The plunge region in frame-based approximation

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Frames

Definition: A set $\Phi = \{\phi_k : k \in I\} \subset \mathcal{H}$ (Hilbert space) is a frame if the *synthesis operator*,

$$\mathcal{T} : \mathbf{c} \mapsto \sum_{k \in I} c_k \phi_k$$

is *continuous and onto* as a linear operator from $\ell^2(I)$ to $\mathcal{H}$.

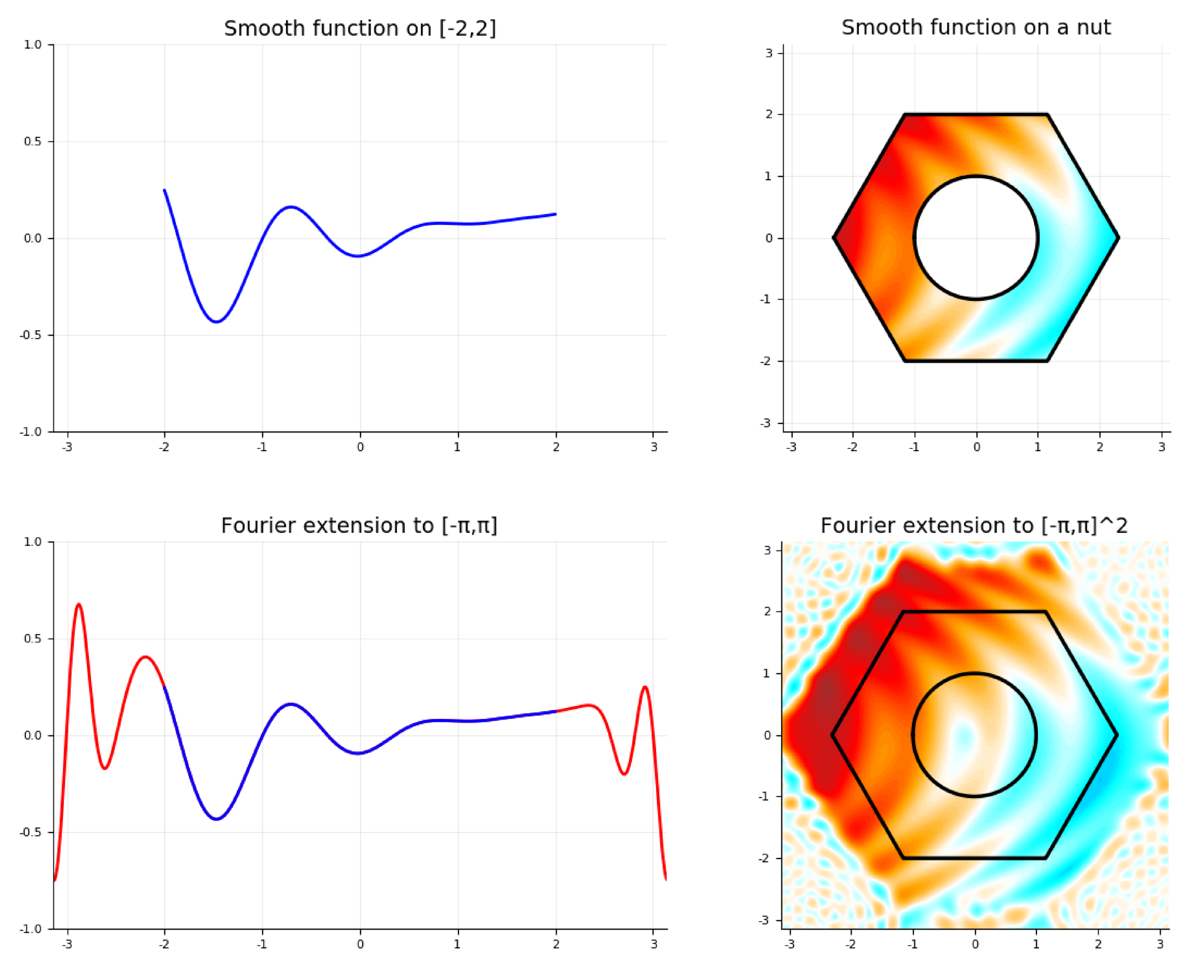
Benefits: A frame generalises an orthonormal basis, but the set Φ does not need to be linearly independent. This gives us *more freedom* to define a frame for our purposes.

Challenges: The redundancy of a frame leads to ill-conditioned linear systems, but despite this, *stable and fast algorithms are possible* for frames with a *plunge region*.

Fourier extension

Fourier extension: To approximate a function on an arbitrary domain Ω, use a Fourier series which is periodic on a larger domain Γ (idea due to Bruno and Boyd '03).

Connection to frames: This is equivalent to approximating using the following (linearly dependent) frame $\Phi = \{\exp(i\pi\mathbf{k}\cdot\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^d\} \subset L^2(\Omega)$ [1].



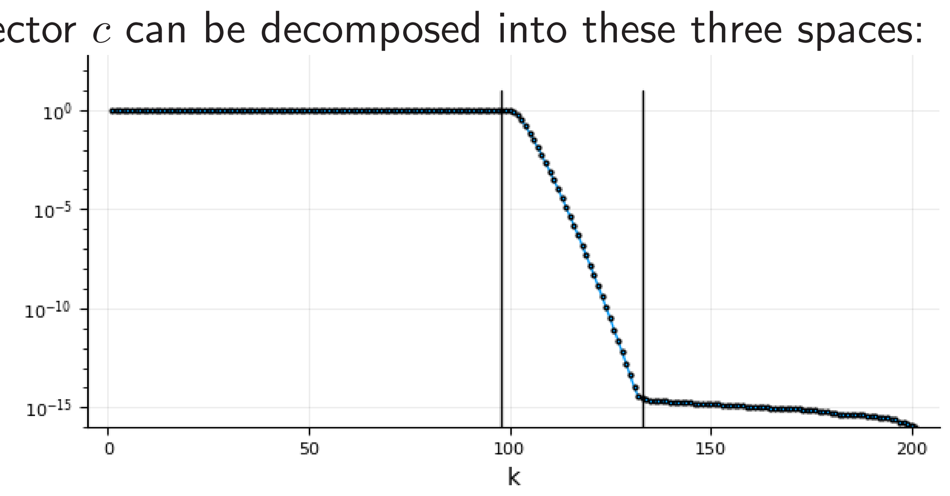
Above: two smooth functions, which each extend to periodic Fourier series on a larger domain.

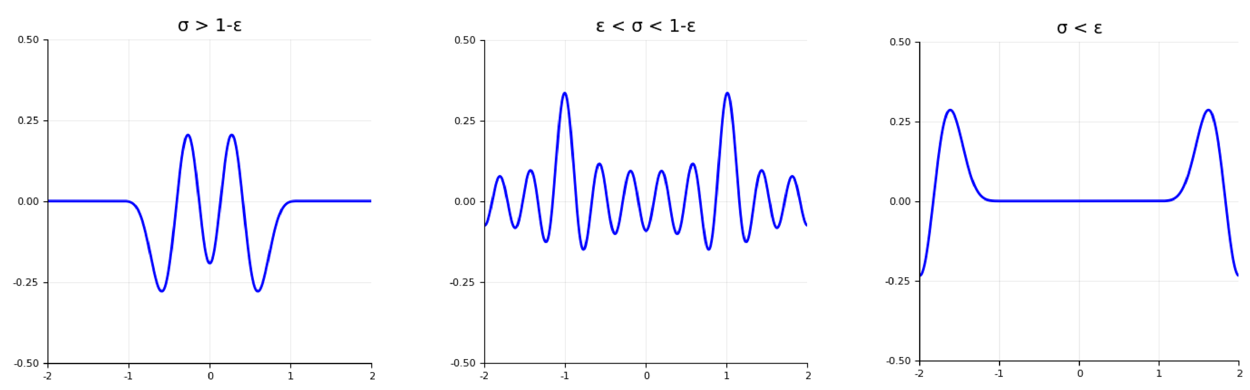
Least-squares interpolation: The underlying linear system is $Ac = b$, where

$$A_{k,j} = \phi_j(x_k), \quad b_k = f(x_k), \quad \{x_k\} \subset \Omega.$$

We use many more samples x_k than desired number of coefficients, which gives a tall, skinny matrix A, and increases the stability of the solution [2].

Plunge region: The collocation matrix A has a distinctive SVD profile, with three parts. Any solution vector c can be decomposed into these three spaces:





Fast algorithm: The modification $(I - AA^*)A$ "kills" all singular vectors of A such $\sigma \approx 1$ and $\sigma \approx 0$. There are the only $\mathcal{O}(\log(N))$ "plunge region" singular vectors remaining [3]!

1. Solve $(I - AA^*)Ap = (I - AA^*)b$
2. Solve $q = A^*(b - Ap)$.
3. Solution: $c = p + q$.

A and A^* can be applied fast using the FFT. A (regularised) pseudoinverse of $(I - AA^*)A$ can be computed fast using randomised SVD with rank $\mathcal{O}(\log(N))$ [4]. A quick calculation shows that $Ac - b = (I - AA^*)(Ap - b)$, which is made small by step 1. Overall complexity: $\mathcal{O}(N\log^2(N))$ for N coefficients in the 1D setting.

Generalisation: Weighted sums of trig-like bases

Generalisation: Fourier extensions is a special case of the following frames [5]:

$$\Phi = \{w_1 \cdot e_0,\, w_1 \cdot e_1,\, w_1 \cdot e_2, \ldots\} \cup \cdots \cup \{w_n \cdot e_0,\, w_n \cdot e_1,\, w_n \cdot e_2, \ldots\},$$

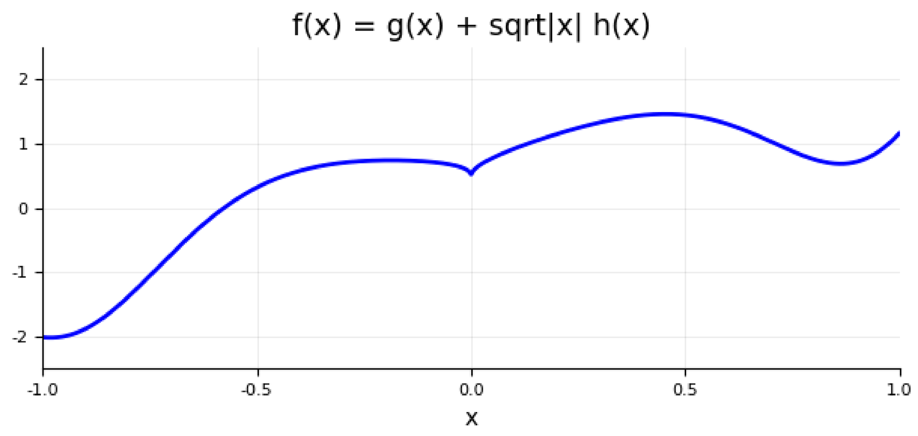
where w_i is a BV function and $\{e_0, e_1, e_2, \ldots\}$ is a "trig-like" basis. Fourier extensions have just $w_1 = \chi_\Omega$ and $\{e_k\}$ = Fourier basis.

Trig-like bases: Technical definition [5]. Examples include Fourier series, Jacobi polynomials, cosine series, sine series.

Application: How can you approximate a function of the form $f(x) = g(x) + |x|^{1/2}h(x)$, where g and h are smooth? Both a polynomial basis, and a weighted polynomial basis will have slow convergence. We suggest you use the frame,

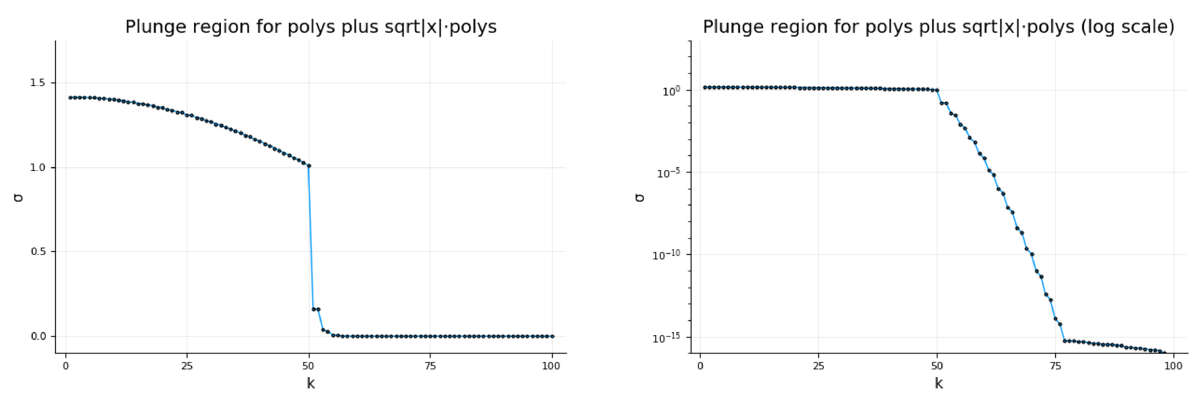
$$\Phi = \{T_0(x), T_1(x), T_2(X), \ldots\} \cup \{|x|^{1/2}T_0(x),\, |x|^{1/2}T_1(x),\, |x|^{1/2}T_2(x)\ldots\},$$

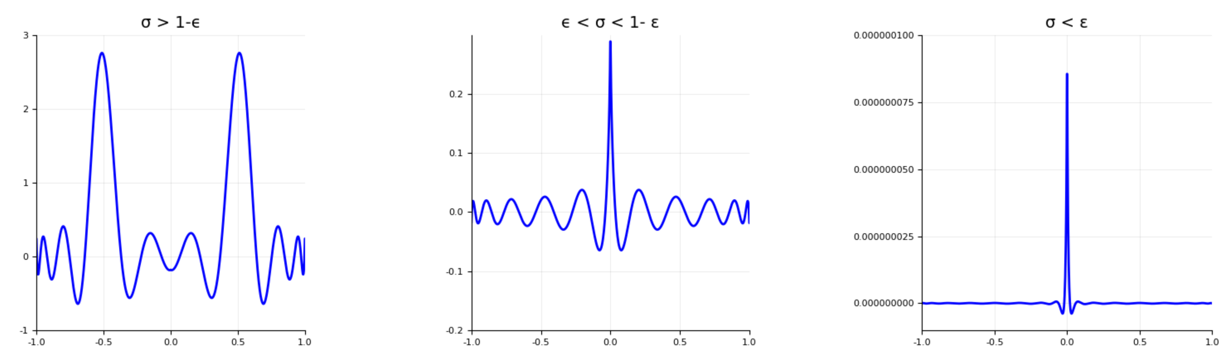
where T_k is the kth Chebyshev polynomial.



Least squares collocation: Just like the Fourier extension, we use oversampled least-squares collocation on a suitable grid.

Plunge region: Just like the Fourier extension, the collocation matrix A has a distinctive SVD profile, with 3 parts:





Fast algorithm: Since the singular values of A do not necessarily cluster at 1 like in the Fourier extension case, we cannot simply use $(I - AA^*)A$ to isolate the plunge region. We can, however, find a matrix Z such that $(I - AZ^*)A$ has approximate rank $\mathcal{O}(\log(N))$. For such Z, we can proceed just as in the Fourier extension case:

1. Solve $(I - AZ^*)Ap = (I - AZ^*)b$ using randomised SVD with rank $\mathcal{O}(\log(N))$.
2. Solve $q = Z^*(b - Ap)$.
3. Solution: $c = p + q$.

This is performed fast in a similar way to the Fourier extension. Overall complexity: $\mathcal{O}(N\log^2(N))$ for N coefficients in the 1D setting [6].

Future goals

Fast algorithm: The generalised fast algorithm is still under investigation [6].

Full generalisation: In [5], only the 1D case is considered. We would like to generalise to higher dimensional weighted sum-frames.

Differential equations: we intend to develop these approximation techniques into spectral methods for problems with complicated geometry and singularities.

References

[1] Huybrechs D., *On the Fourier extension of nonperiodic functions*, SIAM J. Numer. Anal., 2010.
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[3] Matthysen R., Huybrechs D., *Fast Algorithms for the computation of Fourier Extensions of arbitrary length*, SIAM J. Sci. Comput., 2016.
[4] Liberty et al., *Randomized algorithms for the low-rank approximation of matrices*, 2007.
[5] Webb M., *The plunge region in frame-based approximation* (in prep.).
[6] Coppé V., Huybrechs D., Matthysen R., Webb M., *Fast and stable algorithms for frame-based approximation* (in prep.).