Computing Complex Singularities of Differential Equations with Chebfun

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Motivation

- There has been scientific interest over the years in problems associated with complex singularities of solutions to differential equations.
- Weideman 2003 [5] describes a strategy for the computation of singularities of solutions to partial differential equations (PDEs).
- Our investigation was in this vein, but for the case of ordinary differential equations (ODEs), focusing on the numerical methods used.
- Viswanath and Sahutoglu 2010 [3] presents an analytic treatment of the ODE system called the *Lorenz attractor*. They consider time as a complex variable and show the existence of branch point singularities.
- This motivated the consideration of similar examples from dynamical systems for finding complex singularities.

Terminology

Singularity A point in $\mathbb{C}$ where a function is not analytic. The types considered in this investigation are poles and branch points:

Pole A typical blow-up type singularity, for example $1/(z^2+1)$ at $z = \pm i$ and $\tan(z)$ at $z = (2k+1)\frac{\pi}{2}$ for all integers k.

Branch point A singularity which may not necessarily entail blow-up but where one must use a *branch cut* for the function to be well defined (see Figure 2, top-right). E.g. $\log(z)$ and $z^{\frac{1}{2}}$ at $z = 0$.

Chebfun

- Chebfun is an open-source software project lead by Nick Hale and Nick Trefethen at Oxford University and Toby Driscoll at University of Delaware.
- It is an extension of MATLAB which overloads common vector and matrix operations to instead manipulate functions and operators.
- The intention is that the commands should feel symbolic, but that the underlying computations are numeric and therefore fast.
- We used Chebfun's built in *Chebop* system and the overload of `ode113` for solving ODEs.

Rational Interpolation

- A rational function r of type (m, n) is a function such that $r = p/q$ where p and q are polynomials of degrees m and n respectively.
- A rational interpolant of a function f for some points $x_0, \ldots, x_K \in \mathbb{C}$ is a rational function r such that $r(x_j) = f(x_j)$ for each j.
- Two papers in 2011 [2],[1] describe an algorithm for rational interpolation that is fast, stable and robust.
- This is important because rational interpolation and variants are *ill-posed* problems so algorithms for them are usually not robust; one almost always finds spurious singularities in r that have no relevance to f, which is a contributing factor to why rational approximation is not common practice.
- The algorithm is implemented in Chebfun as the function `ratinterp`.

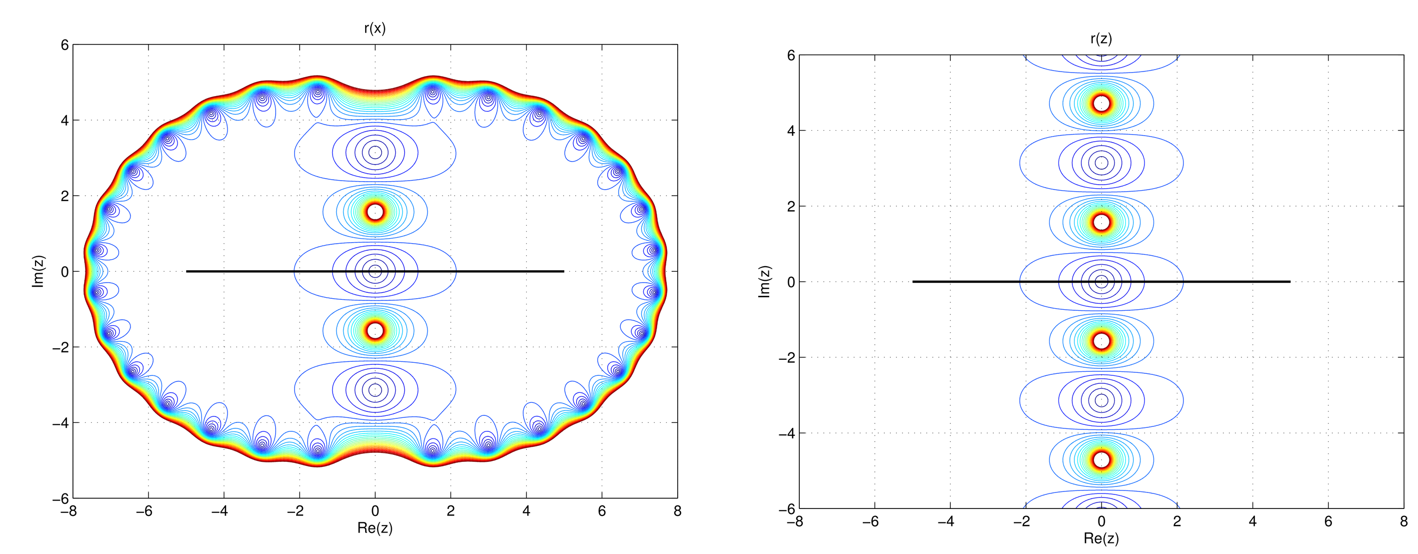
Aims

- Investigate the prospects of finding singularities for ODEs using Chebfun and robust rational interpolation.
- Work with the Chebfun team, debugging and developing `ratinterp`.
- Study examples of ODEs not normally studied with complex variables.

Method

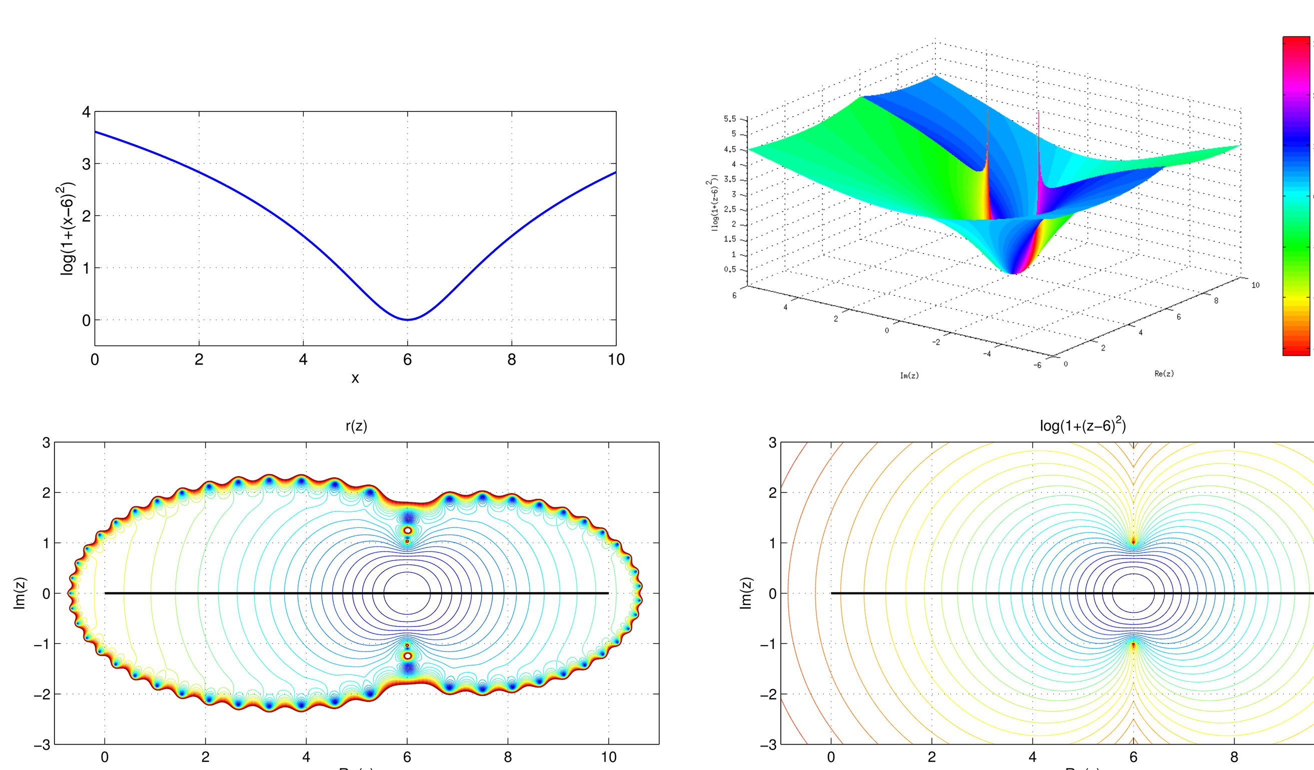
1. Solve ODE on real interval to return a chebfun u with degree N.
2. Use `ratinterp` to compute a type (m, n) rational interpolant $r = p/q$ on $N+1$ points with $m \approx \frac{1}{2}N$ and n sufficiently large.
3. Compute the roots of q, which are precisely the singularities of r. There should be no spurious poles if `ratinterp` is effective.

Elementary Examples



$$\frac{d^2u}{dx^2} + 2u\frac{du}{dx} = 0 \qquad (1)$$

Figure 1: Equation (1) (with appropriate boundary conditions) has solution $\tanh(z) = -i\tan(iz)$, which has poles at $z = (2k+1)\frac{\pi}{2}i$ for all integers k. Contour plots of rational interpolant $|r|$ (left) and the same for $|\tanh|$ (right), coloured blue to red in the range $[0, 5]$.



$$\frac{du}{dx} + 2(6-x)\exp(-u) = 0 \qquad (2)$$

Figure 2: Equation (2) (with appropriate boundary conditions) has solution $f(z) = \log(1+(z-6)^2)$, which has branch point singularities at $z = 6 \pm i$.
Top-left: f on $[0, 10]$. Top-right: 3D plot of the absolute value of f coloured by the argument of f. The lines of disconinuity in colour are the branch cuts. Bottom-left: contour plot of the rational interpolant $|r|$ of the numerical solution to (2). Bottom:right: the same for $|f|$. The contours are coloured blue to red in the range $[0, 5]$.

Lorenz Attractor

This system was introduced by Lorenz, who derived it from the simplified equations of convection rolls arising in the equations of the atmosphere:

$$\frac{dx}{dt} = 10(y-x), \quad \frac{dy}{dt} = 28x - y - xz, \quad \frac{dz}{dt} = -\frac{8}{3}z + xy. \qquad (3)$$

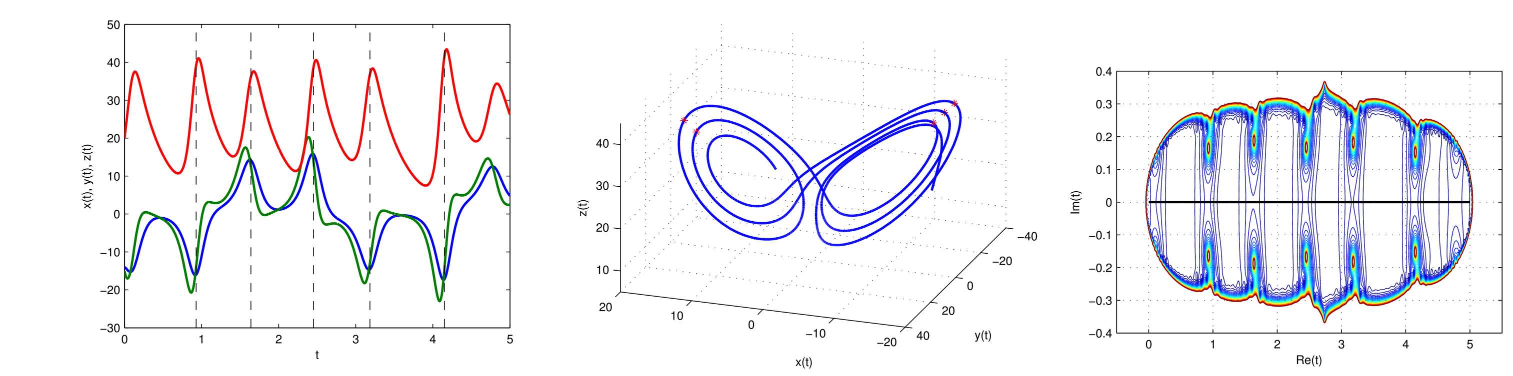


Figure 3: Solution to system (3) with initial conditions: $(x, y, z) = (-14, -15, 20)$. Left: x, y and z on $[0, 5]$ coloured blue, green and red respectively. Middle: Trajectory of the solution in 3D space. Right: Contour plot of a rational interpolant of the computed x-solution.
Dotted lines and stars are at plotted at the real parts of the complex singularities of the system.

Lotka-Volterra Predator Prey Model

A common population model for interacting species is the Lotka-Volterra predator-prey model. u and v represent the populations of prey and predators respectively and $\alpha, \beta, \gamma, \delta$ are positive constants.

$$\frac{du}{dt} = \alpha u - \beta uv, \quad \frac{dv}{dt} = -\gamma v + \delta uv. \qquad (4)$$

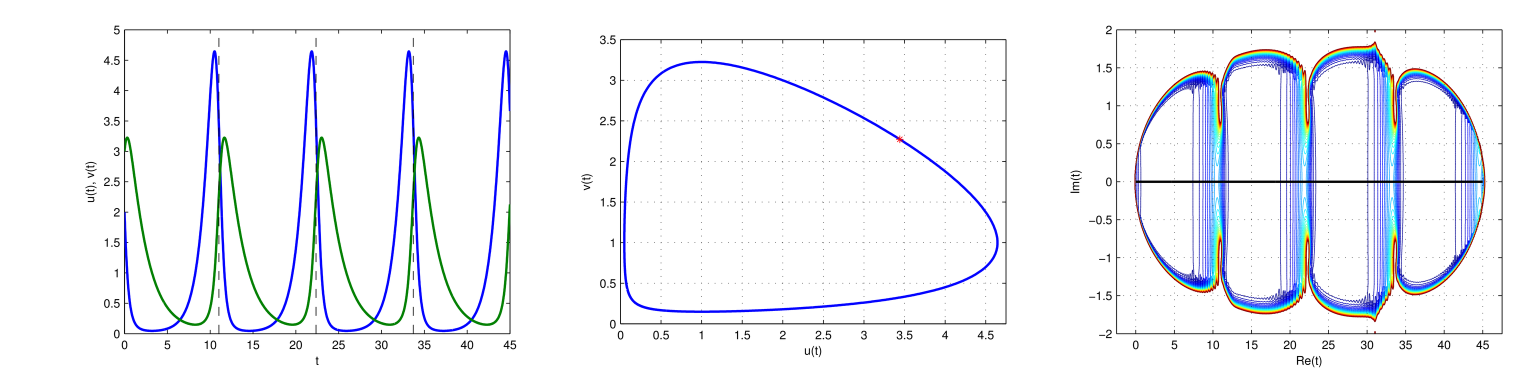


Figure 4: Here we use $(\alpha, \beta, \gamma, \delta) = (1, 1, 0.5, 0.5)$ and initial conditions $(u, v) = (2, 3)$. These figures are analogous to those in Figure 3.

Outcomes

- We developed a method for finding the singularities of a solution to an ODE, while avoiding spurious poles, by using the degree of the chebfun to inform our choice of m and n. The process can to some extent be automated.
- Among the ODEs studied were the three body problem, van der Pol oscillator and Bessel's equation. See [4] for the three body problem.

Acknowledgements & References

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