Classical Young Measures in the Calculus of Variations

Author: Marcus Webb Supervisor: Filip Rindler

Cambridge Centre for Analysis

May 27, 2013

Abstract

Young measures are a way of understanding the limiting behaviour of a sequence of measurable functions which encompasses how the sequence behaves under composition with a continuous function. This is useful in the Calculus of Variations, particularly for the direct method, where the integral functional is not necessarily continuous with respect to the convergence associated with minimising sequences. In this essay Young measures are introduced and used to prove weak lower semi-continuity results for such functionals, with discussion of the rôle of quasiconvexity for functionals defined on Sobolev spaces.

1 Introduction

The Calculus of Variations is a field of mathematical analysis primarily concerned with functionals $I : X \to \mathbb{R} \cup \{+\infty\}$, where X is a Banach space, and their minimisation over a subset $\mathcal{A} \subseteq X$. The so-called "Direct Method" is a way of proving the existence of an element $\bar{u} \in \mathcal{A}$ that minimises I. Specifically, the problem is often reduced to the satisfaction of conditions $(D1), (D2)$ and $(D3)$ for the following theorem, leaving regularity results until later:

Theorem 1.1 (The Direct Method for Existence)**.** *Let $I : X \to \mathbb{R} \cup \{+\infty\}$, where X is a reflexive Banach space, and let $\mathcal{A} \subseteq X$. Suppose the following three conditions hold:*

(D1) $\mathcal{A}$ is a closed, convex subset of X,

(D2) I is **coercive** *i.e. for $u \in \mathcal{A}$, I is bounded below and*

$$\lim_{\|u\| \to \infty} I(u) = +\infty, \tag{1}$$

(D3) I is **lower semi-continuous** *with respect to the weak topology on X i.e.*

$$u_j \rightharpoonup u \implies \liminf_{j \to \infty} I(u_j) \geq I(u). \tag{2}$$

Then there exists $\bar{u} \in \mathcal{A}$ with $I(\bar{u}) \leq I(u)$ for all $u \in \mathcal{A}$.

Proof. Since I is bounded below, we can define $\alpha = \inf_{u \in \mathcal{A}} I(u)$. If $\alpha = +\infty$ then the theorem is trivial, so we may choose a sequence $\{u_j\}_{j \in \mathbb{N}}$ such that $I(u_j) < +\infty$ and

$$\lim_{j \to \infty} I(u_j) = \alpha. \tag{3}$$

We call $\{u_j\}_{j\in\mathbb{N}}$ a **minimising sequence**. We see that $\{I(u_j)\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, so by the coercivity assumption (D2) we have that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X. Since X is separable and reflexive, the sequential version of the Banach-Alaoglu Theorem (Corollary A.6) implies the existence of a weakly convergent subsequence in X:

$$u_{j_k} \rightharpoonup \bar{u}. \tag{4}$$

Mazur's Theorem [5, p.70] states that a subset of a Banach space that is both closed and convex is also weakly closed. Hence by (D1), $\bar{u} \in \mathcal{A}$. By the lower semi-continuity assumption (D3)

$$\alpha = \inf_{u\in\mathcal{A}} I(u) \le I(\bar{u}) \le \liminf_{k\to\infty} I(u_{j_k}) = \alpha. \tag{5}$$

Hence $\bar{u}$ minimises I over $\mathcal{A}$. □

In this essay, we consider functionals $I : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}\cup\{+\infty\}$ of the form

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x. \tag{6}$$

Here Ω is a bounded, open, measurable subset of $\mathbb{R}^n$, and $1 < p < \infty$ so that $X = W^{1,p}(\Omega;\mathbb{R}^m)$ satisfies the conditions in Theorem 1.1. The endpoint cases $p = 1, \infty$ must be treated carefully, and will be discussed later on.

We would like to prove the existence of a minimiser for I over the set

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega;\mathbb{R}^m) : u - g \in W_0^{1,p}(\Omega;\mathbb{R}^m) \right\}, \tag{7}$$

for a given function $g \in W^{1,p}(\Omega;\mathbb{R}^m)$. Here $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is the closure of the space of compactly supported functions in $W^{1,p}(\Omega,\mathbb{R}^m)$. Such a set $\mathcal{A}$ is therefore closed and convex, so satisfies condition (D1) of Theorem 1.1.

We assume that $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$ is a **Carathéodory function**, i.e. it is measurable in its first argument, and continuous in all others. To satisfy condition (D2) of Theorem 1.1, we assume there exists $c > 0$ and $d \ge 0$ such that

$$f(x, \lambda, A) \ge c|A|^p - d. \tag{8}$$

Here $|\cdot|$ denotes the entrywise p-norm on $\mathbb{R}^{m\times n}$. Indeed, then we have that I is bounded below, and by Friedrich's inequality on $W_0^{1,p}(\Omega;\mathbb{R}^m)$ there is a constant $\gamma > 0$ such that

$$\|u\|_p \le \|u-g\|_p + \|g\|_p \le \gamma\|\nabla u - \nabla g\|_p + \|g\|_p \le \gamma\|\nabla u\| + \gamma\|\nabla g\|_p + \|g\|_p.$$

Therefore in $\mathcal{A}$, $\|u\|_{1,p} \to \infty \iff \|\nabla u\|_p \to \infty$, and so,

$$\lim_{\|u\|_{1,p}\to\infty} I(u) = \lim_{\|\nabla u\|_p\to\infty} I(u) \ge \lim_{\|\nabla u\|_p\to\infty} c\|\nabla u\|_p^p - d\cdot|\Omega| = +\infty.$$

Considering the final condition (D3), that of weak lower semi-continuity, it is difficult to see what reasonable condition we can place on f for its satisfaction. It is nontrivial; in fact, the remainder of the essay is devoted to understanding the solution to this problem!

In Section 2, we discuss some of the fine details of weak convergence in Lebesgue spaces. In Section 3, we introduce Young Measures as a way to understand these weak limits under composition by a continuous function f. In Section 4, we prove weak lower semi-continuity results for functionals of the form (6).

2 Oscillations, Concentrations and Weak Convergence

In this section we discuss weak convergence in Lebesgue spaces in order to better understand minimising sequences. We let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions from Ω into $\mathbb{R}^m$ throughout. We must stress the assumption that Ω is of finite measure. Some of the results here can be made to apply to more general domains, but only with distracting technicalities.

Definition 2.1. We say that $\{u_j\}_{j\in\mathbb{N}}$ **converges in measure** to u, written $u_j \xrightarrow{m} u$, if

$$\forall\varepsilon > 0, \quad \lim_{j\to\infty} |\{x \in \Omega : |u_j(x) - u(x)| > \varepsilon\}| = 0. \tag{9}$$

Definition 2.2. We say that $\{u_j\}_{j\in\mathbb{N}}$ is **uniformly integrable**, abbreviated to "UI", if

$$\lim_{M\to\infty} \sup_j \int_{\{|u_j|\geq M\}} |u_j(x)| \,\mathrm{d}x = 0. \tag{10}$$

We say that $\{u_j\}_{j\in\mathbb{N}}$ is **uniformly p-integrable**, abbreviated to "p-UI", if $\{|u_j|^p\}_{j\in\mathbb{N}}$ is UI.

Remark 2.3. Note that if a sequence is p-UI then it is bounded in $L^p(\Omega;\mathbb{R}^m)$. The partial converse is that if a sequence is bounded in $L^p(\Omega;\mathbb{R}^m)$ then it is q-UI for any $1 \leq q < p$.

Theorem 2.4 (Vitali's Convergence Theorem)**.** *Let $\{u_j\}_{j\in\mathbb{N}}$ and u be measurable functions from Ω into $\mathbb{R}^m$, and let $1 \leq p < \infty$. Then $u_j \to u$ in $L^p(\Omega;\mathbb{R}^m)$ if and only if*

(V1) $u_j \xrightarrow{m} u$

(V2) $\{u_j\}_{j\in\mathbb{N}}$ is p-UI

Remark 2.5. For a proof, see [11, p.165]. A corollary is the Dominated Convergence Theorem.

Theorem 2.6 (Characterisations of weak convergence)**.** *Let $\{u_j\}_{j\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 \leq p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following three statements are equivalent:*

1. *$u_j \rightharpoonup u$ ($u_j \overset{\star}{\rightharpoonup} u$ if $p = \infty$)*
2. *$\int_\Omega u_j(x)\cdot g(x)\,\mathrm{d}x \to \int_\Omega u(x)\cdot g(x)\,\mathrm{d}x \quad \forall g \in L^q(\Omega;\mathbb{R}^m)$*
3. *(W1) $\int_B u_j(x)\,\mathrm{d}x \to \int_B u(x)\,\mathrm{d}x \quad \forall B \in \mathcal{B}(\Omega)$ (i.e. all Borel-measurable subsets $B \subseteq \Omega$)*
   (W2) For $p > 1$, $\sup_j \|u_j\|_p < \infty$. For $p = 1$, $\{u_j\}_{j\in\mathbb{N}}$ is UI.

Remark 2.7. Note that by the second statement, weak* convergence in L^∞ is the strongest of these notions of convergence, and weak convergence in L^1 is the weakest.

A consequence of Vitali's Convergence Theorem 2.4 is that if $\{u_j\}_{j\in\mathbb{N}}$ is weakly convergent but not convergent in norm then at least one of conditions (V1) and (V2) above must fail. Informally, if (V1) fails we say that the sequence **oscillates**, and if (V2) fails we say that the sequence **concentrates**. A consequence of Theorem 2.6 is that concentrations do not occur for sequences converging weakly in $L^1(\Omega;\mathbb{R}^m)$, because they are uniformly integrable by (W2). We demonstrate this with the following examples:

Example 2.8 (Oscillation)**.** Define $\chi : \mathbb{R} \to \mathbb{R}$ by

$$\chi(x) = \mathbb{1}_{[0,1/2]}(x) - \mathbb{1}_{[1/2,1]}(x), \quad x \in [0,1], \tag{11}$$

extended 1-periodically to $\mathbb{R}$, and define $u_j(x) = \chi(jx)$ on $(0,1)$.

Then $u_j \rightharpoonup 0$ in $L^p(0,1)$ ($u_j \overset{\star}{\rightharpoonup} 0$ for $p = \infty$), using Theorem 2.6. However, $u_j \not\to 0$ in $L^p(0,1)$, because $\{u_j\}_{j\in\mathbb{N}}$ does not satisfy (V1).

Example 2.9 (Concentration)**.** Define the sequence $\{v_j\}_{j\in\mathbb{N}} \subset L^p(0,1)$ $(1 \le p < \infty)$ by

$$v_j(x) = j^{1/p}\mathbb{1}_{[0,1/j)}(x). \tag{12}$$

Then by Theorem 2.6, $v_j \rightharpoonup 0$ in $L^p(0,1)$ for $p > 1$, but **not** for $p = 1$. If we let $g \in C([0,1])$ then

$$\begin{aligned}\int_0^1 v_j(x)g(x)\,\mathrm{d}x &= \int_0^{1/j} jg(x)\,\mathrm{d}x \\ &= \int_0^1 g(j^{-1}x)\,\mathrm{d}x \\ &\to g(0) \text{ as } j \to \infty.\end{aligned}$$

Here we see that $\{v_j\}_{j\in\mathbb{N}}$ does not converge weakly in $L^1(0,1)$, but it does converge weakly* in the dual space $(C([0,1]))^\star$ to the Dirac delta measure at zero, δ_0. We do not have norm convergence in any of the cases, since $\{v_j\}_{j\in\mathbb{N}}$ is not p-UI.

Example 2.10 (Oscillation and concentration)**.** Consider the sum of the above sequences, $w_j = u_j + v_j$. Then for $1 < p < \infty$, $w_j \rightharpoonup 0$ in $L^p(0,1)$, but the sequence satisfies neither (V1) nor (V2).

Suppose we have a sequence of functions $\{u_j\}_{j\in\mathbb{N}}$ such that $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ ($u_j \overset{\star}{\rightharpoonup} u$ if $p = \infty$) and we take a continuous function $f : \mathbb{R}^m \to \mathbb{R}$. If $\{f(u_j(\cdot))\}_{j\in\mathbb{N}}$ converges weakly in $L^p(\Omega;\mathbb{R}^m)$ too, then what is the limit?

If we consider the oscillation example above, and let $f : \lambda \mapsto \lambda^2$, then $f(u_j(x)) = \mathbb{1}(x)$ so that $f(u_j(\cdot)) \overset{\star}{\rightharpoonup} \mathbb{1}$. Hence, in general $f(u_j(\cdot))$ does not converge weakly to $f(u(\cdot))$. For general $f : \mathbb{R} \to \mathbb{R}$ continuous and g smooth with compact support in $[0,1]$, we have

$$\begin{aligned}\int_0^1 g(x)f(u_j(x))\,\mathrm{d}x &= -\int_0^1 g'(x)\int_0^x f(u_j(y))\,\mathrm{d}y\,\mathrm{d}x \\ &= -\int_0^1 g'(x)j^{-1}\int_0^{jx} f(\chi(y))\,\mathrm{d}y\,\mathrm{d}x \\ &= -\int_0^1 g'(x)\left(j^{-1}\lceil jx\rceil\frac{1}{2}\left(f(1)+f(-1)\right) - j^{-1}\int_{jx}^{\lceil jx\rceil} f(\chi(y))\,\mathrm{d}y\right)\mathrm{d}x \\ &\to -\int_0^1 g'(x)x\frac{1}{2}\left(f(1)+f(-1)\right)\mathrm{d}x \\ &= \int_0^1 g(x)\frac{1}{2}\left(f(1)+f(-1)\right)\mathrm{d}x \text{ as } j\to\infty.\end{aligned}$$

Since the space of all such g is dense in $L^1(0,1)$ we have that $f(u_j(\cdot)) \overset{\star}{\rightharpoonup} \frac{1}{2}(f(1)+f(-1))$ in $L^\infty(0,1)$.

Now consider the concentration example for a fixed $p \in [1,\infty)$. If $f : \mathbb{R} \to \mathbb{R}$ continuous then it is **not** necessarily true that $f(v_j(\cdot)) = f(j^{1/p})\mathbb{1}_{[0,1/j)} + f(0)\mathbb{1}_{[1/j,1]}$ is uniformly integrable. Take for example $f(\lambda) = |\lambda|^p$. By Theorem 2.6 then, it is not even necessarily true that the sequence converges weakly in $L^1(0,1)$. But if we assume that $\{f(v_j(\cdot))\}_{j\in\mathbb{N}}$ is uniformly integrable, then since $f(v_j(\cdot)) \overset{m}{\to} f(0)$ as $j \to \infty$, we can deduce from Vitali's Convergence Theorem 2.4 that $f \to f(0)$ in $L^1(0,1)$.

Put plainly, in the UI case, $f(v_j(\cdot)) \to f(v(\cdot))$, where v is the limit in measure of $\{v_j\}_{j\in\mathbb{N}}$. The concentration is either ignored completely in the limit, or behaves badly because $\{f(v_j(\cdot))\}_{j\in\mathbb{N}}$ is not uniformly integrable. We will see that the behaviour exhibited here is typical of oscillations and concentrations in general.

3 Weak Convergence, Duality and Young Measures

In this section, we seek to understand how $f(\cdot, u_j(\cdot)) : \Omega \to \mathbb{R}$ behaves for a Carathéodory function f and a weakly convergent sequence $\{u_j\}_{j\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$. Our approach here will be to construct a Banach space that is a subset of the Carathéodory functions, for which we can use theorems of functional analysis, then extend our results to the space of all Carathéodory functions.

Definition 3.1 (Simple functions)**.** For a Banach space X, a function $S : \Omega \to X$ is said to be **simple**, if it is of the form

$$S(x) = \sum_{k=1}^{K} \mathbb{1}_{B_k}(x) \cdot \varphi_k, \tag{13}$$

where $K \in \mathbb{N}$, and $\varphi_k \in X$ and $B_k \subset \mathcal{B}(\Omega)$ for each k. They are a generalisation of step functions.

Definition 3.2 (Strongly measurable functions)**.** A function $F : \Omega \to X$ is said to be **strongly measurable** if there exists a sequence of simple functions $\{S_i\}_{i\in\mathbb{N}}$ such that

$$\|S_i(x) - F(x)\|_X \to 0 \text{ as } i \to \infty \text{ for a.e. } x \in \Omega. \tag{14}$$

The vector space of strongly measurable functions is denoted $L^0(\Omega; X)$.

Definition 3.3 (Lebesgue spaces for Banach space valued functions)**.** For a Banach space X and $1 \leq p \leq \infty$, we define the space

$$L^p(\Omega; X) = \left\{F \in L^0(\Omega; X) : (x \mapsto \|F(x)\|_X) \in L^p(\Omega;\mathbb{R})\right\} \tag{15}$$

Remark 3.4. When endowed with the norm $\|F\|_{p,X} = \|x \mapsto \|F(x)\|_X\|_p$, the space is a Banach space. It is separable if and only if X is separable and $p < \infty$.

For any $E \subset \mathbb{R}^m$ define the space of continuous functions that vanish at infinity:

$$C_0(E) = \left\{\varphi : E \to \mathbb{R} \text{ continuous and bounded } : \lim_{|\lambda|\to\infty} \varphi(\lambda) = 0\right\}. \tag{16}$$

We endow $C_0(E)$ with the supremum norm, denoted $\|\cdot\|_\infty$, to make it a separable Banach space. By Remark 3.4, $L^1(\Omega; C_0(\mathbb{R}^m))$ is also a separable Banach space. Note that we can consider this space as a subset of the Carathéodory functions by defining $f(x, \lambda) = (F(x))(\lambda)$ for each $F \in L^1(\Omega; C_0(\mathbb{R}^m))$. We alert the reader to the fact that we will abuse notation and switch between F and f without mention.

Now, for a fixed $u \in L^p(\Omega;\mathbb{R}^m)$ consider the following functional ν over $f \in L^1(\Omega; C_0(\mathbb{R}^m))$:

$$\nu(f) = \int_\Omega f(x, u(x))\,\mathrm{d}x. \tag{17}$$

We can see that ν is a bounded linear functional with norm 1; it is therefore a member of the dual space $(L^1(\Omega; C_0(\mathbb{R}^m)))^\star$. Then for the weakly convergent sequence $\{u_j\}_{j\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ (weakly* convergent for $p = \infty$), we can associate functionals $\{\nu^{(j)}\}_{j\in\mathbb{N}}$ to see the limit in a more functional analytic setting:

$$\lim_{j\to\infty} \int_\Omega f(x, u_j(x))\,\mathrm{d}x = \lim_{j\to\infty} \nu^{(j)}(f). \tag{18}$$

Our goal now is to understand this dual space $(L^1(\Omega; C_0(\mathbb{R}^m)))^\star$.

Definition 3.5 (Radon measures)**.** Let $E \in \mathcal{B}(\mathbb{R}^m)$. A **Radon measure** on E is a measure over $\mathcal{B}(E)$ with values in $[-\infty, \infty]$, such that every compact set has finite measure. For every Radon measure μ there is a positive Radon measure $|\mu|$ called the total variation measure:

$$|\mu|(B) = \sup \left\{ \sum_{i \in \mathbb{N}} |\mu(B_i)| : \{B_i\}_{i \in \mathbb{N}} \subset \mathcal{B}(E) \text{ partitions } B \right\} \tag{19}$$

The Banach space of all Radon measures on E with $\|\mu\|_{\mathbf{M}} := |\mu|(E) < \infty$ is denoted $\mathbf{M}(E)$.

The cone of all **positive Radon measures** is denoted $\mathbf{M}^+(E)$. The set of all **probability measures** over $\mathcal{B}(E)$, which are necessarily Radon measures, is denoted $\mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$.

Theorem 3.6 (Riesz-Alexandrov Representation Theorem)**.** *There is an isometric isomorphism between $\mathbf{M}(E)$ and the dual space $(C_0(E))^\star$, with the duality pairing given by [1, p.61]:*

$$\langle \mu, \varphi \rangle = \int_E \varphi \, \mathrm{d}\mu, \quad \mu \in \mathbf{M}(E), \, \varphi \in C_0(E). \tag{20}$$

Definition 3.7 (Weakly* measurable functions)**.** For a normed space X, a function $\nu : \Omega \to X^\star$ is said to be **weakly* measurable** if the function

$$x \mapsto \langle \nu(x), \varphi \rangle \tag{21}$$

is measurable from Ω into $\mathbb{R}$ for every $\varphi \in X$. The vector space of weakly* measurable functions is denoted $L^0_{w^\star}(\Omega; X^\star)$. We define $L^p_{w\star}(\Omega; X^\star)$ in the obvious way.

Remark 3.8. To make the notation clearer, for a weakly* measurable function ν and $x \in \Omega$ we will often denote $\nu(x)$ by ν_x.

Theorem 3.9 (General duality theorem for Lebesgue spaces)**.** *Let X be a separable Banach space, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(L^p(\Omega; X))^\star \cong L^q_{w\star}(\Omega; X^\star) \tag{22}$$

with equal norms, by the duality relation

$$\langle\langle \nu, F \rangle\rangle = \int_\Omega \langle \nu(x), F(x) \rangle \, \mathrm{d}x, \tag{23}$$

for $\nu \in L^q_{w\star}(\Omega; X^\star)$ and $F \in L^p(\Omega; X)$.

Remark 3.10. A reference for this powerful theorem is [3]. The weak* measurable condition can be dropped if (but not only if) X is reflexive.

The isomorphism that interests us is: $(L^1(\Omega; C_0(\mathbb{R}^m))^\star \cong L^\infty_{w\star}(\Omega; \mathbf{M}(\mathbb{R}^m))$. With this we are finally ready to prove the theorem introducing Young Measures.

Theorem 3.11 (Fundamental theorem for Young measures)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and measurable, and let $\{u_j\}_{j \in \mathbb{N}}$ be measurable functions from Ω into $\mathbb{R}^m$ satisfying the* **tightness condition**

$$\lim_{M \to \infty} \sup_j |\{x \in \Omega : |u_j(x)| \geq M\}| = 0. \tag{24}$$

Then there exists a subsequence $\{u_{j_k}\}_{k \in \mathbb{N}}$ and a weakly* measurable function $\nu : \Omega \to \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$ such that: If f is a Carathéodory function and $\{f(\cdot, u_{j_k}(\cdot))\}_{k \in \mathbb{N}}$ is uniformly integrable, then

$$\lim_{k \to \infty} \int_\Omega f(x, u_{j_k}(x)) \, \mathrm{d}x = \int_\Omega \bar{f}(x) \, \mathrm{d}x, \tag{25}$$

where

$$\bar{f}(x) = \langle \nu_x, f(x,\cdot)\rangle = \int_{\mathbb{R}^m} f(x,\lambda)\,\mathrm{d}\nu_x(\lambda). \tag{26}$$

Remarks 3.12. 1. The map $\nu : \Omega \to \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$ is called the **Young measure** generated by (or associated to) the sequence $\{u_{j_k}\}_{k\in\mathbb{N}}$.

2. Condition (24) is very weak, and is equivalent to there being a continuous, non-decreasing function $g : [0,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} g(t) = +\infty$ such that $\sup_j \int_\Omega g(|u_{j_k}(x)|)\,\mathrm{d}x < \infty$ [2].

Proof. To each u_j assign a $\nu^{(j)} \in L^\infty_{w^\star}(\Omega; \mathbf{M}(\mathbb{R}^m))$ with

$$\nu^{(j)}_x = \delta_{u_j(x)} \text{ for a.e. } x \in \Omega. \tag{27}$$

Then $\|\nu^{(j)}\|_{\infty,\mathbf{M}} = \operatorname{ess\,sup}_{x\in\Omega} \|\delta_{u_j(x)}\|_{\mathbf{M}} = 1$. The sequence is therefore bounded in $L^\infty_{w^\star}(\Omega; \mathbf{M}(\mathbb{R}^m))$, which is the dual of the separable Banach space $L^1(\Omega; C_0(\mathbb{R}^m))$ (Theorem 3.9). By the sequential version of the Banach-Alaoglu Theorem (Theorem A.3), there exists a weakly* convergent subsequence:

$$\nu^{(j_k)} \overset{\star}{\rightharpoonup} \nu \text{ in } L^\infty_{w^\star}(\Omega; \mathbf{M}(\mathbb{R}^m)). \tag{28}$$

This is precisely statement (25), but only for $f \in L^1(\Omega; C_0(\mathbb{R}^m))$. In particular, if we let $f(x,\lambda) = g(x)\varphi(\lambda)$ for $g \in L^1(\Omega; \mathbb{R}^m)$ and $\varphi \in C_0(\mathbb{R}^m)$ with $g, \varphi \geq 0$, then:

$$0 \leq \lim_{k\to\infty} \int_\Omega g(x)\varphi(u_{j_k}(x))\,\mathrm{d}x = \int_\Omega g(x) \int_{\mathbb{R}^m} \varphi(\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x. \tag{29}$$

This shows that $\nu_x \geq 0$ for a.e. $x \in \Omega$. We will show that $\nu_x \in \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$ at the end of the proof.

Suppose that f is Carathéodory and $\{f(\cdot, u_{j_k}(\cdot))\}_{k\in\mathbb{N}}$ is UI. Without loss of generality we may assume that $f \geq 0$, because if f is UI then its positive and negative parts are also.

Now, by (24), we can find a sequence $\{M_i\}_{i\in\mathbb{N}}$ such that

$$\sup_k |\{|u_{j_k}| \geq M_i\}| \leq i^{-2}. \tag{30}$$

For each $\alpha > 0$ define the continuous cutoff function $\Gamma_\alpha : [0,\infty) \to [0,1]$ by

$$\Gamma_\alpha(t) = \mathbb{1}_{[0,\alpha)}(t) + (\alpha + 1 - t)\mathbb{1}_{[\alpha,\alpha+1)}(t). \tag{31}$$

Now for each $i \in \mathbb{N}$ consider the $L^1(\Omega; C_0(\mathbb{R}^m))$ functions:

$$f_i(x,\lambda) = \Gamma_i(f(x,\lambda))\Gamma_{M_i}(|\lambda|)f(x,\lambda). \tag{32}$$

Then

$$\begin{aligned}
0 &\leq \int_\Omega f(x, u_{j_k}(x)) - f_i(x, u_{j_k}(x))\,\mathrm{d}x \\
&\leq \int_{\{|u_{j_k}| \geq M_i\} \cup \{f(x,u_{j_k}(x)) \geq i\}} f(x, u_{j_k}(x))\,\mathrm{d}x \\
&= \int_{\{f(x,u_{j_k}(x)) \geq i\}} f(x, u_{j_k}(x))\,\mathrm{d}x + \int_{\{|u_{j_k}| \geq M_i\} \cap \{f(x,u_{j_k}(x)) \leq i\}} f(x, u_{j_k}(x))\,\mathrm{d}x \\
&\leq \sup_k \int_{\{f(x,u_{j_k}(x)) \geq i\}} f(x, u_{j_k}(x))\,\mathrm{d}x + i \sup_k |\{|u_{j_k}| \geq M_i\}|\,.
\end{aligned}$$

The right hand side converges to 0 as $i \to \infty$, uniformly in k. This uniformity allows us to change the order of limits in the following sequence:

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega f(x,u_{j_k}(x))\,\mathrm{d}x &= \lim_{k\to\infty}\lim_{i\to\infty}\int_\Omega f_i(x,u_{j_k}(x))\,\mathrm{d}x \\ &= \lim_{i\to\infty}\lim_{k\to\infty}\int_\Omega f_i(x,u_{j_k}(x))\,\mathrm{d}x \\ &= \lim_{i\to\infty}\int_\Omega\int_{\mathbb{R}^m} f_i(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x.\end{aligned}$$

Note that $0 \le f_i(x,\lambda) \nearrow f(x,\lambda)$ as $i \to \infty$ for a.e. $(x,\lambda) \in \Omega \times \mathbb{R}^m$, and $\nu_x \ge 0$, so by the Monotone Convergence Theorem we have (25) for the required f.

Now we show that $\nu_x \in \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$. Let B be a measurable subset of Ω and consider (25) for the function $f(x,\lambda) = \mathbb{1}_B(x)$. Then we have

$$\begin{aligned}\int_B \|\nu_x\|_{\mathbf{M}}\,\mathrm{d}x &= \int_\Omega \langle \nu_x, \mathbb{1}_B(x)\rangle\,\mathrm{d}x \\ &= \lim_{k\to\infty}\int_\Omega \mathbb{1}_B(x)\,\mathrm{d}x \\ &= |B|.\end{aligned}$$

Since B was arbitrary, $\|\nu_x\|_{\mathbf{M}} = 1$ for a.e. $x \in \Omega$. □

We denote the set of all Young measures by $\mathbf{Y}(\Omega;\mathbb{R}^m)$, and if $\{u_j\}_{j\in\mathbb{N}}$ has a *subsequence* that generates a Young measure ν, we write $u_j \xrightarrow{Y} \nu$. The fundamental property of Young measures is given in (25), but ν is determined uniquely by its action on a dense subset of $L^1(\Omega;C_0(\mathbb{R}^m))$, so to find ν generated by $\{u_j\}_{j\in\mathbb{N}}$ we need only compute the limits:

$$\lim_{j\to\infty}\int_B \varphi(u_j(x))\,\mathrm{d}x \tag{33}$$

for all $B \in \mathcal{B}(\Omega)$ and $\varphi \in C_0(\mathbb{R}^m)$. In fact, it can be shown that all weakly* measurable functions $\nu : \Omega \to \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m)$ are Young measures in $\mathbf{Y}(\Omega;\mathbb{R}^m)$. In other words, there exists a sequence of measurable functions $\{u_j\}_{j\in\mathbb{N}}$ such that $u_j \xrightarrow{Y} \nu$. This can be shown by construction of the sequence, or using the Hahn-Banach Theorem [9].

Corollary 3.13 (Young measures capture weak limits)**.** *Let $1 \le p \le \infty$ and suppose that $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ ($u_j \overset{\star}{\rightharpoonup} u$ if $p = \infty$). Then $u_j \xrightarrow{Y} \nu$ for some $\nu \in \mathbf{Y}(\Omega;\mathbb{R}^m)$, and for any such Young measure generated, $u(x)$ is the expected value of ν_x for almost every $x \in \Omega$:*

$$u(x) = \int_{\mathbb{R}^m} \lambda\,\mathrm{d}\nu_x(\lambda). \tag{34}$$

Proof. The sequence $\{u_j\}_{j\in\mathbb{N}}$ satisfies the tightness condition by Remark 3.12, so let $\{u_{j_k}\}_{k\in\mathbb{N}}$ be a subsequence generating a Young measure ν.

By Theorem 2.6 and Remark 2.3 $\{u_{j_k}\}_{k\in\mathbb{N}}$ is UI, so for any $g \in L^q(\Omega;\mathbb{R}^m)$, $\{g \cdot u_{j_k}\}_{k\in\mathbb{N}}$ is UI. So if we let $f(x,\lambda) = g(x)\cdot\lambda$, then by the Fundamental Theorem 3.11, we have:

$$\lim_{k\to\infty}\int_\Omega g(x)\cdot u_{j_k}(x)\,\mathrm{d}x = \int_\Omega g(x)\cdot\int_{\mathbb{R}^m}\lambda\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x. \tag{35}$$

By uniqueness of weak and weak* limits, $u(x) = \int_{\mathbb{R}^m}\lambda\,\mathrm{d}\nu_x(\lambda)$ for a.e. $x \in \Omega$. □

Corollary 3.14 (Young measures ignore concentrations)**.** *Let* $\{u_j\}_{j\in\mathbb{N}}$ *be measurable functions from* Ω *into* $\mathbb{R}^m$*. Then* $u_j \overset{m}{\to} u$ *if and only if* $\{u_j\}_{j\in\mathbb{N}}$ *generates the Young measure* $\nu_x = \delta_{u(x)}$*.*

Proof. First suppose that $u_j \overset{m}{\to} u$ and let $\varphi \in C_0(\mathbb{R}^m)$. Note that φ is uniformly continuous, so that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|\{|\varphi(u_j) - \varphi(u)| \geq \varepsilon\}| \leq |\{|u_j - u| \geq \delta\}| \to 0$ as $j \to 0$. Hence $\varphi(u_j(\cdot)) \overset{m}{\to} \varphi(u(\cdot))$ and so by the Dominated Convergence Theorem, for any $B \in \mathcal{B}(\Omega)$,

$$\lim_{j\to\infty} \int_B \varphi(u_j(x))\,\mathrm{d}x = \int_B \varphi(u(x))\,\mathrm{d}x. \tag{36}$$

Hence $\{u_j\}_{j\in\mathbb{N}}$ generates the Young measure $\nu_x = \delta_{u(x)}$.

Now suppose that $\{u_j\}_{j\in\mathbb{N}}$ generates $\nu_x = \delta_{u(x)}$. By Markov's inequality, for any $\varepsilon > 0$,

$$|\{x \in \Omega : |u_j(x) - u(x)| \geq \varepsilon\}| \leq \varepsilon^{-1} \int_\Omega \min\{|u_j(x) - u(x)|, \varepsilon\}\,\mathrm{d}x. \tag{37}$$

If we consider the Carathéodory function $f(x,\lambda) = \min\{|\lambda - u(x)|, \varepsilon\}$, for which $\{f(\cdot, u_j(\cdot))\}_{j\in\mathbb{N}}$ is bounded and therefore UI, we see that this right hand side converges to zero as $j \to \infty$, so that $u_j \overset{m}{\to} u$. □

Now we may ask, what is it that Young measures actually "measure"? Corollary 3.14 shows that Young measures completely ignore concentrations. On the other hand, the following proposition shows that Young measures do capture oscillations:

Proposition 3.15 (Young measure for Riemann-Lebesgue lemma)**.** *Let* $Q = (0,1)^n$ *and* $u \in L^p(Q;\mathbb{R}^m)$ *for* $1 \leq p \leq \infty$*. Define the sequence* $\{u_j\}_{j\in\mathbb{N}} \subset L^p(Q;\mathbb{R}^m)$ *by* $u_j(x) = u(jx)$*, where* u *is extended* Q*-periodically throughout* $\mathbb{R}^n$*. Then* $\{u_j\}_{j\in\mathbb{N}}$ *generates the Young measure* ν *satisfying:*

$$\langle \nu_x, \varphi\rangle = \int_Q \varphi(u(y))\,\mathrm{d}y, \quad \forall \varphi \in C_0(\mathbb{R}^m). \tag{38}$$

Proof. Let g be smooth with compact support contained in Q and let $\varphi \in C_0(\mathbb{R}^m)$. Then since u is Q-periodic we have:

$$\begin{aligned}
\int_Q g(x)\varphi(u_j(x))\,\mathrm{d}x &= \int_Q g(x)\varphi(u(jx))\,\mathrm{d}x \\
&= (-1)^n \int_Q \partial_1 \cdots \partial_n g(x) \left(\int_{xQ} \varphi(u(jy))\,\mathrm{d}y \right) \mathrm{d}x \\
&= (-1)^n \int_Q \partial_1 \cdots \partial_n g(x) \left(j^{-n} \int_{jxQ} \varphi(u(y))\,\mathrm{d}y \right) \mathrm{d}x \\
&= (-1)^n \int_Q \partial_1 \cdots \partial_n g(x) \Bigg(j^{-n} \lceil jx_1 \rceil \cdots \lceil jx_n \rceil \int_Q \varphi(u(y))\,\mathrm{d}y \\
&\qquad\qquad - j^{-n} \int_{(\lceil jx\rceil - jx)Q} \varphi(u(y))\,\mathrm{d}y \Bigg)\,\mathrm{d}x \\
&\to (-1)^n \int_Q \partial_1 \cdots \partial_n g(x) x_1 \cdots x_n \int_Q \varphi(u(y))\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_Q g(x) \int_Q \varphi(u(y))\,\mathrm{d}y\,\mathrm{d}x \text{ as } j \to \infty.
\end{aligned}$$

By the density of the linear span of functions of the form $(x,\lambda) \mapsto g(x)\varphi(\lambda)$ in $L^1(\Omega; C_0(\mathbb{R}^m))$, we have that $\{u_j\}_{j\in\mathbb{N}}$ generates the Young measure in (38). □

Remark 3.16. Note that this Young measure does not depend on x. When this happens, we say that ν is a **homogeneous Young measure**.

Example 3.17 (Oscillation revisited)**.** The sequence in Example 2.8 generates the homogeneous Young measure $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$.

Example 3.18 (Riemann-Lebesgue lemma)**.** Let $\Omega = (0,1)$ and define $u_j(x) = \sin(2\pi jx)$. Then $\{u_j\}_{j\in\mathbb{N}}$ generates the homogeneous Young measure ν satisfying:

$$\begin{aligned}\langle \nu, \varphi\rangle &= \int_0^1 \varphi(\sin(2\pi y))\,\mathrm{d}y \\ &= 2\int_{-1}^1 \varphi(y)\frac{1}{2\pi}(\sin^{-1})'(y)\,\mathrm{d}y \\ &= \int_{-1}^1 \varphi(y)\,\frac{\mathrm{d}y}{\pi\sqrt{1-y^2}}.\end{aligned}$$

In particular, $\sin(2\pi jx) \overset{\star}{\rightharpoonup} 0$ in $L^\infty(0,1)$.

The approach we have taken so far has seen Young measures as being precisely the elements of $L^\infty_{w^\star}(\Omega;\mathbf{M}(\mathbb{R}^m))$ that are almost everywhere probability measures. An issue with this setting is that $\mathbf{Y}(\Omega;\mathbb{R}^m)$ is not a closed subset of $L^\infty_{w^\star}(\Omega;\mathbf{M}(\mathbb{R}^m))$ when equipped with the weak* topology. Consider the sequence of Young measures $\{\nu_x^{(j)}\}_{j\in\mathbb{N}}$ where $\nu_x^{(j)} = \delta_j$, the Dirac delta measure at $j \in \mathbb{R}$, which converges weakly* to $0 \notin \mathbf{Y}(\Omega;\mathbb{R})$.

A helpful alternative is to view Young measures as Radon measures on the product space $\Omega \times \mathbb{R}^m$. Specifically, for a given $\nu \in \mathbf{Y}(\Omega;\mathbb{R}^m)$ and any $B_1 \in \mathcal{B}(\Omega)$, $B_2 \in \mathcal{B}(\mathbb{R}^m)$ we can define

$$\kappa(B_1 \times B_2) = \int_{B_1} \nu_x(B_2)\,\mathrm{d}x. \tag{39}$$

By unique extension, this associates a $\kappa \in \mathbf{M}^+(\Omega\times\mathbb{R}^m)$ to each ν in an injective way with the property

$$\kappa(B_1 \times \mathbb{R}^m) = |B_1|, \tag{40}$$

because ν_x is a probability measure for almost every $x \in \Omega$. In other words, the Ω-marginal of κ is the Lebesgue measure on Ω, which we write as $\kappa = |\cdot|_\Omega \otimes \nu$ where $|\cdot|_\Omega$ is the Lebesgue measure on Ω. The following theorem, which we do not prove here (see [1, Ch. 4]), describes the converse.

Theorem 3.19 (Disintegration of measures)**.** *Let $\kappa \in \mathbf{M}^+(\Omega\times E)$ for $\Omega \in \mathcal{B}(\mathbb{R}^n)$, $E \in \mathcal{B}(\mathbb{R}^m)$ and define the Ω-marginal $\mu(B_1) = \kappa(B_1 \times E)$ for all $B_1 \in \mathcal{B}(\Omega)$. Then there exists a unique weakly* μ-measurable function $\nu : \Omega \to \mathbf{M}^{\mathbb{P}}(E)$ such that for all $B_1 \in \mathcal{B}(\Omega)$, $B_2 \in \mathcal{B}(E)$*

$$\kappa(B_1 \times B_2) = \int_{B_1} \nu_x(B_2)\,\mathrm{d}\mu(x). \tag{41}$$

An immediate consequence of Theorem 3.19 and the preceding discussion is that Young measures $\nu \in \mathbf{Y}(\Omega;\mathbb{R}^m)$ and product measures $\kappa \in \mathbf{M}^+(\Omega\times\mathbb{R}^m)$ whose Ω-marginals are the Lebesgue measure are in one-to-one correspondence.

This leads to the probabilist's approach to Young Measures, mirroring the discussion in Appendix A.2. The space of Young measures inherits the notion of **narrow convergence** from the ambient space $\mathbf{M}^+(\Omega\times\mathbb{R}^m)$. Marginals are preserved under narrow convergence, so $\mathbf{Y}(\Omega;\mathbb{R}^m)$ is closed when viewed a subset of $\mathbf{M}^+(\Omega\times\mathbb{R}^m)$ in the narrow topology. We now use this different perspective to show the reader an alternative proof of the Fundamental Theorem of Young Measures 3.11.

Definition 3.20 (Tightness for Young measures). A sequence of Young measures $\{\nu^{(j)}\}_{j\in\mathbb{N}}$ is said to be **tight** if the associated measures $\{\kappa_j\}_{j\in\mathbb{N}} \subset \mathbf{M}^+(\Omega \times \mathbb{R}^m)$ are tight in the sense of Definition A.7

It is straightforward to see that a sequence of Young measures is tight if and only if for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \mathbb{R}^m$ such that

$$\sup_j \kappa_j(\Omega \times (\mathbb{R}^m \setminus K_\varepsilon)) < \varepsilon. \tag{42}$$

Further, this notion of tightness for Young measures corresponds precisely with the tightness property in the conditions for the Fundamental Theorem of Young measures 3.11, a fact we prove below:

Proposition 3.21. *A sequence of measurable functions $\{u_j\}_{j\in\mathbb{N}}$ satisfies the tightness property if and only if the associated Young measures $\nu^{(j)} : x \mapsto \delta_{u_j(x)}$ are tight.*

Proof. First suppose that $\{u_j\}_{j\in\mathbb{N}}$ has the tightness property. Then for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\sup_j |\{x \in \Omega : |u_j(x)| \geq M_\varepsilon\}| < \varepsilon. \tag{43}$$

Let $K_\varepsilon = \Omega \times \{\lambda \in \mathbb{R}^m : |\lambda| < M_\varepsilon\}$. Then if κ_j is the associated product measure to u_j, we have

$$\begin{aligned} \sup_j \kappa_j(\Omega \times (\mathbb{R}^m \setminus K_\varepsilon)) &= \sup_j \int_\Omega \mathbb{1}_{u_j(x)\notin K_\varepsilon}(x)\,\mathrm{d}x \\ &= \sup_j |\{x \in \Omega : |u_j(x)| \geq M_\varepsilon\}| \\ &< \varepsilon, \end{aligned}$$

so the associated sequence of Young measures is tight. Now suppose that $\{\nu^{(j)}\}_{j\in\mathbb{N}}$ associated with $\{u_j\}_{j\in\mathbb{N}}$ is tight. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}^m$ such that

$$\sup_j \kappa_j(\Omega \times (\mathbb{R}^m \setminus K_\varepsilon)) < \varepsilon. \tag{44}$$

The tightness property for $\{u_j\}_{j\in\mathbb{N}}$ can now be demonstrated by selecting M_ε such that $K_\varepsilon \subseteq \{\lambda \in \mathbb{R}^m : |\lambda| < M_\varepsilon\}$. □

Now let us reconsider the Fundamental Theorem 3.11. Prokhorov's Theorem A.9 combined with the above theorem implies that a sequence of measurable functions $\{u_j\}_{j\in\mathbb{N}}$ satisfying the tightness property has a subsequence $\{u_{j_k}\}_{k\in\mathbb{N}}$ and a Young measure $\nu \in \mathbf{Y}(\Omega; \mathbb{R}^m)$ such that:

$$\lim_{k\to\infty} \int_\Omega f(x, u_{j_k}(x))\,\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^m} f(x, \lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x \tag{45}$$

for all $f \in C_b(\Omega \times \mathbb{R}^m)$. In order to extend the result to all Carathéodory functions f such that $\{f(\cdot, u_{j_k}(\cdot))\}_{k\in\mathbb{N}}$ is uniformly integrable, one can use the Scorza-Dragoni Theorem. We leave the details to the reader.

Theorem 3.22 (Scorza-Dragoni). *Let Ω be bounded and measurable, $S \subset \mathbb{R}^m$ be compact and $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq \Omega$ such that $|\Omega \setminus K_\varepsilon| \leq \varepsilon$ and f restricted to $K_\varepsilon \times S$ is continuous.*

Remark 3.23. For a proof, see [4, p.76]. This theorem is a generalisation of Lusin's theorem: for a given measurable function $u : \Omega \to \mathbb{R}^m$ and all $\varepsilon > 0$, there exists a compact set K_ε with $|\Omega \setminus K_\varepsilon| \leq \varepsilon$, upon which u is continuous.

4 Young Measures, Lower Semi-Continuity and Convexity

In this section we return to the question posed in the Introduction: What condition on a Carathéodory function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ can be imposed so that the integral functional

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x \tag{46}$$

is weakly lower semi-continuous in $W^{1,p}(\Omega;\mathbb{R}^m)$? Young measures reveal a link between convex functions and lower semicontinuity, leading to a weaker form of convexity known as quasiconvexity.

Theorem 4.1. *Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function that is* **bounded below**. *Then for any sequence $\{u_j\}_{j\in\mathbb{N}}$ generating a Young measure ν, the following holds:*

$$\liminf_{j\to\infty} \int_\Omega f(x, u_j(x))\,\mathrm{d}x \geq \int_\Omega \int_{\mathbb{R}^m} f(x, \lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x. \tag{47}$$

Proof. Without loss of generality we may assume that f is non-negative. For each $i \in \mathbb{N}$ define $f_i(x, \lambda) = \min\{f(x, \lambda), i\}$, which is Carathéodory and $0 \leq f_i \nearrow f$ a.e. in $\mathbb{R}^m$. We also have that $0 \leq f_i \leq i$, so that $\{f_i(\cdot, u_j(\cdot))\}_{j\in\mathbb{N}}$ is UI. Therefore, by the Fundamental Theorem 3.11,

$$\begin{aligned}
\liminf_{j\to\infty} \int_\Omega f(x, u_j(x))\,\mathrm{d}x &\geq \sup_i \liminf_{j\to\infty} \int_\Omega f_i(x, u_j(x))\,\mathrm{d}x \\
&= \sup_i \int_\Omega \int_{\mathbb{R}^m} f_i(x, \lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x \\
&= \int_\Omega \int_{\mathbb{R}^m} f(x, \lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x,
\end{aligned}$$

by the Monotone Convergence Theorem, since $0 \leq f_i(x, \lambda) \nearrow f(x, \lambda)$ for a.e. $(x, \lambda) \in \Omega \times \mathbb{R}^m$. □

Theorem 4.2 (Jensen's inequality)**.** *Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous and convex. Then*

$$\int_{\mathbb{R}^m} f(\lambda)\,\mathrm{d}\mu(\lambda) \geq f\left(\int_{\mathbb{R}^m} \lambda\,\mathrm{d}\mu(\lambda)\right) \quad \forall \mu \in \mathbf{M}^{\mathbb{P}}(\mathbb{R}^m). \tag{48}$$

Remark 4.3. The theorem is easy to prove from the existence of a subgradient at each $x = \int_{\mathbb{R}^m} \lambda\,\mathrm{d}\mu(\lambda)$. To prove the converse: For any $a, b \in \mathbb{R}^m$, $\theta \in (0, 1)$ take $\mu = \theta\delta_a + (1-\theta)\delta_b$.

Theorem 4.4 (First lower semi-continuity result)**.** *Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function that is bounded below, and for $1 \leq p \leq \infty$ define $I : L^p(\Omega;\mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\}$ by*

$$I(u) = \int_\Omega f(x, u(x))\,\mathrm{d}x. \tag{49}$$

I is weakly lower semi-continuous (weakly* for $p = \infty$) if and only if $\lambda \mapsto f(x, \lambda)$ is convex.

Proof. First assume that f is convex in its second argument. Suppose that $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ ($u_j \overset{\star}{\rightharpoonup} u$ if $p = \infty$) and take a subsequence so that $\liminf_{j\to\infty} I(u_j) = \lim_{j\to\infty} I(u_j)$. Take a further subsequence so that $\{u_j\}_{j\in\mathbb{N}}$ generates $\nu \in \mathbf{Y}(\Omega;\mathbb{R}^m)$. Note that by Corollary 3.13 $u(x) = \int_{\mathbb{R}^m} \lambda\,\mathrm{d}\nu_x(\lambda)$, so by Theorem 4.1 and Jensen's inequality we have

$$\begin{aligned}
\liminf_{j\to\infty} \int_\Omega f(x, u_j(x))\,\mathrm{d}x &\geq \int_\Omega \int_{\mathbb{R}^m} f(x, \lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x \\
&\geq \int_\Omega f\left(x, \int_{\mathbb{R}^m} \lambda\,\mathrm{d}\nu_x(\lambda)\right)\mathrm{d}x \\
&= \int_\Omega f(x, u(x))\,\mathrm{d}x.
\end{aligned}$$

Now assume that I is weakly lower semi-continuous (weakly* if $p=\infty$). Let $Q \subset \Omega$ be a hyper-cube with an affine bijection $\phi : Q \to (0,1)^n$. Fix $y \in \Omega$, then for any $a, b \in \mathbb{R}^m$, $\theta \in (0,1)$, define $\xi \in L^\infty(Q;\mathbb{R}^m)$ by:

$$\xi(x) = \begin{cases} a & \text{if } [\phi(x)]_1 \in [0,\theta), \\ b & \text{if } [\phi(x)]_1 \in [\theta,1). \end{cases}$$

Now extend ξ Q-periodically to $\mathbb{R}^n$ and define the sequence $\{u_j\}_{j\in\mathbb{N}} \subset L^\infty(\Omega;\mathbb{R}^m)$ by

$$u_j(x) = \begin{cases} \xi(jx) & \text{if } x \in Q, \\ \theta a + (1-\theta)b & \text{if } x \in \Omega \setminus Q. \end{cases} \tag{50}$$

Then by Proposition 3.15 $\{u_j\}_{j\in\mathbb{N}}$ generates the Young measure $\nu_x = \theta\delta_a + (1-\theta)\delta_b$ for $x \in Q$ and $\nu_x = \delta_{\theta a+(1-\theta)b}$ for $x \in \Omega \setminus Q$. Also, $u_j \rightharpoonup u \equiv \theta a + (1-\theta)b$ in $L^p(\Omega;\mathbb{R}^m)$ for all p ($\overset{\star}{\rightharpoonup}$ for $p=\infty$), so by the lower semi-continuity assumption we have:

$$\begin{aligned} \int_Q f(x,\theta a+(1-\theta)b)\,\mathrm{d}x &= \int_\Omega f(x,u(x))\,\mathrm{d}x - \int_{\Omega\setminus Q} f(x,\theta a+(1-\theta)b)\,\mathrm{d}x \\ &\le \liminf_{j\to\infty} \int_\Omega f(x,u_j(x))\,\mathrm{d}x - \int_{\Omega\setminus Q} f(x,\theta a+(1-\theta)b)\,\mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^m} f(x,\lambda)\,\mathrm{d}\nu_x(\lambda)\,\mathrm{d}x - \int_{\Omega\setminus Q} f(x,\theta a+(1-\theta)b)\,\mathrm{d}x \\ &= \int_Q \theta f(x,a) + (1-\theta) f(x,b)\,\mathrm{d}x. \end{aligned}$$

Since Q is an arbitrary cube, we have that $f : \lambda \mapsto f(x,\lambda)$ is convex for a.e. $x \in \Omega$. □

What happens if we replace $u(x)$ with $\nabla u(x)$? Consider the following example:

Example 4.5. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $I : W^{1,\infty}(\Omega;\mathbb{R}^2) \to \mathbb{R}$ be defined by

$$I(u) = \int_\Omega \det(\nabla u(x))\,\mathrm{d}x. \tag{51}$$

Then I is weakly* continuous, but det is not a convex function on $\mathbb{R}^{2\times 2}$.

Proof. Suppose that $u_j \overset{\star}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega;\mathbb{R}^2)$ and define the sequence $v_j \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ by

$$v_j(x) = u(x) + (u_j(x) - u(x))\Gamma_{\frac{1}{j}}(1 + \operatorname{dist}(x,\partial\Omega)), \tag{52}$$

where Γ_α is the continuous cutoff function defined in (31). This choice of v_j enjoys the property that $u_j - v_j \in W_0^{1,\infty}(\Omega;\mathbb{R}^2)$ and $\nabla v_j \overset{m}{\to} \nabla u$. Combining the former property with the result in Lemma 4.6 and the latter property with Corollary 3.14 gives the following:

$$\begin{aligned} \int_\Omega \det(\nabla u(x))\,\mathrm{d}x &= \lim_{j\to\infty} \int_\Omega \det(\nabla v_j(x))\,\mathrm{d}x \\ &= \lim_{j\to\infty} \int_\Omega \det(\nabla u_j(x))\,\mathrm{d}x, \end{aligned}$$

and we see that I is weakly* continuous. However,

$$\det\left(\frac{1}{2}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}\right) = \frac{3}{2} > 1 = \frac{1}{2}\det\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\det\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix},$$

so det is not convex. □

Lemma 4.6. *For $u \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ where Ω is a Lipschitz domain, the integral of its Jacobian is dependent only on boundary values of u, i.e.*

$$\int_\Omega \det(\nabla u(x))\,\mathrm{d}x = \int_\Omega \det(\nabla v(x))\,\mathrm{d}x \tag{53}$$

for all $v \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ such that $u - v \in W_0^{1,\infty}(\Omega;\mathbb{R}^n)$.

Proof. We first restrict ourselves to the case where $u, v \in C^\infty(\Omega;\mathbb{R}^m)$. Using the language of exterior algebras and differential forms, we have

$$\begin{aligned} \int_\Omega \det(\nabla u(x))\,\mathrm{d}x &= \int_\Omega \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n \\ &= \int_\Omega \mathrm{d}(u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n). \end{aligned}$$

By Stokes' Theorem, this is equal to

$$\begin{aligned} \int_{\partial\Omega} u_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n &= \int_{\partial\Omega} v_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n \\ &= \int_\Omega \mathrm{d}v_1 \wedge \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n. \end{aligned}$$

By the anticommutativity of the wedge product, we can continue in the same way:

$$\begin{aligned} \int_\Omega \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}u_n &= (-1)^{n-1} \int_\Omega \mathrm{d}u_2 \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}v_1 \\ &= (-1)^{n-1} \int_\Omega \mathrm{d}v_2 \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}v_1 \\ &= (-1)^{2(n-1)} \int_\Omega \mathrm{d}u_3 \wedge \cdots \wedge \mathrm{d}u_n \wedge \mathrm{d}v_1 \wedge \mathrm{d}v_2 \\ &\cdots \\ &= (-1)^{n(n-1)} \int_\Omega \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n \\ &= \int_\Omega \det(\nabla v(x))\,\mathrm{d}x. \end{aligned}$$

The full result follows by a density argument. □

Remark 4.7. This result is related to the fact that the Jacobian is a **null Lagrangian**.

The underlying issue in Example 4.5 is that in general, not every function in $L^p(\Omega;\mathbb{R}^{m\times n})$ is the gradient of a $W^{1,p}(\Omega;\mathbb{R}^m)$ function. Further, the class of Young measures generated by sequences of $W^{1,p}(\Omega;\mathbb{R}^m)$ gradients is in general strictly smaller than $\mathbf{Y}(\Omega;\mathbb{R}^{m\times n})$. We denote the set of such **gradient Young measures** by $\mathbf{GY}_p(\Omega;\mathbb{R}^{m\times n})$. The analogous proof of Theorem 4.4 would fail because we wouldn't be able to generate the appropriate Young measure in $\mathbf{GY}_p(\Omega;\mathbb{R}^{m\times n})$ to show the convexity of f.

It turns out that we *would* be able to show that $f(\theta a + (1-\theta)b) \le \theta f(a) + (1-\theta)f(b)$ for all a, b such that $a - b$ is a matrix of rank 1 i.e. that f is **rank-one convex**, but in general, rank-one convexity is not sufficient for lower semi-continuity. It was Morrey in 1952 [7], who discovered the necessary and sufficient condition for lower semi-continuity:

Definition 4.8 (Quasiconvexity)**.** A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called **quasiconvex** if *for every* open, bounded set $U \subset \mathbb{R}^n$ with $|\partial U| = 0$, we have:

$$f(A) \leq \frac{1}{|U|}\int_U f(A + \nabla\xi(y))\,\mathrm{d}y \quad \forall \xi \in W_0^{1,\infty}(U;\mathbb{R}^m). \tag{54}$$

Lemma 4.9. *A function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if and only if* there exists *an open, bounded set $U \subset \mathbb{R}^n$ with $|\partial U| = 0$ such that* (54) *holds.*

Proof. The case "only if" follows from the definition. The "if" case is a technical scaling and covering argument, which can be found in [4, p.172]. □

Example 4.10 (Determinant is quasiaffine)**.** The function $f(A) = \det(A)$ for $A \in \mathbb{R}^{n\times n}$ is **quasiaffine** i.e it satisfies the quasiconvex condition (54) with equality.

Proof. This is a direct consequence of Lemma 4.6 with $u(y) = Ay$ and $v(y) = Ay + \xi(y)$. □

The following theorem demonstrates the intimate relationship between gradient Young measures and quasiconvexity.

Theorem 4.11 (Jensen's inequality for gradient Young measures)**.** *Let $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ for $1 \leq p \leq \infty$, with $\{\nabla u_j\}_{j\in\mathbb{N}}$ generating $\nu \in \mathbf{GY}_p(\Omega;\mathbb{R}^{m\times n})$. If $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous, quasiconvex, and if $p < \infty$ there exists $C > 0$ with $0 \leq f(A) \leq C(1 + |A|^p)$, then we have:*

$$\int_{\mathbb{R}^{m\times n}} f(A)\,\mathrm{d}\nu_x(A) \geq f\left(\int_{\mathbb{R}^{m\times n}} A\,\mathrm{d}\nu_x(A)\right) \text{ for a.e } x \in \Omega. \tag{55}$$

Remark 4.12. This theorem is proved using some sophisticated techniques for Young measures, called **homogenisation** and **localisation**. A reference for this theorem and its converse is [6].

Lemma 4.13. *Let $u_j : \Omega \to \mathbb{R}^m$, $v_j : \Omega \to \mathbb{R}^M$ be a measurable functions such that $u_j \xrightarrow{m} u$ and v_j generates $\nu \in \mathbf{Y}(\Omega;\mathbb{R}^M)$. Then (u_j, v_j) generates $\left(x \mapsto \delta_{u(x)} \otimes \nu_x\right) \in \mathbf{Y}(\Omega;\mathbb{R}^{m+M})$.*

Proof. Let $\varphi \in C_0(\mathbb{R}^m)$, $\psi \in C_0(\mathbb{R}^M)$ and $B \in \mathcal{B}(\Omega)$. Then since by Corollary 3.14, $\{u_j\}_{j\in\mathbb{N}}$ generates the Young measure $x \mapsto \delta_{u(x)}$, we have:

$$\begin{aligned}
&\left|\int_B \varphi(u_j(x))\psi(v_j(x))\,\mathrm{d}x - \int_B \varphi(u(x))\langle\nu_x,\psi\rangle\,\mathrm{d}x\right| \\
\leq\; &\left|\int_B \left(\varphi(u_j(x)) - \varphi(u(x))\right)\psi(v_j(x))\,\mathrm{d}x\right| + \left|\int_B \varphi(u(x))\left(\psi(v_j(x)) - \langle\nu_x,\psi\rangle\right)\,\mathrm{d}x\right| \\
\leq\; &\|\psi\|_\infty \left|\int_B \varphi(u_j(x)) - \varphi(u(x))\,\mathrm{d}x\right| + \|\varphi\|_\infty \left|\int_B \psi(v_j(x)) - \langle\nu_x,\psi\rangle\,\mathrm{d}x\right| \to 0 \text{ as } j \to 0.
\end{aligned}$$

Since the linear span of functions of the form $(\lambda,\xi) \mapsto \varphi(\lambda)\psi(\xi)$ is dense in $C_0(\mathbb{R}^m \times \mathbb{R}^M)$, we have the desired result. □

Theorem 4.14 (Second lower semi-continuity result)**.** *Let $1 \leq p \leq \infty$ and let $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times n} \to \mathbb{R}$ be a Carathéodory function such that if $p < \infty$ there exists $C > 0$ with*

$$0 \leq f(x,\lambda,A) \leq C(1 + |A|^p) \text{ for a.e. } x \in \Omega,\ \forall\lambda \in \mathbb{R}^m,\ \forall A \in \mathbb{R}^{m\times n}. \tag{56}$$

Then $I : u \mapsto \int_\Omega f(x,u(x),\nabla u(x))\,\mathrm{d}x$ is weakly lower semi-continuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ (weakly* for $p = \infty$) if and only if $A \mapsto f(x,\lambda,A)$ is quasiconvex.

Proof. First assume that $A \mapsto f(x,\lambda,A)$ is quasiconvex. Suppose that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ ($u_j \overset{\star}{\rightharpoonup} u$ if $p=\infty$) and take a subsequence so that $\liminf_{j\to\infty} I(u_j) = \lim_{j\to\infty} I(u_j)$. Take a further subsequence so that $\{\nabla u_j\}_{j\in\mathbb{N}}$ generates $\nu \in \mathbf{GY}_p(\Omega;\mathbb{R}^{m\times n})$. By Lemma 4.13, $\{(u_j,\nabla u_j)\}_{j\in\mathbb{N}}$ generates $\big(x\mapsto \delta_{u(x)}\otimes \nu_x\big) \in \mathbf{Y}(\Omega;\mathbb{R}^{m+mn})$, so that by Theorem 4.1:

$$\liminf_{j\to\infty} \int_\Omega f(x,u_j(x),\nabla u_j(x))\,\mathrm{d}x \geq \int_\Omega\int_{\mathbb{R}^m} f(x,u(x),A)\,\mathrm{d}\nu_x(A)\,\mathrm{d}x. \tag{57}$$

By Theorem 4.11 for the continuous function $A\mapsto f(x,u(x),A)$, we have:

$$\int_{\mathbb{R}^m} f(x,u(x),A)\,\mathrm{d}\nu_x(A) \geq f(x,u(x),\nabla u(x)). \tag{58}$$

Now assume that I is weakly lower semi-continuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ (weakly* if $p=\infty$). For $M>0$ define the set S_M to be the closed ball of radius M in $\mathbb{R}^m\times\mathbb{R}^{m\times n}$. Then by Theorem 3.22, for every $i\in\mathbb{N}$ there exists a compact set $K_i^M\subseteq\Omega$ such that $|\Omega\setminus K_i^M|\leq \frac{1}{i}$ and f is continuous on $K_i^M\times S_M$. Define the set:

$$\Omega_0 = \bigcap_{M\in\mathbb{N}}\bigcup_{i\in\mathbb{N}} K_i^M. \tag{59}$$

Then $|\Omega\setminus\Omega_0|=0$. Now let $x_0\in\Omega_0$, $u_0\in\mathbb{R}^m$ be fixed and let $U=(0,1)^n$. We are going to show, by defining an appropriate gradient Young measure, that for every $A_0\in\mathbb{R}^{m\times n}$ and every $\xi\in W_0^{1,\infty}(U;\mathbb{R}^m)$ the function defined by:

$$F(x) = \int_U f(x,A_0(x-x_0)+u_0,A_0+\nabla\xi(y))\,\mathrm{d}y - f(x,A_0(x-x_0)+u_0,A_0) \tag{60}$$

is non-negative at $x=x_0$. This will show that $A\mapsto f(x_0,u_0,A)$ is quasiconvex for almost every $x_0\in\Omega$ and every $u_0\in\mathbb{R}^m$, by Lemma 4.9.

Now, let $Q\subset\Omega$ be a hyper-cube with affine bijection $\phi: Q\to U$ whose gradient is $\Lambda\in\mathbb{R}^{n\times n}$. Then, extending ξ periodically throughout $\mathbb{R}^n$, define $\{u_j\}_{j\in\mathbb{N}}\subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ by:

$$u_j(x) = \begin{cases} A_0(x-x_0)+u_0+\frac{1}{j}\xi(j\phi(x))\Lambda^{-1} & \text{if } x\in Q,\\ A_0(x-x_0)+u_0 & \text{if } x\in\Omega\setminus Q.\end{cases} \tag{61}$$

Then we have:

$$\nabla u_j(x) = \begin{cases} A_0+\nabla\xi(j\phi(x)) & \text{if } x\in Q,\\ A_0 & \text{if } x\in\Omega\setminus Q.\end{cases} \tag{62}$$

We see that $u_j\to A_0(\cdot-x_0)+u_0$ in $L^\infty(\Omega;\mathbb{R}^m)$ and

$$\nabla u_j \overset{\star}{\rightharpoonup} \left\{\begin{matrix} A_0+\int_U\nabla\xi(y)\,\mathrm{d}y & \text{if } x\in Q,\\ A_0 & \text{if } x\in\Omega\setminus Q.\end{matrix}\right\} = A_0, \tag{63}$$

by the Divergence Theorem, because $\xi\in W_0^{1,\infty}(U;\mathbb{R}^m)$. So $u_j\rightharpoonup u = A_0(\cdot-x_0)+u_0$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ for all p ($\overset{\star}{\rightharpoonup}$ if $p=\infty$). By Proposition 3.15, $\{\nabla u_j\}_{j\in\mathbb{N}}$ generates $\nu\in\mathbf{GY}_\infty(\Omega;\mathbb{R}^{m\times n})$ such that:

$$\langle\nu_x,\varphi\rangle = \begin{cases} \int_U\varphi(A_0+\nabla\xi(y))\,\mathrm{d}y & \text{if } x\in Q,\\ A_0 & \text{if } x\in\Omega\setminus Q,\end{cases} \tag{64}$$

for all $\varphi \in C_0(\mathbb{R}^{m\times n})$. Then by lower semi-continuity we have:

$$\begin{aligned}\int_Q f(x, A_0(x-x_0)+u_0, A_0)\,\mathrm{d}x &= \int_Q f(x, u(x), \nabla u(x))\,\mathrm{d}x \\ &\leq \liminf_{j\to\infty} \int_Q f(x, u_j(x), \nabla u_j(x))\,\mathrm{d}x \\ &= \int_Q \int_{\mathbb{R}^{m\times n}} f(x, A_0(x-x_0)+u_0, A)\,\mathrm{d}\nu_x(A)\,\mathrm{d}x \\ &= \int_Q \int_U f(x, A_0(x-x_0)+u_0, A_0+\nabla\xi(y))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Since Q was an arbitrary cube, $F(x) \geq 0$ for almost all $x \in \Omega$. For any A_0 and ξ, there exists an $M \in \mathbb{N}$ such that $(A_0(x-x_0)+u_0, A_0+\nabla\xi(y)) \in S_M$ for all $x \in \Omega$ and $y \in U$. Since $x_0 \in K_i^M$ for some i, x_0 is a point of continuity of F and so $F(x_0) \geq 0$ as desired. □

Theorem 4.15 (Existence of minimisers)**.** *Let* $1 < p < \infty$ *and let* $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ *be a quasiconvex Carathéodory function such that there exists* $c, C > 0$, $d \geq 0$ *with:*

$$c|A|^p - d \leq f(x, \lambda, A) \leq C(1+|A|^p). \tag{65}$$

Then

$$I : u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x \tag{66}$$

has at least one minimiser over the set

$$\mathcal{A} = \left\{ u \in W^{1,p}(\Omega;\mathbb{R}^m) : u - g \in W_0^{1,p}(\Omega;\mathbb{R}^m) \right\}. \tag{67}$$

Proof. Combine Theorem 1.1, the discussion in the Introduction, and Theorem 4.14. □

It is possible to use the same argument to prove Theorem 4.15 for $p = \infty$ (without any bounds on f). Noting that the sequential version of the Banach-Alaoglu Theorem (Theorem A.3) holds in $W^{1,\infty}(\Omega;\mathbb{R}^m)$, just with weak* convergence, and that $\mathcal{A}$ is weakly* closed, we can modify the Direct Method Theorem 1.1 to show that Theorem 4.14 is sufficient for our needs.

The case $p = 1$ is tricky. If we just assume that for some $c > 0$, $d \geq 0$:

$$f(x, \lambda, A) \geq c|A| - d, \tag{68}$$

then we deduce in the same way as in the Introduction that minimising sequences are bounded in $W^{1,1}(\Omega;\mathbb{R}^m)$. However, this does not imply the existence of a weakly convergence subsequence. Indeed, consider an antiderivative of our concentration example (Example 2.9).

As compensation we found that the sequence converged weakly* in $(C([0,1]))^\star = \mathbf{M}([0,1])$, which leads to a possible solution: Consider the sequence in the space of **bounded variation**, which consists of $L^1(\Omega;\mathbb{R}^m)$ functions whose distributional derivatives are in $\mathbf{M}(\Omega;\mathbb{R}^m)$, denoted $BV(\Omega;\mathbb{R}^m)$. We then have weak* compactness, but the theory of Young measures for sequences in $BV(\Omega;\mathbb{R}^m)$ is much more complicated than that for sequences in $W^{1,1}(\Omega;\mathbb{R}^m)$. The definition has to be extended to what are called generalised Young measures, which not only measure oscillations, but also concentrations [10]. This is the reason why we used the word "classical" in the title.

A Some Compactness Results

Compactness is one of the most important concepts in mathematical analysis. The form of compactness we care about most, at least in this essay, is *sequential* compactness, that every sequence has a convergent subsequence. In a metric space this distinction does not exist, but for general topologies neither property implies the other in general.

A.1 The Banach-Alaoglu Theorem

The Banach-Alaoglu Theorem is usually referred to as "weak* compactness" for a Banach space, or "weak compactness" in the case of a reflexive Banach space. There are several subtle variations on the theorem which are clarified below.

Theorem A.1 (General Banach-Alaoglu)**.** *Let X be a Banach space and let $\mathcal{A} \subset X$ be uniformly bounded in the norm on X. Then $\mathcal{A}$ is relatively compact for the weak* topology on X.*

Remark A.2. Note that this theorem does not assert relative *sequential* compactness.

Theorem A.3 (Sequential Version of Banach-Alaolgu)**.** *Let X be a* **separable** *normed space. Then any bounded sequence in $X^\star$ has a weakly* convergent subsequence.*

Corollary A.4. *Let X be a separable, reflexive Banach space. Then any bounded sequence in X has a weakly convergent subsequence.*

Theorem A.5 (Eberlein-Šmulian)**.** *Let $\mathcal{A}$ be a subset of a Banach space X. Then $\mathcal{A}$ is relatively weakly compact if and only if every sequence $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ has a weakly convergent subsequence.*

Corollary A.6 (Sequential Version of Banach-Alaoglu 2)**.** *Let X be a reflexive space. Then any bounded sequence in X has a weakly convergent subsequence.*

A.2 Prokhorov's Theorem

If a functional analyst were asked to give a topology on the space of probability measures on $E \subseteq \mathbb{R}^n$, $\mathbf{M}^{\mathbb{P}}(E)$, with good compactness properties, he/she may be inclined to say the induced weak* topology from the space of Radon measures, $\mathbf{M}(E)$ and appeal to the Riesz-Alexandrov Theorem 3.6 and the Banach-Alaoglu Theorem A.3. Then for any sequence of probability measures $\{\mu_j\}_{j\in\mathbb{N}}$, since they all have norm 1, we have a subsequence converging weakly* in $\mathbf{M}(E)$:

$$\langle f, \mu_n \rangle \to \langle f, \mu \rangle \text{ as } n \to \infty, \text{ for all } f \in C_0(E). \tag{69}$$

However, the limit μ is not necessarily a probability measure. Take $E = \mathbb{R}$ and $\mu_j = \delta_j$, whose weak* limit is the zero measure. We call this "mass escaping to infinity".

If a probabilist were asked of the same thing, he/she might say the *weak* topology, much to the confusion of the functional analyst. The probabilist's notion of weak convergence is not induced from $\mathbf{M}(E)$ through its dual space; it is the convergence described by:

$$\langle f, \mu_j \rangle \to \langle f, \mu \rangle \text{ as } j \to \infty, \text{ for all } f \in C_b(E), \tag{70}$$

where $C_b(E)$ is the vector space of all bounded continuous functions on E. The space of probability measures is closed in the weak topology, but it also has some good compactness properties:

Definition A.7 (Tightness)**.** A set of positive measures $M \subseteq \mathbf{M}^+(E)$ is said to be tight if for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subseteq E$ such that

$$\sup_{\mu \in M} \mu(E \setminus K_\varepsilon) < \varepsilon \tag{71}$$

Remark A.8. If E is compact, then every collection of probability measures is tight, since we can take $K_\varepsilon = E$.

Theorem A.9 (Prokhorov)**.** *A set of probability measures $M \subseteq \mathbf{M}^{\mathbb{P}}(E)$ is relatively sequentially compact in the weak topology if and only if it is tight.*

The probabilist's weak topology is sometimes called the **narrow** topology, in an attempt to reduce confusion. A reference for this subsection is [1, Ch. 4].

Acknowledgements

The author would like to thank his advisor Filip for not only being a helpful and encouraging project supervisor but also a mentor and friend. This work was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/H023348/1.

References

[1] H. Attouch, G. Buttazzo, and G. Michaille, *Variational analysis in Sobolev and BV spaces: applications to PDEs and optimization*, SIAM, 2006.

[2] J. Ball, *A version of the fundamental theorem for young measures*, PDEs and continuum models of phase transitions, (1989), pp. 207–215.

[3] P. Cembranos and J. Mendoza, *Banach Spaces of Vector-Valued Functions*, vol. 1676, Springer-Verlag, Berlin, 1997.

[4] B. Dacorogna, *Direct methods in the calculus of variations*, vol. 78, Springer, 2007.

[5] M. Fabian, P. Habala, P. Hájek, V. Santalucía, J. Pelant, and V. Zizler, *Functional analysis and infinite-dimensional geometry*, vol. 8, Springer, 2001.

[6] D. Kinderlehrer and P. Pedregal, *Gradient young measures generated by sequences in sobolev spaces*, Journal of Geometric Analysis, 4 (1994), pp. 59–90.

[7] C. Morrey, *Quasi-convexity and the lower semicontinuity of multiple integrals.*, Pacific Journal of Mathematics, 2 (1952), pp. 25–53.

[8] S. Müller, *Variational models for microstructure and phase transitions*, Calculus of variations and geometric evolution problems, (1999), pp. 85–210.

[9] P. Pedregal, *Parametrized measures and variational principles*, vol. 30, Birkhauser, 1997.

[10] F. Rindler, *Lower Semicontinuity and Young Measures for Integral Functionals with Linear Growth*, PhD thesis, Oxford University, 2011.

[11] R. Schilling, *Measures, integrals and martingales*, vol. 13, Cambridge University Press, 2005.