_{18}) =$ | V_1 | + | V_2 | | | + | $6V_8$ | + | $596V_{16}$ | + | $518V_{32}$ | 26163 |
| $L^4(V_{19}) =$ | | | V_2 | + | $4V_4$ | + | $5V_8$ | + | $443V_{16}$ | + | $792V_{32}$ | 32490 |
| $L^4(V_{20}) =$ | | | $2V_2$ | + | $14V_4$ | + | $4V_8$ | + | $320V_{16}$ | + | $1084V_{32}$ | 39900 |
| $L^4(V_{21}) =$ | | | V_2 | + | $5V_4$ | + | $19V_8$ | + | $221V_{16}$ | + | $1400V_{32}$ | 48510 |
| $L^4(V_{22}) =$ | V_1 | + | V_2 | + | $2V_4$ | + | $40V_8$ | + | $140V_{16}$ | + | $1746V_{32}$ | 58443 |
| $L^4(V_{23}) =$ | | | | | $3V_4$ | + | $73V_8$ | + | $71V_{16}$ | + | $2128V_{32}$ | 69828 |
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| $L^4(V_{25}) =$ | | | | | $3V_4$ | + | $72V_8$ | + | $9V_{16}$ | + | $3024V_{32}$ | 97500 |

Another conjecture for cyclic 2-groups

Let C be a cyclic 2-group of order $q > 1$, K a field of characteristic 2 and let $q/2 < r \leq q$. Then for $m > 1$

$$L^{2^m}(V_r) = L^{2^m}(V_{q-r}) \bmod V_q, V_{q/2}.$$

(Note: $V_0 = 0$.)

Thus for $r = q$ the conjecture is consistent with the fact that Lie powers of the regular KC -module V_q only involve terms of the form V_q and $V_{q/2}$ (Bryant and Michos).

Similarly, for $r = q - 1$, the conjecture is consistent with the fact that Lie powers of the augmentation ideal V_{q-1} only involve terms of the form V_q and $V_{q/2}$ (Kovács and Stöhr)

In fact, this conjecture implies Bryant's conjecture in the case of cyclic 2-groups.

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| $L^4(V_5) =$ | | | V_2 | + | $5V_4$ | + | $16V_8$ | | | 150 | | |
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| $L^4(V_7) =$ | | | | | $3V_4$ | + | $72V_8$ | | | 588 | | |
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| $L^4(V_{18}) =$ | V_1 | + | V_2 | | | + | $6V_8$ | + | $596V_{16}$ | + | $518V_{32}$ | 26163 |
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| $L^4(V_{23}) = 3V_4 + 73V_8 + 71V_{16} + 2128V_{32}$ | 69828 |
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| $L^4(V_{25}) = 3V_4 + 72V_8 + 9V_{16} + 3024V_{32}$ | 97500 |

Example

| | Dimension |
|---|-----------|
| $L^4(V_2) = V_1 + V_2$ | 3 |
| $L^4(V_3) = V_2 + 4V_4$ | 18 |
| $L^4(V_4) = 2V_2 + 14V_4$ | 60 |
| $L^4(V_5) = V_2 + 5V_4 + 16V_8$ | 150 |
| $L^4(V_6) = V_1 + V_2 + 2V_4 + 38V_8$ | 315 |
| $L^4(V_7) = 3V_4 + 72V_8$ | 588 |
| $L^4(V_8) = 4V_4 + 124V_8$ | 1008 |
| $L^4(V_9) = 3V_4 + 73V_8 + 64V_{16}$ | 1620 |
| $L^4(V_{10}) = V_1 + V_2 + 2V_4 + 40V_8 + 134V_{16}$ | 2475 |
| $L^4(V_{11}) = V_2 + 5V_4 + 19V_8 + 216V_{16}$ | 3630 |
| $L^4(V_{12}) = 2V_2 + 14V_4 + 4V_8 + 316V_{16}$ | 5148 |
| $L^4(V_{13}) = V_2 + 4V_4 + 5V_8 + 440V_{16}$ | 7098 |
| $L^4(V_{14}) = V_1 + V_2 + 6V_8 + 594V_{16}$ | 9555 |
| $L^4(V_{15}) = 7V_8 + 784V_{16}$ | 12600 |
| $L^4(V_{16}) = 8V_8 + 1016V_{16}$ | 16320 |
| $L^4(V_{17}) = 7V_8 + 785V_{16} + 256V_{32}$ | 20808 |
| $L^4(V_{18}) = V_1 + V_2 + 6V_8 + 596V_{16} + 518V_{32}$ | 26163 |
| $L^4(V_{19}) = V_2 + 4V_4 + 5V_8 + 443V_{16} + 792V_{32}$ | 32490 |
| $L^4(V_{20}) = 2V_2 + 14V_4 + 4V_8 + 320V_{16} + 1084V_{32}$ | 39900 |
| $L^4(V_{21}) = V_2 + 5V_4 + 19V_8 + 221V_{16} + 1400V_{32}$ | 48510 |
| $L^4(V_{22}) = V_1 + V_2 + 2V_4 + 40V_8 + 140V_{16} + 1746V_{32}$ | 58443 |
| $L^4(V_{23}) = 3V_4 + 73V_8 + 71V_{16} + 2128V_{32}$ | 69828 |
| $L^4(V_{24}) = 4V_4 + 124V_8 + 8V_{16} + 2552V_{32}$ | 82800 |
| $L^4(V_{25}) = 3V_4 + 72V_8 + 9V_{16} + 3024V_{32}$ | 97500 |

Remarks

- ▶ Recall that $L^2(V) \cong \Lambda^2(V)$.
- ▶ The conjectured recursion is reminiscent of a theorem of Gow and Laffey on the exterior square:
Let C be a cyclic 2-group of order $q > 1$, K a field of characteristic 2 and let $q/2 < r \leq q$. Then

$$\Lambda^2(V_r) = \Lambda^2(V_{q-r}) \pmod{V_q, V_{3q/2}}.$$

- ▶ The proof of this result does not seem to shed any light on how to prove our conjecture (moreover, the authors ask for a ‘more conceptual proof’ of their result.)

For your consideration...

- ▶ Try to find a ‘conceptual proof’ of the Gow and Laffey result, with a view to extending this in some way to prove our conjectured recursion.

- ▶ Concentrate on the case $n = 4$.

Can **you** prove that for any cyclic 2-group:

- ▶ $\Phi^4 = 0$? (Equivalently, that $L^4(V) \oplus L^4(V) \cong L^2(V^{\otimes 2})$ for all (indecomposable) KC -modules V ?)
 - ▶ $L^4(V_r) = L^4(V_{q-r}) \bmod V_q, V_{q/2}$, for $r = 1, \dots, q$?
 - ▶ The fourth Lie power of an indecomposable KC -module is always a permutation module?
- ▶ Alternatively, concentrate on the case $|C| = 8$.
Can you show that $\Phi^4 = \Phi^8 = \Phi^{16} = \dots = 0$?
(The difficult case here seems to be proving that $\Phi^4(V_6) = \Phi^8(V_6) = \Phi^{16}(V_6) = \dots = 0$.)