Lie representations of $GL(V)$

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Outline

- Tensor representations of $GL(V)$
- Lie representations of $GL(V)$
- Klyachko's Theorem
- A combinatorial proof

Tensor representations of $GL(V)$

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- V is the natural module for $GL(V)$
- T is a $GL(V)$-module
- Each T_n is a $GL(V)$-submodule of T called the nth tensor representation.

Tensor representations of $GL(V)$

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e.g. $\lambda = (4, 2, 2, 1, 1) = (4, 2^2, 1^2) \vdash 10$

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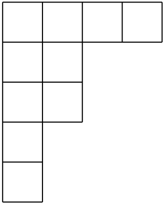

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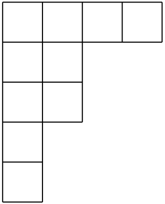

A standard tableau of shape λ is a Young diagram numbered with $\{1, \ldots, n\}$ such that the entries increase along every row and down every column.

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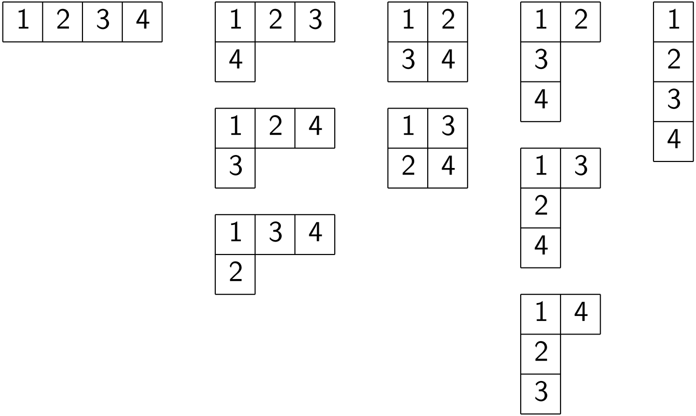
- It turns out that
  $t_\lambda =$ number of standard tableaux of shape λ.

Tensor representations of $GL(V)$

Example:

$$T_4 \cong [4] \oplus 3\,[3,1] \oplus 2\,[2^2] \oplus 3\,[2,1^2] \oplus [1^4]$$



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- What is l_λ?

Lie representations of $GL(V)$

Decomposition into irreducibles			Missing
L_1	$\cong$	$[1]$	$-$
L_2	$\cong$	$[1^2]$	$[2]$
L_3	$\cong$	$[2,1]$	$[3],[1^3]$
L_4	$\cong$	$[3,1]\oplus[2,1^2]$	$[4],[2^2],[1^4]$
L_5	$\cong$	$[4,1]\oplus[3,2]\oplus[3,1^2]\oplus[2^2,1]\oplus[2,1^3]$	$[5],[1^5]$
L_6	$\cong$	$[5,1]\oplus[4,2]\oplus 2[4,1^2]\oplus[3^2]\oplus 3[3,2,1]$ $\oplus[3,1^3]\oplus 2[2^2,1^2]\oplus[2,1^4]$	$[6],[2^3],[1^6]$

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It is difficult to see in general which modules actually occur in the decomposition of L_n, that is, for which λ we have $l_\lambda > 0$.

Klyachko's Theorem (1974)

Let $n \geq 3$ and let $\lambda \vdash n$ with no more than $\dim(V)$ parts. Then

$$l_\lambda > 0 \Leftrightarrow \lambda \neq (1^n), (n), (2^2), (2^3).$$

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In other words, almost every irreducible $GL(V)$ module occurs in the Lie representation.

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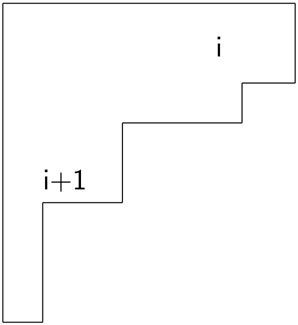

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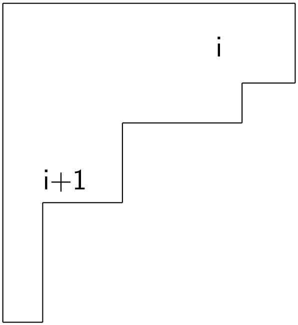

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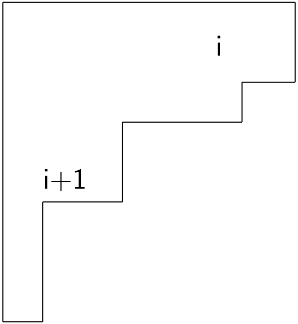

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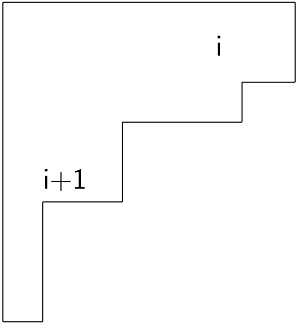

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$\mathrm{maj}(T) = \sum_{i \in D(T)} i$

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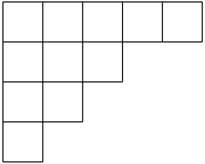

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Remarks:

- $D(T) \subseteq \{1,\ldots,n-1\}$

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- $D(T) \subseteq \{1, \ldots, n-1\}$
- $k - 1 \leq |D(T)| \leq n - \lambda_1$

Kraśkiewicz-Weyman Theorem (1987)

Let i and n be coprime.

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- Note that i can be any fixed number which is coprime to n.
- It is natural to try to prove Klyachko's Theorem using the Kraśkiewicz-Weyman Theorem

Theorem

Let $n \geq 3$, $\lambda \vdash n$.
$\exists$ a standard tableau of shape λ with major index coprime to n

$$\iff \lambda \neq (1^n), (n), (2^2), (2^3)$$

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1	2
3	4

2

1	3
2	4

4

1	2
3	4
5	6

6

1	2
3	5
4	6

10

1	3
2	4
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- Rectangles.
- Non-rectangular partitions into more than two parts.

Two part partitions

$n = 2m + 1$, $\lambda = (n - s, s)$:

1	...	s	...	m	m+s+1	...	2m+1
m+1	...	m+s					

$n = 2m$, $\lambda = (n - s, s)$, $1 < s < m$:

1	2	...	s	...	m-1	m+1	m+2	m+s+2	...	2m
m	m+3	...	m+s+1							

1	3	...	2m
2			

1	2	3	...	m-1	m+2
m	m+1	m+3	...	2m-1	2m

Rectangles

Let $n = mk$, $\lambda = (m^k) \vdash n$ $\qquad 0 \le i \le k-2 \qquad 1 \le s \le m-1.$

$T =$

1	...					m
⋮						⋮
(i-1)m+1	...					im
im+1	im+2	...	im+s	im+s+2	...	(i+1)m+1
im+s+1	(i+1)m+2	...				(i+2)m
⋮						⋮
(k-2)m+1	...					(k-1)m
(k-1)m+1	...					km

- $\mathrm{maj}(T) = \frac{mk(k-1)}{2} + im + s + 1$
- Show that one of these is coprime to n (technical)

The rest

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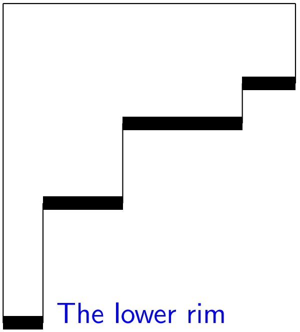

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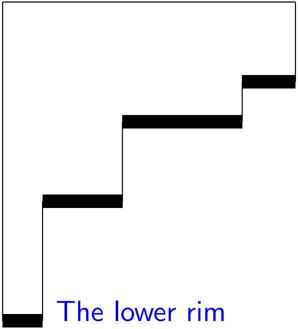

Can remove m_i boxes from the lower rim of $\lambda^{(i)}$ to obtain a Young diagram $\lambda^{(i-1)}$ which has $i - 1$ rows.

The rest

- For every choice $m_1, \ldots, m_k$ we can construct a standard tableau T of shape λ with descent set

$$D(T) = \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_{k-1}\}$$

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- Put the entries

$$m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m_i$$

from left to right in $\lambda^{(i)} \setminus \lambda^{(i-1)}$

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  from left to right in $\lambda^{(i)} \setminus \lambda^{(i-1)}$
- It can be shown that one of these descent sets gives major index which is coprime to n.