Orthogonal systems with a skew-symmetric differentiation matrix

Marcus Webb

KU Leuven, Belgium

joint work with Arieh Iserles (Cambridge)

2 May 2018

Cambridge Analysists' Knowledge Exchange

Motivation: Time-dependent PDEs

$$u \in \mathrm{C}^\infty([0,\infty);\mathrm{H}^1(\mathbb{R})), \qquad t \in [0,\infty), \quad x \in \mathbb{R}.$$

- Diffusion:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a(t,x,u)\frac{\partial u}{\partial x}\right], \qquad a \geq 0$$

- Semi-classical Schrödinger:

$$\mathrm{i}\varepsilon\frac{\partial u}{\partial t} = -\varepsilon^2\frac{\partial^2 u}{\partial x^2} + V(t,x,u)u, \qquad 0 < \varepsilon \ll 1, \quad \mathrm{Imag}(V) = 0$$

- Nonlinear advection:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + f(u), \qquad v \cdot f(v) \leq 0$$

Motivation: Time-dependent PDEs

$$u \in \mathrm{C}^\infty([0,\infty);\mathrm{H}^1(\mathbb{R})), \qquad t \in [0,\infty), \quad x \in \mathbb{R}.$$

- Diffusion:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a(t,x,u)\frac{\partial u}{\partial x}\right], \qquad a \geq 0$$

- Semi-classical Schrödinger:

$$\mathrm{i}\varepsilon\frac{\partial u}{\partial t} = -\varepsilon^2\frac{\partial^2 u}{\partial x^2} + V(t,x,u)u, \qquad 0 < \varepsilon \ll 1, \quad \mathrm{Imag}(V) = 0$$

- Nonlinear advection:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + f(u), \qquad v \cdot f(v) \leq 0$$

- Common property?

Motivation: Time-dependent PDEs

$$u \in \mathrm{C}^\infty([0,\infty);\mathrm{H}^1(\mathbb{R})), \qquad t \in [0,\infty), \quad x \in \mathbb{R}.$$

- Diffusion:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a(t,x,u)\frac{\partial u}{\partial x}\right], \qquad a \geq 0$$

- Semi-classical Schrödinger:

$$\mathrm{i}\varepsilon\frac{\partial u}{\partial t} = -\varepsilon^2\frac{\partial^2 u}{\partial x^2} + V(t,x,u)u, \qquad 0 < \varepsilon \ll 1, \quad \mathrm{Imag}(V) = 0$$

- Nonlinear advection:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + f(u), \qquad v \cdot f(v) \leq 0$$

- Common property? L_2 **stability**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|u(t,x)|^2\,\mathrm{d}x \leq 0, \text{ for all } t \geq 0.$$

L_2 stability of these PDEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u(t,x)|^2 \,\mathrm{d}x = \int \frac{\partial}{\partial t}|u(t,x)|^2 \,\mathrm{d}x = 2\mathrm{Re}\int \overline{u(t,x)}\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x$$

L_2 stability of these PDEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u(t,x)|^2\,\mathrm{d}x = \int \frac{\partial}{\partial t}|u(t,x)|^2\,\mathrm{d}x = 2\mathrm{Re}\int \overline{u(t,x)}\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x$$

- Diffusion:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|u|^2\,\mathrm{d}x &= 2\int_{-\infty}^{\infty} u\frac{\partial}{\partial x}\left(a(t,x,u)\frac{\partial u}{\partial x}\right)\,\mathrm{d}x \\ &= -2\int_{-\infty}^{\infty} a(t,x,u)\left(\frac{\partial u}{\partial x}\right)^2\,\mathrm{d}x \leq 0\end{aligned}$$

L_2 stability of these PDEs

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u(t,x)|^2\,\mathrm{d}x = \int \frac{\partial}{\partial t}|u(t,x)|^2\,\mathrm{d}x = 2\mathrm{Re}\int \overline{u(t,x)}\frac{\partial u}{\partial t}(t,x)\,\mathrm{d}x$$

- Diffusion:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|u|^2\,\mathrm{d}x &= 2\int_{-\infty}^{\infty} u\frac{\partial}{\partial x}\left(a(t,x,u)\frac{\partial u}{\partial x}\right)\,\mathrm{d}x \\ &= -2\int_{-\infty}^{\infty} a(t,x,u)\left(\frac{\partial u}{\partial x}\right)^2\,\mathrm{d}x \leq 0\end{aligned}$$

- Schrödinger:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}|u|^2\,\mathrm{d}x &= 2\mathrm{Re}\int_{-\infty}^{\infty} \overline{u}\left(\mathrm{i}\varepsilon\frac{\partial^2 u}{\partial x^2} - \mathrm{i}\varepsilon^{-1}V(t,x,u)u\right)\,\mathrm{d}x \\ &= -2\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{i}\varepsilon\left|\frac{\partial u}{\partial x}\right|^2 + \mathrm{i}\varepsilon^{-1}V(t,x,u)|u|^2\,\mathrm{d}x = 0\end{aligned}$$

Numerical solution of these PDEs

- Suppose we want a **numerical solution** to the diffusion equation $\partial_t u = \partial_x(a(x) \cdot \partial_x u)$.

Numerical solution of these PDEs

- Suppose we want a **numerical solution** to the diffusion equation $\partial_t u = \partial_x (a(x) \cdot \partial_x u)$.
- Obtain a semi-discretised PDE:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{C}^N$$

- E.g. **finite difference method** on a grid $x_1, x_2, \ldots, x_N$

$$\mathbf{u}(t) = (u(t, x_1), u(t, x_2), \ldots, u(t, x_N))^T \in \mathbb{C}^N$$

- $\mathcal{D}$ is a matrix encoding a finite-difference approximation to the partial derivative ∂_x.
- $\mathcal{A}$ is a diagonal matrix with entries $(a(x_1), \ldots, a(x_N))$.

Numerical solution of these PDEs

- Suppose we want to approximate the solution to the diffusion equation $\partial_t u = \partial_x(a(x) \cdot \partial_x u)$.
- Obtain a semi-discretised PDE:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t), \quad \mathbf{u}(0) = \mathbf{u}_0 \in \ell_2$$

- E.g. **spectral method** for a basis $\Phi = \{\varphi_n\}_{n \in \mathbb{Z}_+}$ of $\mathrm{L}_2(\mathbb{R})$

$$u(t, \cdot) = \sum_{n=0}^{\infty} u_n(t)\varphi_n$$

- $\mathcal{D}$ and $\mathcal{A}$ are infinite-dimensional matrices encoding:

$$\varphi_k'(x) = \sum_{j=0}^{\infty} D_{k,j}\varphi_j(x), \qquad a(x)\varphi_k(x) = \sum_{j=0}^{\infty} A_{k,j}\varphi_j(x)$$

ℓ_2 stability of (semi-)discretised PDEs

- Diffusion:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = 2(\mathcal{D}^T\mathbf{u})\mathcal{A}(\mathcal{D}\mathbf{u}),$$

ℓ_2 stability of (semi-)discretised PDEs

- Diffusion:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = 2(\mathcal{D}^T\mathbf{u})\mathcal{A}(\mathcal{D}\mathbf{u}),$$

- Nonlinear advection:

$$\mathbf{u}'(t) = \mathcal{D}\mathbf{u}(t) + \mathbf{f}(\mathbf{u}(t))$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathbf{u} + 2\mathbf{u}^T\mathbf{f}(\mathbf{u}) \leq 2\mathbf{u}^T\mathcal{D}\mathbf{u},$$

ℓ_2 stability of (semi-)discretised PDEs

- Diffusion:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = 2(\mathcal{D}^T\mathbf{u})\mathcal{A}(\mathcal{D}\mathbf{u}),$$

- Nonlinear advection:

$$\mathbf{u}'(t) = \mathcal{D}\mathbf{u}(t) + \mathbf{f}(\mathbf{u}(t))$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathbf{u} + 2\mathbf{u}^T\mathbf{f}(\mathbf{u}) \leq 2\mathbf{u}^T\mathcal{D}\mathbf{u},$$

- We want $\mathcal{D}$ to be a skew-symmetric matrix.

ℓ_2 stability of (semi-)discretised PDEs

- Diffusion:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = 2(\mathcal{D}^T\mathbf{u})\mathcal{A}(\mathcal{D}\mathbf{u}),$$

- Nonlinear advection:

$$\mathbf{u}'(t) = \mathcal{D}\mathbf{u}(t) + \mathbf{f}(\mathbf{u}(t))$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathbf{u} + 2\mathbf{u}^T\mathbf{f}(\mathbf{u}) \leq 2\mathbf{u}^T\mathcal{D}\mathbf{u},$$

- We want $\mathcal{D}$ to be a skew-symmetric matrix. Differential operator is skew-Hermitian.

ℓ_2 stability of (semi-)discretised PDEs

- Diffusion:

$$\mathbf{u}'(t) = \mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u}(t)$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathcal{A}\mathcal{D}\mathbf{u} = 2(\mathcal{D}^T\mathbf{u})\mathcal{A}(\mathcal{D}\mathbf{u}),$$

- Nonlinear advection:

$$\mathbf{u}'(t) = \mathcal{D}\mathbf{u}(t) + \mathbf{f}(\mathbf{u}(t))$$

$$\frac{\mathrm{d}\|\mathbf{u}\|_{\ell^2}^2}{\mathrm{d}t} = 2\mathbf{u}^T\mathbf{u}' = 2\mathbf{u}^T\mathcal{D}\mathbf{u} + 2\mathbf{u}^T\mathbf{f}(\mathbf{u}) \leq 2\mathbf{u}^T\mathcal{D}\mathbf{u},$$

- We want $\mathcal{D}$ to be a skew-symmetric matrix. Differential operator is skew-Hermitian.

Geometric Numerical Integration

The field of research on **discretisation** of differential equations which **respects qualitative properties** of the analytical solution (see Hairer-Lubich-Wanner 2006)

The joy and pain of skew-symmetry

- The simplest second-order finite difference scheme gives

$$\mathcal{D} = \frac{1}{2\Delta x} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

The joy and pain of skew-symmetry

- The simplest second-order finite difference scheme gives

$$\mathcal{D} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

- Looking good!

The joy and pain of skew-symmetry

- The simplest second-order finite difference scheme gives

$$\mathcal{D} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

- Looking good! This is the **highest order** skew-symmetric differentiation matrix on an **equispaced grid** (Iserles 2014)

The joy and pain of skew-symmetry

- The simplest second-order finite difference scheme gives

$$\mathcal{D} = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

- Looking good! This is the **highest order** skew-symmetric differentiation matrix on an **equispaced grid** (Iserles 2014)
- Higher-order skew-symmetric differentiation matrices on **special grids** are possible but complicated (Hairer-Iserles 2016,2017).

Known example: Fourier spectral methods

Take the Fourier basis:

$$\varphi_0(x) \equiv \frac{1}{(2\pi)^{1/2}}, \qquad \varphi_{2n}(x) = \frac{\cos nx}{\pi^{1/2}}, \quad \varphi_{2n+1}(x) = \frac{\sin nx}{\pi^{1/2}}, \quad n \in \mathbb{N}$$

– note that the basis is orthonormal.

Known example: Fourier spectral methods

Take the Fourier basis:

$$\varphi_0(x) \equiv \frac{1}{(2\pi)^{1/2}}, \qquad \varphi_{2n}(x) = \frac{\cos nx}{\pi^{1/2}}, \quad \varphi_{2n+1}(x) = \frac{\sin nx}{\pi^{1/2}}, \quad n \in \mathbb{N}$$

– note that the basis is orthonormal. The differentiation matrix is

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Known example: Fourier spectral methods

Take the Fourier basis:

$$\varphi_0(x) \equiv \frac{1}{(2\pi)^{1/2}}, \qquad \varphi_{2n}(x) = \frac{\cos nx}{\pi^{1/2}}, \quad \varphi_{2n+1}(x) = \frac{\sin nx}{\pi^{1/2}}, \quad n \in \mathbb{N}$$

– note that the basis is orthonormal. The differentiation matrix is

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

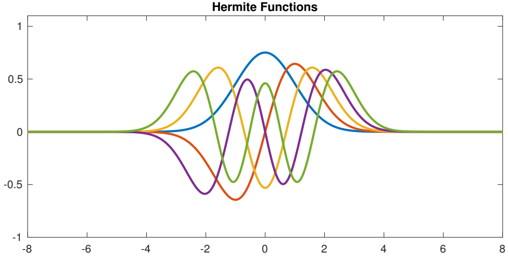
For **periodic boundary conditions** only.

Known example: Hermite spectral methods

- Hermite functions are familiar in mathematical physics:

$$\varphi_n(x) = \frac{(-1)^n}{(2^n n!)^{1/2}\pi^{1/4}} \mathrm{e}^{-x^2/2}\mathrm{H}_n(x), \qquad n \in \mathbb{Z}_+, \quad x \in \mathbb{R},$$

  where H_n is the nth *Hermite polynomial.*
- Orthonormal basis for $\mathrm{L}_2(\mathbb{R})$
- Uniformly bounded, and smooth
- Eigenfunctions of the Fourier transform



Known example: Hermite spectral methods

Hermite functions obey the ODE

$$\begin{aligned}
\varphi_0'(x) &= -\sqrt{\frac{1}{2}}\varphi_1(x), \\
\varphi_n'(x) &= \sqrt{\frac{n}{2}}\varphi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\varphi_{n+1}(x), \qquad n \in \mathbb{N}.
\end{aligned}$$

In other words,

$$\mathcal{D} = \begin{pmatrix} 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ \sqrt{\frac{1}{2}} & 0 & -\sqrt{\frac{2}{2}} & 0 & \cdots \\ 0 & \sqrt{\frac{2}{2}} & 0 & -\sqrt{\frac{3}{2}} & \ddots \\ \vdots & 0 & \sqrt{\frac{3}{2}} & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

a **skew-symmetric, tridiagonal** differentiation matrix.

Nonstandard example: Spherical Bessel Functions

Solutions to the ODE $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n(n+1))y = 0$, for each $n \in \mathbb{Z}_+$, are the **spherical Bessel functions** $\mathrm{j}_n(x)$.

$$\mathrm{j}_0(x) = \frac{\sin(x)}{x}, \quad \mathrm{j}_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}, \quad \mathrm{j}_2(x) = \left(\frac{3}{x^2} - 1\right) \frac{\sin(x)}{x} - \frac{3\cos(x)}{x^2}.$$

Writing $\varphi_n(x) = \sqrt{\frac{2n+1}{\pi}} \mathrm{j}_n(x)$, one can obtain the known result

$$\varphi_n'(x) = -\frac{n}{\sqrt{(2n-1)(2n+1)}} \varphi_{n-1}(x) + \frac{n+1}{\sqrt{(2n+1)(2n+3)}} \varphi_{n+1}(x)$$

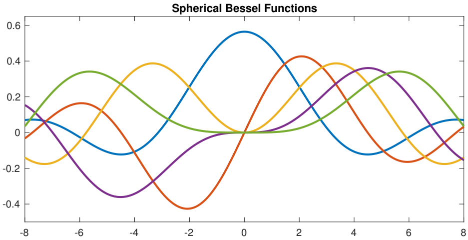
Nonstandard example: Spherical Bessel Functions

Solutions to the ODE $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x \frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n(n+1))y = 0$, for each $n \in \mathbb{Z}_+$, are the **spherical Bessel functions** $\mathrm{j}_n(x)$.

$$\mathrm{j}_0(x) = \frac{\sin(x)}{x}, \quad \mathrm{j}_1(x) = \frac{\sin(x)}{x^2} - \frac{\cos(x)}{x}, \quad \mathrm{j}_2(x) = \left(\frac{3}{x^2} - 1\right)\frac{\sin(x)}{x} - \frac{3\cos(x)}{x^2}.$$

Writing $\varphi_n(x) = \sqrt{\frac{2n+1}{\pi}}\mathrm{j}_n(x)$, one can obtain the known result

$$\varphi_n'(x) = -\frac{n}{\sqrt{(2n-1)(2n+1)}}\varphi_{n-1}(x) + \frac{n+1}{\sqrt{(2n+1)(2n+3)}}\varphi_{n+1}(x)$$



Aims of the talk

Aim 1

Find a system of functions $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$, and nonzero scalars $\{b_n\}_{n\in\mathbb{Z}_+}$ such that

$$\begin{aligned}\varphi_0'(x) &= b_0\varphi_1(x),\\ \varphi_n'(x) &= -b_{n-1}\varphi_{n-1}(x) + b_n\varphi_{n+1}(x), \qquad n\in\mathbb{N}.\end{aligned}$$

Φ has **real, skew-symmetric, tridiagonal irreducible differentiation matrix**

Aims of the talk

Aim 1

Find a system of functions $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$, and nonzero scalars $\{b_n\}_{n\in\mathbb{Z}_+}$ such that

$$\begin{aligned} \varphi_0'(x) &= b_0\varphi_1(x), \\ \varphi_n'(x) &= -b_{n-1}\varphi_{n-1}(x) + b_n\varphi_{n+1}(x), \qquad n \in \mathbb{N}. \end{aligned}$$

Φ has **real, skew-symmetric, tridiagonal irreducible differentiation matrix**

Aim 2

Determine systems which are also **orthonormal** in $\mathrm{L}_2(\mathbb{R})$:

$$u(x) = \sum_{n=0}^{\infty} u_k\varphi_k(x) \implies \|\mathbf{u}\|_{\ell_2} = \|u\|_{\mathrm{L}_2(\mathbb{R})} \tag{1}$$

Aims of the talk

Aim 1

Find a system of functions $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$, and nonzero scalars $\{b_n\}_{n\in\mathbb{Z}_+}$ such that

$$\begin{aligned} \varphi_0'(x) &= b_0\varphi_1(x), \\ \varphi_n'(x) &= -b_{n-1}\varphi_{n-1}(x) + b_n\varphi_{n+1}(x), \qquad n \in \mathbb{N}. \end{aligned}$$

Φ has **real, skew-symmetric, tridiagonal irreducible differentiation matrix**

Aim 2

Determine systems which are also **orthonormal** in $\mathrm{L}_2(\mathbb{R})$:

$$u(x) = \sum_{n=0}^{\infty} u_k\varphi_k(x) \implies \|\mathbf{u}\|_{\ell_2} = \|u\|_{\mathrm{L}_2(\mathbb{R})} \tag{1}$$

Our continuing mission: to explore strange new bases, to seek out new methods and new special functions, to boldly go...

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n \in \mathbb{Z}_+}$ be given.

$$n = 0 : \qquad \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x),$$

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ be given.

$$n = 0: \qquad \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x),$$

$$n = 1: \qquad \varphi_2(x) = \frac{1}{b_1}[\varphi_1'(x) + b_0\varphi_0(x)] = \frac{1}{b_0 b_1}[b_0^2\varphi_0(x) + \varphi_0''(x)],$$

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ be given.

$$
\begin{aligned}
n = 0: &\qquad \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x),\\
n = 1: &\qquad \varphi_2(x) = \frac{1}{b_1}[\varphi_1'(x) + b_0\varphi_0(x)] = \frac{1}{b_0 b_1}[b_0^2\varphi_0(x) + \varphi_0''(x)],\\
n = 2: &\qquad \varphi_3(x) = \frac{1}{b_2}[\varphi_2'(x) + b_1\varphi_1(x)] = \frac{1}{b_0 b_1 b_2}[(b_0^2 + b_1^2)\varphi_0'(x) + \varphi_0'''(x)]
\end{aligned}
$$

and so on.

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ be given.

$$\begin{aligned}
n = 0: \qquad & \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x), \\
n = 1: \qquad & \varphi_2(x) = \frac{1}{b_1}[\varphi_1'(x) + b_0\varphi_0(x)] = \frac{1}{b_0 b_1}[b_0^2\varphi_0(x) + \varphi_0''(x)], \\
n = 2: \qquad & \varphi_3(x) = \frac{1}{b_2}[\varphi_2'(x) + b_1\varphi_1(x)] = \frac{1}{b_0 b_1 b_2}[(b_0^2 + b_1^2)\varphi_0'(x) + \varphi_0'''(x)]
\end{aligned}$$

and so on. Easy **induction** confirms that

$$\varphi_n(x) = \frac{1}{b_0 b_1 \cdots b_{n-1}} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \alpha_{n,\ell}\varphi_0^{(n-2\ell)}(x), \qquad n \in \mathbb{N},$$

$$\alpha_{n+1,0} = 1, \qquad \alpha_{n+1,\ell} = b_{n-1}^2\alpha_{n-1,\ell-1} + \alpha_{n,\ell}, \quad \ell = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ be given.

$$n = 0: \qquad \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x),$$

$$n = 1: \qquad \varphi_2(x) = \frac{1}{b_1}[\varphi_1'(x) + b_0\varphi_0(x)] = \frac{1}{b_0 b_1}[b_0^2\varphi_0(x) + \varphi_0''(x)],$$

$$n = 2: \qquad \varphi_3(x) = \frac{1}{b_2}[\varphi_2'(x) + b_1\varphi_1(x)] = \frac{1}{b_0 b_1 b_2}[(b_0^2 + b_1^2)\varphi_0'(x) + \varphi_0'''(x)]$$

and so on. Easy **induction** confirms that

$$\varphi_n(x) = \frac{1}{b_0 b_1 \cdots b_{n-1}} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \alpha_{n,\ell}\varphi_0^{(n-2\ell)}(x), \qquad n \in \mathbb{N},$$

$$\alpha_{n+1,0} = 1, \qquad \alpha_{n+1,\ell} = b_{n-1}^2\alpha_{n-1,\ell-1} + \alpha_{n,\ell}, \quad \ell = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

This method works in some sense.

Elementary construction

Let $\varphi_0 \in \mathrm{C}^\infty(\mathbb{R})$ and $\{b_n\}_{n\in\mathbb{Z}_+}$ be given.

$$n = 0: \qquad \varphi_1(x) = \frac{1}{b_0}\varphi_0'(x),$$

$$n = 1: \qquad \varphi_2(x) = \frac{1}{b_1}[\varphi_1'(x) + b_0\varphi_0(x)] = \frac{1}{b_0 b_1}[b_0^2\varphi_0(x) + \varphi_0''(x)],$$

$$n = 2: \qquad \varphi_3(x) = \frac{1}{b_2}[\varphi_2'(x) + b_1\varphi_1(x)] = \frac{1}{b_0 b_1 b_2}[(b_0^2 + b_1^2)\varphi_0'(x) + \varphi_0'''(x)]$$

and so on. Easy **induction** confirms that

$$\varphi_n(x) = \frac{1}{b_0 b_1 \cdots b_{n-1}} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \alpha_{n,\ell}\varphi_0^{(n-2\ell)}(x), \qquad n \in \mathbb{N},$$

$$\alpha_{n+1,0} = 1, \qquad \alpha_{n+1,\ell} = b_{n-1}^2\alpha_{n-1,\ell-1} + \alpha_{n,\ell}, \quad \ell = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

This method works in some sense. How to tell if orthogonal?

The Fourier transform

The unitary Fourier transform and its inverse:

$$\mathcal{F}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x, \quad \mathcal{F}^{-1}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}x$$

The Fourier transform

The unitary Fourier transform and its inverse:

$$\mathcal{F}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x, \quad \mathcal{F}^{-1}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}x$$

Well known differentiation formula:

$$\mathcal{F}[\varphi'](\xi) = \mathrm{i}\xi \mathcal{F}[\varphi](\xi).$$

The Fourier transform

The unitary Fourier transform and its inverse:

$$\mathcal{F}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}x\xi} \, \mathrm{d}x, \quad \mathcal{F}^{-1}[\varphi](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}x$$

Well known differentiation formula:

$$\mathcal{F}[\varphi'](\xi) = \mathrm{i}\xi \mathcal{F}[\varphi](\xi).$$

Define the transformed functions

$$\psi_n(\xi) = (-\mathrm{i})^n \mathcal{F}[\varphi_n](\xi).$$

Then

$$\xi \psi_n(\xi) = (-\mathrm{i})^n \xi \mathcal{F}[\varphi_n](\xi) = (-\mathrm{i})^{n+1} (\mathrm{i}\xi) \mathcal{F}[\varphi_n](\xi) = (-\mathrm{i})^{n+1} \mathcal{F}[\varphi_n'](\xi).$$

The transformed functions

- Fourier differentiation formula implies

$$\psi_n(\xi) = (-\mathrm{i})^n \mathcal{F}[\varphi_n](\xi), \qquad \xi\psi_n(\xi) = (-\mathrm{i})^{n+1} \mathcal{F}[\varphi_n'](\xi), \qquad n \in \mathbb{Z}_+.$$

The transformed functions

- Fourier differentiation formula implies

$$\psi_n(\xi) = (-\mathrm{i})^n \mathcal{F}[\varphi_n](\xi), \qquad \xi\psi_n(\xi) = (-\mathrm{i})^{n+1}\mathcal{F}[\varphi_n'](\xi), \qquad n \in \mathbb{Z}_+.$$

- Using the skew-symmetric differentiation formula,

$$\begin{aligned} \xi\psi_0(\xi) &= b_0(-\mathrm{i})\mathcal{F}[\varphi_1](\xi) = b_0\psi_1(\xi), \\ \xi\psi_n(\xi) &= -b_{n-1}(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n-1}](\xi) + b_n(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n+1}](\xi) \\ &= b_{n-1}\psi_{n-1}(\xi) + b_n\psi_{n+1}(\xi). \end{aligned}$$

The transformed functions

- Fourier differentiation formula implies

$$\psi_n(\xi) = (-\mathrm{i})^n \mathcal{F}[\varphi_n](\xi), \qquad \xi\psi_n(\xi) = (-\mathrm{i})^{n+1}\mathcal{F}[\varphi_n'](\xi), \qquad n \in \mathbb{Z}_+.$$

- Using the skew-symmetric differentiation formula,

$$\begin{aligned}
\xi\psi_0(\xi) &= b_0(-\mathrm{i})\mathcal{F}[\varphi_1](\xi) = b_0\psi_1(\xi), \\
\xi\psi_n(\xi) &= -b_{n-1}(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n-1}](\xi) + b_n(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n+1}](\xi) \\
&= b_{n-1}\psi_{n-1}(\xi) + b_n\psi_{n+1}(\xi).
\end{aligned}$$

- They satisfy a symmetric recurrence!

The transformed functions

- Fourier differentiation formula implies

$$\psi_n(\xi) = (-\mathrm{i})^n \mathcal{F}[\varphi_n](\xi), \qquad \xi\psi_n(\xi) = (-\mathrm{i})^{n+1}\mathcal{F}[\varphi_n'](\xi), \qquad n \in \mathbb{Z}_+.$$

- Using the skew-symmetric differentiation formula,

$$\begin{aligned} \xi\psi_0(\xi) &= b_0(-\mathrm{i})\mathcal{F}[\varphi_1](\xi) = b_0\psi_1(\xi), \\ \xi\psi_n(\xi) &= -b_{n-1}(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n-1}](\xi) + b_n(-\mathrm{i})^{n+1}\mathcal{F}[\varphi_{n+1}](\xi) \\ &= b_{n-1}\psi_{n-1}(\xi) + b_n\psi_{n+1}(\xi). \end{aligned}$$

- They satisfy a symmetric recurrence!
- Therefore, they are of the form $\psi_n(\xi) = p_n(\xi)\psi_0(\xi)$, where

$$\begin{aligned} p_0(\xi) &= 1, \qquad p_1(\xi) = b_0^{-1}\xi \\ p_{n+1}(\xi) &= \frac{\xi}{b_n}p_n(\xi) - \frac{b_{n-1}}{b_n}p_{n-1}(\xi), \qquad n \in \mathbb{N}. \end{aligned}$$

Favard's Theorem

Theorem (Favard)

Let $P = \{p_n\}_{n\in\mathbb{Z}_+}$ *be a sequence of real polynomials such that* $\deg(p_n) = n$*.* P *is an* **orthogonal system** *with respect to the inner product* $\langle f, g\rangle_\mu = \int \overline{f(\xi)} g(\xi)\,\mathrm{d}\mu(\xi)$ *for some* **probability measure** $\mathrm{d}\mu$ *on the real line if and only if the polynomials satisfy the three-term recurrence,*

$$p_{n+1}(\xi) = (\alpha_n - \beta_n\xi)p_n(\xi) - \gamma_n p_{n-1}(\xi), \qquad n \in \mathbb{Z}_+,$$

for some real sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, $\{\beta_n\}_{n\in\mathbb{Z}_+}$, $\{\gamma_n\}_{n\in\mathbb{Z}_+}$ *with* $\gamma_0 = 0$ *and* $\gamma_n\beta_{n-1}/\beta_n > 0$ *for all* $n \in \mathbb{N}$*.*

Favard's Theorem

Theorem (Favard)

Let $P = \{p_n\}_{n\in\mathbb{Z}_+}$ *be a sequence of real polynomials such that* $\deg(p_n) = n$*.* P *is an* **orthogonal system** *with respect to the inner product* $\langle f, g\rangle_\mu = \int \overline{f(\xi)} g(\xi)\,\mathrm{d}\mu(\xi)$ *for some* **probability measure** $\mathrm{d}\mu$ *on the real line if and only if the polynomials satisfy the three-term recurrence,*

$$p_{n+1}(\xi) = (\alpha_n - \beta_n\xi)p_n(\xi) - \gamma_n p_{n-1}(\xi), \qquad n \in \mathbb{Z}_+,$$

for some real sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, $\{\beta_n\}_{n\in\mathbb{Z}_+}$, $\{\gamma_n\}_{n\in\mathbb{Z}_+}$ *with* $\gamma_0 = 0$ *and* $\gamma_n\beta_{n-1}/\beta_n > 0$ *for all* $n \in \mathbb{N}$*.*

- $\mathrm{d}\mu$ is symmetric (i.e. $\mathrm{d}\mu(-\xi) = \mathrm{d}\mu(\xi)$) if and only if $\alpha_n = 0$

Favard's Theorem

Theorem (Favard)

Let $P = \{p_n\}_{n\in\mathbb{Z}_+}$ be a sequence of real polynomials such that $\deg(p_n) = n$. P is an **orthogonal system** *with respect to the inner product $\langle f, g\rangle_\mu = \int \overline{f(\xi)} g(\xi)\,\mathrm{d}\mu(\xi)$ for some* **probability measure** *$\mathrm{d}\mu$ on the real line if and only if the polynomials satisfy the three-term recurrence,*

$$p_{n+1}(\xi) = (\alpha_n - \beta_n \xi) p_n(\xi) - \gamma_n p_{n-1}(\xi), \qquad n \in \mathbb{Z}_+,$$

for some real sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, $\{\beta_n\}_{n\in\mathbb{Z}_+}$, $\{\gamma_n\}_{n\in\mathbb{Z}_+}$ with $\gamma_0 = 0$ and $\gamma_n\beta_{n-1}/\beta_n > 0$ for all $n \in \mathbb{N}$.

- $\mathrm{d}\mu$ is symmetric (i.e. $\mathrm{d}\mu(-\xi) = \mathrm{d}\mu(\xi)$) if and only if $\alpha_n = 0$
- P is orthonormal if and only if $\gamma_n\beta_{n-1}/\beta_n = 1$

Favard's Theorem

Theorem (Favard)

Let $P = \{p_n\}_{n\in\mathbb{Z}_+}$ be a sequence of real polynomials such that $\deg(p_n) = n$. P is an **orthogonal system** *with respect to the inner product $\langle f, g\rangle_\mu = \int \overline{f(\xi)} g(\xi)\,\mathrm{d}\mu(\xi)$ for some* **probability measure** *$\mathrm{d}\mu$ on the real line if and only if the polynomials satisfy the three-term recurrence,*

$$p_{n+1}(\xi) = (\alpha_n - \beta_n\xi)p_n(\xi) - \gamma_n p_{n-1}(\xi), \qquad n \in \mathbb{Z}_+,$$

for some real sequences $\{\alpha_n\}_{n\in\mathbb{Z}_+}$, $\{\beta_n\}_{n\in\mathbb{Z}_+}$, $\{\gamma_n\}_{n\in\mathbb{Z}_+}$ with $\gamma_0 = 0$ and $\gamma_n\beta_{n-1}/\beta_n > 0$ for all $n \in \mathbb{N}$.

- $\mathrm{d}\mu$ is symmetric (i.e. $\mathrm{d}\mu(-\xi) = \mathrm{d}\mu(\xi)$) if and only if $\alpha_n = 0$
- P is orthonormal if and only if $\gamma_n\beta_{n-1}/\beta_n = 1$

For us: $$p_{n+1}(\xi) = \frac{\xi}{b_n} p_n(\xi) - \frac{b_{n-1}}{b_n} p_{n-1}(\xi)$$

Fourier characterisation for Φ

We can now deduce for our $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$:

$$\mathrm{i}^n\mathcal{F}[\varphi_n](\xi) = \psi_n(\xi) = \psi_0(\xi)p_n(\xi), \quad \text{so} \quad \varphi_n(x) = (-\mathrm{i})^n\mathcal{F}^{-1}[\psi_0 \cdot p_n]$$

The mapping can be (carefully) followed both ways:

$$(\{\varphi_n\}_{n\in\mathbb{Z}_+}, \{b_n\}_{n\in\mathbb{Z}_+}) \leftrightarrow (\{p_n\}_{n\in\mathbb{Z}_+}, \psi_0)$$

Theorem (Iserles-Webb 2018)

The sequence $\Phi = \{\varphi_n\}_{n\in\mathbb{Z}_+}$ has a real, skew-symmetric, tridiagonal, irreducible, differentiation matrix if and only if

$$\varphi_n(x) = (-\mathrm{i})^n\mathcal{F}^{-1}[g \cdot p_n],$$

where $P = \{p_n\}_{n\in\mathbb{Z}_+}$ is an orthonormal polynomial system on the real line with respect to a symmetric probability measure $\mathrm{d}\mu$, and $g = \psi_0 = \mathcal{F}[\varphi_0]$.

Legendre and Bessel functions

The Legendre polynomials $P = \{P_0, P_1, \ldots\}$ satisfy

$$\int_{-1}^{1} P_n(\xi) P_m(\xi) \, \mathrm{d}\xi = \left(n + \frac{1}{2} \right)^{-1} \delta_{n,m}$$

Legendre and Bessel functions

The Legendre polynomials $P = \{P_0, P_1, \ldots\}$ satisfy

$$\int_{-1}^{1} P_n(\xi) P_m(\xi) \,\mathrm{d}\xi = \left(n + \frac{1}{2}\right)^{-1} \delta_{n,m}$$

It is known (see DLMF) that the Fourier transform of the normalised Legendre polynomials (denoted p_n) is

$$\frac{(-\mathrm{i})^n}{\sqrt{2\pi}} \int_{-1}^{1} p_n(x) \mathrm{e}^{\mathrm{i}x\xi} \,\mathrm{d}x =$$

Legendre and Bessel functions

The Legendre polynomials $P = \{P_0, P_1, \dots\}$ satisfy

$$\int_{-1}^{1} P_n(\xi) P_m(\xi) \, \mathrm{d}\xi = \left(n + \frac{1}{2} \right)^{-1} \delta_{n,m}$$

It is known (see DLMF) that the Fourier transform of the normalised Legendre polynomials (denoted p_n) is

$$\frac{(-\mathrm{i})^n}{\sqrt{2\pi}} \int_{-1}^{1} p_n(x) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}x = \sqrt{\frac{2n+1}{\pi}} \mathrm{j}_n(\xi).$$

We obtain the **spherical Bessel functions** again!

Legendre and Bessel functions

The Legendre polynomials $P = \{P_0, P_1, \ldots\}$ satisfy

$$\int_{-1}^{1} P_n(\xi) P_m(\xi) \, \mathrm{d}\xi = \left(n + \frac{1}{2}\right)^{-1} \delta_{n,m}$$

It is known (see DLMF) that the Fourier transform of the normalised Legendre polynomials (denoted p_n) is

$$\frac{(-\mathrm{i})^n}{\sqrt{2\pi}} \int_{-1}^{1} p_n(x) \mathrm{e}^{\mathrm{i}x\xi} \, \mathrm{d}x = \sqrt{\frac{2n+1}{\pi}} \mathrm{j}_n(\xi).$$

We obtain the **spherical Bessel functions** again! $(g(\xi) = \chi_{[-1,1]}(\xi))$

Orthogonal systems

- We have the formula, $\varphi_n(x) = (-\mathrm{i})^n \mathcal{F}^{-1}[g \cdot p_n]$, where $g = \mathcal{F}[\varphi_0]$ and $P = \{p_n\}_{n \in \mathbb{Z}_+}$ are orthonormal with respect to a symmetric measure.

Orthogonal systems

- We have the formula, $\varphi_n(x) = (-\mathrm{i})^n \mathcal{F}^{-1}[g \cdot p_n]$, where $g = \mathcal{F}[\varphi_0]$ and $P = \{p_n\}_{n \in \mathbb{Z}_+}$ are orthonormal with respect to a symmetric measure.
- How can we tell if Φ is an **orthogonal system**?

Orthogonal systems

- We have the formula, $\varphi_n(x) = (-\mathrm{i})^n \mathcal{F}^{-1}[g \cdot p_n]$, where $g = \mathcal{F}[\varphi_0]$ and $P = \{p_n\}_{n\in\mathbb{Z}_+}$ are orthonormal with respect to a symmetric measure.
- How can we tell if Φ is an **orthogonal system**?
- **Parseval's Theorem**: For all $\varphi, \psi \in \mathrm{L}_2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \overline{\mathcal{F}[\varphi](\xi)}\mathcal{F}[\psi](\xi)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} \overline{\varphi(x)}\psi(x)\,\mathrm{d}x$$

- Simple!

$$\int_{-\infty}^{\infty} \varphi_n(x)\varphi_m(x)\,\mathrm{d}x = (-\mathrm{i})^{m-n} \int p_n(\xi)p_m(\xi)|g(\xi)|^2\,\mathrm{d}\xi$$

Orthogonal systems

- We have the formula, $\varphi_n(x) = (-\mathrm{i})^n \mathcal{F}^{-1}[g \cdot p_n]$, where $g = \mathcal{F}[\varphi_0]$ and $P = \{p_n\}_{n \in \mathbb{Z}_+}$ are orthonormal with respect to a symmetric measure.
- How can we tell if Φ is an **orthogonal system**?
- **Parseval's Theorem**: For all $\varphi, \psi \in \mathrm{L}_2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \overline{\mathcal{F}[\varphi](\xi)} \mathcal{F}[\psi](\xi) \, \mathrm{d}\xi = \int_{-\infty}^{\infty} \overline{\varphi(x)} \psi(x) \, \mathrm{d}x$$

- Simple!

$$\int_{-\infty}^{\infty} \varphi_n(x) \varphi_m(x) \, \mathrm{d}x = (-\mathrm{i})^{m-n} \int p_n(\xi) p_m(\xi) |g(\xi)|^2 \, \mathrm{d}\xi$$

Theorem (Iserles-Webb 2018)

Φ is orthogonal in $\mathrm{L}_2(\mathbb{R})$ if and only if P is orthogonal with respect to the measure $|g(\xi)|^2 \mathrm{d}\xi$. Note, $g = \mathcal{F}[\varphi_0]$.

Hermite revisited

$$\varphi_n(x) = \frac{(-1)^n}{(2^n n!)^{1/2}\pi^{1/4}} \mathrm{e}^{-x^2/2} \mathrm{H}_n(x), \qquad n \in \mathbb{Z}_+, \quad x \in \mathbb{R},$$

- As mentioned earlier, the Hermite functions are **eigenfunctions** of the Fourier transform:

$$\mathcal{F}[\varphi_n](\xi) = (-i)^n \varphi_n(\xi) \tag{2}$$

- Therefore, the Hermite functions are, in a sense, a **fixed point** of our correspondence

Theorem (Iserles-Webb 2018)

Up to trivial rescaling, the only orthogonal system that consists of "quasi-polynomials" is the Hermite system.

Transformed Chebyshev functions

- The Chebyshev polynomials of the second kind, $U_0, U_1, U_2, \ldots$ are **orthonormal** with respect to the measure

$$\mathrm{d}\mu(\xi) = \frac{2}{\pi}\chi_{[-1,1]}(\xi)\sqrt{1-\xi^2}\,\mathrm{d}\xi.$$

Transformed Chebyshev functions

- The Chebyshev polynomials of the second kind, $U_0, U_1, U_2, \ldots$ are **orthonormal** with respect to the measure

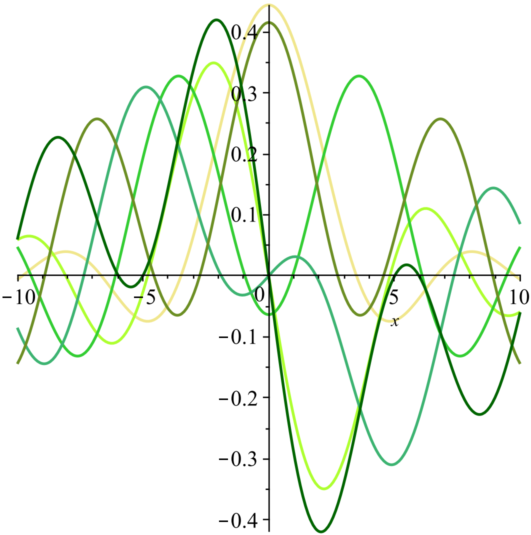
$$\mathrm{d}\mu(\xi) = \frac{2}{\pi}\chi_{[-1,1]}(\xi)\sqrt{1-\xi^2}\,\mathrm{d}\xi.$$

- We have $b_n = \frac{1}{2}$ for all $n \in \mathbb{Z}_+$ (so $\mathcal{D}$ is also a **Toeplitz matrix**)

$$\begin{aligned} \varphi_0(x) &\propto \int_{-1}^{1} (1-\xi^2)^{1/4} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}\xi \propto \frac{\mathrm{J}_1(x)}{x} \\ \varphi_1(x) &\propto \int_{-1}^{1} \xi(1-\xi^2)^{1/4} \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}\xi \propto \frac{\mathrm{J}_2(x)}{x}, \end{aligned}$$

- Here $\mathrm{J}_n(x)$ is the Bessel function of degree n.
- The expressions get more complicated...

Transformed Chebyshev functions



The generated Hilbert space

- Well known that Hermite functions are **complete** in $\mathrm{L}_2(\mathbb{R})$

The generated Hilbert space

- Well known that Hermite functions are **complete** in $\mathrm{L}_2(\mathbb{R})$
- What about transformed Legendre functions (spherical Bessel functions) or transformed Chebyshev functions?

The generated Hilbert space

- Well known that Hermite functions are **complete** in $\mathrm{L}_2(\mathbb{R})$
- What about transformed Legendre functions (spherical Bessel functions) or transformed Chebyshev functions?
- Clearly, their Fourier transforms are **compactly supported** in $[-1, 1]$

The generated Hilbert space

- Well known that Hermite functions are **complete** in $\mathrm{L}_2(\mathbb{R})$
- What about transformed Legendre functions (spherical Bessel functions) or transformed Chebyshev functions?
- Clearly, their Fourier transforms are **compactly supported** in $[-1, 1]$
- The **Paley-Wiener spaces** are closed subspaces of $\mathrm{L}_2(\mathbb{R})$ obtained by **restricting Fourier transforms** to a set $\Omega \subset \mathbb{R}$:

$$\mathcal{PW}_\Omega(\mathbb{R}) := \{\varphi \in \mathrm{L}^2(\mathbb{R}) : \mathcal{F}[\varphi](\xi) = 0 \text{ for a.e. } \xi \in \mathbb{R} \setminus \Omega\},$$

- Keywords: **Band-limiting**, **band-limited function spaces**. Numerous applications and relevance in signal processing

Transformed Carlitz functions

Consider the **hyperbolic secant** measure

$$\mathrm{d}\mu(\xi) = \mathsf{sech}^2(\pi\xi)\,\mathrm{d}\xi$$

This measure (after heroic algebra) is related to the **Carlitz polynomials** on a line in the complex plane.

Transformed Carlitz functions

Consider the **hyperbolic secant** measure

$$\mathrm{d}\mu(\xi) = \mathrm{sech}^2(\pi\xi)\,\mathrm{d}\xi$$

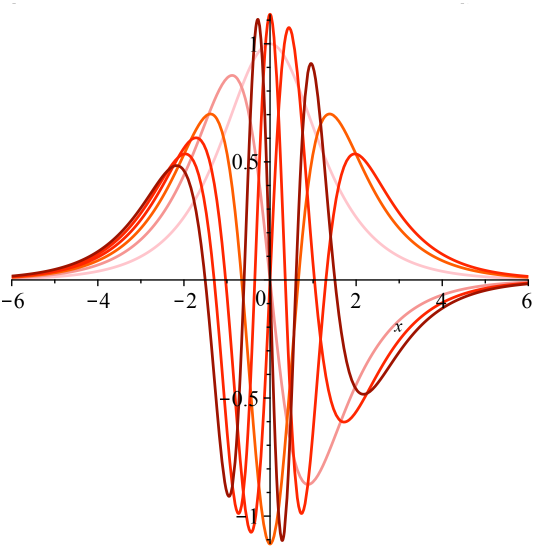
This measure (after heroic algebra) is related to the **Carlitz polynomials** on a line in the complex plane.

$$b_n = \frac{(n+1)^2}{\sqrt{(2n+1)(2n+3)}}$$

Up to a constant scaling,

$$\begin{aligned}
\varphi_0(x) &= \mathrm{sech}(x) \\
\varphi_1(x) &= -\sqrt{3}\,\tanh(x)\mathrm{sech}(x) \\
\varphi_2(x) &= \frac{\sqrt{5}}{2}\left(2\mathrm{sech}(x) - 3\mathrm{sech}^3(x)\right) \\
\varphi_3(x) &= \frac{\sqrt{7}}{2}\tanh(x)\left(2\mathrm{sech}^2(x) - 5\mathrm{sech}^4(x)\right)
\end{aligned}$$

Transformed Carlitz functions

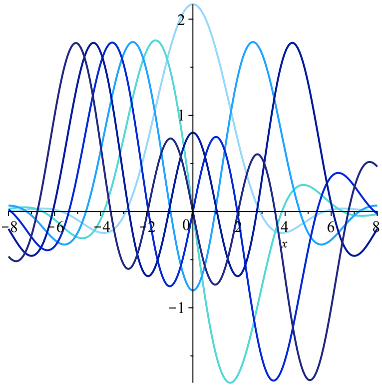


Transformed Freud functions

Polynomials orthogonal with respect to the measure $\mathrm{d}\mu(\xi) = \mathrm{e}^{-\xi^4}\,\mathrm{d}\xi$ are a particular instance of **Freud polynomials**.

$$\varphi_0(x) = \frac{2^{\frac{3}{4}}}{4\Gamma(\frac{3}{4})}\left\{2\pi\,{}_0F_2\left[\begin{matrix}-; \\ \frac{1}{2},\frac{3}{4};\end{matrix}\ \frac{x^4}{128}\right] - x^2\Gamma^2\left(\frac{3}{4}\right){}_0F_2\left[\begin{matrix}-; \\ \frac{5}{4},\frac{3}{2};\end{matrix}\ \frac{x^4}{128}\right]\right\},$$

The coefficients $\{b_n\}_{n\in\mathbb{Z}_+}$ satisfy so-called **string relations** (see Clarkson 2016).



Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:
  - Approximation properties of Φ?

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:
    - Approximation properties of Φ?
    - Interesting features? E.g. interlacing roots

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:
  - Approximation properties of Φ?
  - Interesting features? E.g. interlacing roots
  - Can expansions be computed rapidly and stably?

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:
  - Approximation properties of Φ?
  - Interesting features? E.g. interlacing roots
  - Can expansions be computed rapidly and stably?
  - Can $\mathrm{e}^{a\mathcal{D}}$ be effectively approximated?

Summary and future directions

- There is a **one-to-one correspondence** between orthonormal systems with a real, skew-symmetric, tridiagonal, irreducible differentiation matrix and orthonormal polynomials with respect to a symmetric probability measure* $\mathrm{d}\mu(\xi) = w(\xi)\mathrm{d}\xi$.
- The orthonormal systems generated are **complete in the Paley-Wiener space** for the support of the measure $\mathrm{d}\mu$.
- Plethora of possibilities and questions for Φ:
  - Approximation properties of Φ?
  - Interesting features? E.g. interlacing roots
  - Can expansions be computed rapidly and stably?
  - Can $\mathrm{e}^{a\mathcal{D}}$ be effectively approximated?
  - Can new, improved, practical, L_2 stable spectral methods for time-dependent PDEs be developed following this work?