Lie powers of modules for cyclic p-groups

Marianne Johnson

12th May 2010

The Green ring of a cyclic p-group

Let K be a field of prime characteristic p,
$C = \langle g \rangle$ a cyclic p-group of order q, where $q > 1$.

Consider finite-dimensional right KC-modules.

The **Green ring** R_{KC} has $\mathbb{Z}$-basis consisting of the indecomposable KC-modules, with multiplication coming from tensor product. We write:

$$\begin{array}{rccc} KC\text{-modules:} & U \oplus V & U \otimes_K V & V^{\otimes n} \\ \text{Elements of } R_{KC}: & U + V & UV & V^n \end{array}$$

The Green ring of a cyclic p-group

There are q indecomposable KC-modules up to isomorphism.
For $r = 1, \ldots, q$, let $V_r = KC/KC(g-1)^r$.
Then V_r is indecomposable of dimension r.

Thus R_{KC} has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$.

Each indecomposable V_r has basis $\{y_1, \ldots, y_r\}$ and the action of g on V_r with respect to this basis is given by the Jordan block

$$\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

(Notice that V_1 is the one-dimensional trivial module and V_q is the regular KC-module.)

Lie powers

Let V be a KC-module with K-basis $\{x_1, \ldots, x_r\}$.
Write $L(V)$ for the free Lie algebra on $x_1, \ldots, x_r$ over K.
Take decompositions into homogeneous components:

$$L(V) = L^1(V) \oplus L^2(V) \oplus \cdots \oplus L^n(V) \oplus \cdots$$

- Each $L^n(V)$ is a finite-dimensional KC-module, called the nth **Lie power** of V, where g acts by linear substitutions.
- The dimension of $L^n(V)$ is given by

$$\dim L^n(V) = \frac{1}{n} \sum_{d|n} \mu(d) r^{n/d},$$

  where μ denotes the Möbius function.

The decomposition problem. For each positive integer n determine $L^n(V)$ up to isomorphism, i.e. as elements of R_{KC}.

Examples

Let $K = \mathbb{F}_3$, $C = \langle g : g^3 = 1 \rangle$, and let $V = V_3$.
Then V_3 has basis $\{y_1, y_2, y_3\}$, where

$$\begin{aligned} y_1 g &= y_1 + y_2, \\ y_2 g &= y_2 + y_3, \\ y_3 g &= y_3. \end{aligned}$$

We want to compute Lie powers $L^n(V_3)$ as elements of R_{KC}.

$L^2(V_3)$ has dimension 3 and basis:

$$[y_1, y_2], [y_1, y_3], [y_2, y_3].$$

The action of g on $L^2(V_3)$ with respect to this basis is given by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, it is easy to see that $L^2(V_3) = V_3$.

Examples

$L^3(V_3)$ has dimension 8 and basis:

$$[y_1, y_2, y_1], [y_1, y_2, y_2], [y_1, y_3, y_1], [y_1, y_3, y_2],$$
$$[y_1, y_3, y_3], [y_2, y_3, y_1], [y_2, y_3, y_2], [y_2, y_3, y_3].$$

The action of g on $L^3(V_3)$ with respect to this basis is given by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be shown that $L^3(V_3) = V_2 + 2V_3$
(e.g. compute the Jordan canonical form of the above matrix).

Lie resolvents

There exist $\mathbb{Z}$-linear functions $\Phi^n : R_{KC} \to R_{KC}$ called the **Lie resolvents** on R_{KG} such that for every KC-module V,

$$L^n(V) = \frac{1}{n} \sum_{d|n} \Phi^d(V^{n/d}).$$

(Note: $\sum_{d|n} \Phi^d(V^{n/d}) \in nR_{KG}$, so division by n makes sense.)
Möbius inversion gives

$$\Phi^n(V) = \sum_{d|n} \mu(n/d) d L^d(V^{n/d}),$$

for all KC-modules V.
Thus we see that Φ^1 is the identity map on R_{KG}.

A factorisation theorem

By a theorem of Bryant and Schocker we have

$$\Phi^{p^m k} = \Phi^{p^m} \circ \Phi^k,$$

for all $m \geqslant 0$ with $p \nmid k$, where $\circ$ denotes composition.

In fact, Φ^k for $p \nmid k$ is well understood and the real interest is in Φ^p, Φ^{p^2}, Φ^{p^3}, ...

Aside:
For $p \nmid k$, $\Phi^k = \mu(k)\psi^k$, where ψ^k denotes the kth Adams operation on R_{KC}.
In recent joint work with Roger Bryant, we gave a description of the maps ψ^k for $p \nmid k$.

The cyclic group of order p

When C is cyclic of order p, the decomposition problem has been solved by Bryant, Kovács and Stöhr.

By our previous remarks, it is enough to describe the Lie resolvents Φ^p, Φ^{p^2}, Φ^{p^3}, ...
It turns out that

$$\Phi^p(V_r) = \begin{cases} V_1 & \text{if } r = 1, \\ p(V_p - V_{p-1}) - (p-r)V_1 & \text{if } r \text{ is odd, } 1 < r \leqslant p, \\ r(V_p - V_{p-1}) & \text{if } r \text{ is even, } 1 < r \leqslant p, \end{cases}$$

whilst $\Phi^{p^2} = \Phi^{p^3} = \cdots = 0$.

(Thus we see that $\Phi^n = 0$ if n is not square-free.)

A conjecture of R. M. Bryant

Let C be a cyclic p-group, K a field of characteristic p. Then

$$\Phi^{p^2} = \Phi^{p^3} = \cdots = 0.$$

Since the Lie resolvents are $\mathbb{Z}$-linear maps, it suffices to show that $\Phi^{p^m}(V_r) = 0$ for $r = 1 \ldots, q$ and all $m > 1$.

Thus, the conjecture is equivalent to the following recursions:

$$p^m L^{p^{m+1}}(V_r) = L^p(V_r^{p^m}), \text{ for } r = 1 \ldots, q,\ m > 0.$$

We have seen that the conjecture holds for C cyclic of order p.

It can also be shown that the conjecture holds for $p = 2$ and C cyclic of order 4. Very little is known for other cyclic p-groups.

Some properties of indecomposable KC-modules

- V_q is the unique projective indecomposable KC-module, up to isomorphism.
- $V_r \otimes V_q \cong rV_q$ for $r = 1, \ldots, q$.
- For $r = 1, \ldots, q$ there is a short exact sequence

$$0 \to V_{q-r} \to V_q \to V_r \to 0$$

  of KC-modules, where $V_{q-r} = \Omega(V_r)$.
- The modules V_{p^i} are **permutation modules** for $1 \leqslant p^i \leqslant q$ and every permutation module can be written as a sum of the V_{p^i}.

A consequence of Bryant's conjecture for $p = 2$

Suppose that we have

$$2^m L^{2^{m+1}}(V_r) = L^2(V_r^{2^m}),$$

for $r = 1 \ldots, q$, $m > 0$.

It can be shown that:

- $V_r^{\otimes 2^m}$ is a permutation module for $r = 1 \ldots, q$, $m > 0$.
- If W is a permutation module then $L^2(W)$ is also a permutation module.

Hence we see that if the conjecture holds then $L^{2^{m+1}}(V_r)$ is a permutation module, for $r = 1 \ldots, q$, $m > 0$.

(Note: This argument is specific to $p = 2$.)

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Another conjecture for cyclic 2-groups

Let C be a cyclic 2-group of order $q > 1$, K a field of characteristic 2 and let $q/2 < r \leqslant q$. Then for $m > 1$

$$L^{2^m}(V_r) = L^{2^m}(V_{q-r}) \bmod V_q, V_{q/2}.$$

(Note: $V_0 = 0$.)

Thus for $r = q$ the conjecture is consistent with the fact that Lie powers of the regular KC-module V_q only involve terms of the form V_q and $V_{q/2}$ (Bryant and Michos).

Similarly, for $r = q - 1$, the conjecture is consistent with the fact that Lie powers of the augmentation ideal V_{q-1} only involve terms of the form V_q and $V_{q/2}$ (Kovács and Stöhr)

In fact, this conjecture implies Bryant's conjecture in the case of cyclic 2-groups.

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Example

												Dimension
$L^4(V_2) =$	V_1	$+$	V_2									3
$L^4(V_3) =$			V_2	$+$	$4V_4$							18
$L^4(V_4) =$			$2V_2$	$+$	$14V_4$							60
$L^4(V_5) =$			V_2	$+$	$5V_4$	$+$	$16V_8$					150
$L^4(V_6) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$					315
$L^4(V_7) =$					$3V_4$	$+$	$72V_8$					588
$L^4(V_8) =$					$4V_4$	$+$	$124V_8$					1008
$L^4(V_9) =$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$			1620
$L^4(V_{10}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$			2475
$L^4(V_{11}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$			3630
$L^4(V_{12}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$			5148
$L^4(V_{13}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$			7098
$L^4(V_{14}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$			9555
$L^4(V_{15}) =$							$7V_8$	$+$	$784V_{16}$			12600
$L^4(V_{16}) =$							$8V_8$	$+$	$1016V_{16}$			16320
$L^4(V_{17}) =$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$	20808
$L^4(V_{18}) =$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$	26163
$L^4(V_{19}) =$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$	32490
$L^4(V_{20}) =$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$	39900
$L^4(V_{21}) =$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$	48510
$L^4(V_{22}) =$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$	58443
$L^4(V_{23}) =$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$	69828
$L^4(V_{24}) =$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$	82800
$L^4(V_{25}) =$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$	97500

Remarks

- Recall that $L^2(V) \cong \Lambda^2(V)$.
- The conjectured recursion is reminiscent of a theorem of Gow and Laffey on the exterior square:
  Let C be a cyclic 2-group of order $q > 1$, K a field of characteristic 2 and let $q/2 < r \leqslant q$. Then
  $$\Lambda^2(V_r) = \Lambda^2(V_{q-r}) \bmod V_q, V_{3q/2-r}.$$
- The proof of this result does not seem to shed any light on how to prove our conjecture (moreover, the authors ask for a 'more conceptual proof' of their result.)

For your consideration...

- Try to find a 'conceptual proof' of the Gow and Laffey result, with a view to extending this in some way to prove our conjectured recursion.
- Concentrate on the case $n = 4$.
  Can **you** prove that for any cyclic 2-group:
  - $\Phi^4 = 0$? (Equivalently, that $L^4(V) \oplus L^4(V) \cong L^2(V^{\otimes 2})$ for all (indecomposable) KC-modules V?)
  - $L^4(V_r) = L^4(V_{q-r}) \bmod V_q, V_{q/2}$, for $r = 1, \ldots, q$?
  - The fourth Lie power of an indecomposable KC-module is always a permutation module?
- Alternatively, concentrate on the case $|C| = 8$.
  Can you show that $\Phi^4 = \Phi^8 = \Phi^{16} = \cdots = 0$?
  (The difficult case here seems to be proving that $\Phi^4(V_6) = \Phi^8(V_6) = \Phi^{16}(V_6) = \cdots = 0$.)