On the Maximal Subgroups of $\mathrm{E}_8(3)$ and $\mathrm{E}_8(2)$

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Abstract

This thesis forms part of a larger project which aims to complete the classification of the maximal subgroups of the finite simple exceptional groups of Lie type $E_8(3)$ and $E_8(2)$. Using the work of Liebeck and Seitz [50], one can form a list of possible candidates for the maximal subgroups of $E_8(q)$. In this thesis, we predominantly deal with the cases $L_2(3^n)$ for $n \in \{3, 4, 5, 6, 7\}$, $L_3(5)$, $L_2(11)$ and M_{11} inside $E_8(3)$. For the groups $H \cong L_2(81)$ and $H \cong L_3(5)$, we construct representatives for all conjugacy classes of subgroups inside $E_8(3)$ isomorphic to H. For all groups mentioned here, we find that none of them exhibit maximal embeddings into $E_8(3)$ apart from $L_3(5)$ (and $L_2(27)$, where this work is unfinished). In the case of $H \cong L_3(5)$, we find that $E_8(3)$ contains one conjugacy class of subgroups isomorphic to H and that these subgroups are maximal.

In addition to this work, we also consider the case $L_2(8) < E_8(2)$; this is a continuation of the work of Javed [41] and Neuhaus [62]. Brauer characters and modular representation theory play a key role in this research. As such, Brauer character tables (over characteristic 3) for all groups mentioned here and more are given.

Furthermore, a comprehensive report on the maximal tori of the finite exceptional simple groups of Lie type is given; this aligns with the work shown in my publication with Javed, Parkin and Rowley [42]. In this thesis, we give extra examples which are not present in the paper.

The computer package Magma was used extensively throughout this research and the appendix contains all supporting code for this work.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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1 Introduction

Classifying maximal subgroups has been a prominent problem since the dawn of group theory. To understand the maximal subgroups of a given group G is to understand G itself. By knowing the maximal subgroups, through a recursive study, one could theoretically find all there is to know about the subgroups and elements of G. Given that the finite simple groups are the building blocks of finite group theory, there is a natural desire to understand the maximal subgroups of these structures.

Since the completion of the classification of the finite simple groups, group theorists from all over the world have tackled classifying their maximal subgroups. The maximal subgroups of the alternating groups $\mathrm{Alt}(n)$ (and the symmetric groups $\mathrm{Sym}(n)$) were classified using the O'Nan Scott theorem by Saxl, Liebeck and Praeger in [54]. The classification of the maximal subgroups of the sporadic groups was a longstanding project that was finally completed upon the completion of the classification for the Monster group in 2023 by Dietrich, Lee, and Popiel [28]. We refer the reader to Wilson's paper [76] which gives an excellent historical survey regarding the classification of the maximal subgroups of the sporadic simple groups.

Significant effort has also been put into understanding the maximal subgroups of the finite simple groups of Lie type. The classification for groups of classical type was completed by Aschbacher in 1984 [1]; this was later followed by a more comprehensive study by Liebeck and Kleidman in 1990 [46]. Suzuki found the maximal subgroups for the infinite family of simple groups which share his name $\mathrm{Sz}(n)$ (or equivalently ${}^2\mathrm{B}_2(q)$) in 1962 [71]. The maximal subgroups of the Ree groups ${}^2\mathrm{G}_2(q)$ were classified by Kleidman in 1988 [45]; this paper also includes a classification for the Chevalley groups $\mathrm{G}_2(q)$ for q odd and all related automorphism groups. Kleidman also provides a list of maximal subgroups for the Steinberg triality groups ${}^3\mathrm{D}_4(q)$ and their automorphism groups in [44]. Malle classifies the maximal subgroups for all groups of the form ${}^2\mathrm{F}_4(2^{2n+1})$ in [60]. Furthermore, the Tits group ${}^2\mathrm{F}_4(2)'$ has been completed independently by Tchakerian [72] and Wilson [74].

Now moving on to the finite simple groups of exceptional Lie type. As mentioned previously, Kleidman deals with the case of $\mathrm{G}_2(q)$ for q odd in [45]; the case of q even is dealt with by Cooperstein in [21]. We remark that Aschbacher gives details for all prime powers q in his paper [2]. The maximal subgroups of $\mathrm{F}_4(2)$ (and its automorphism group) were found in 1989 by Wilson and Norton [63]. Others have also had success by considering $q = 2$ for other exceptional groups, such as Kleidman and Wilson with $\mathrm{E}_6(2)$ [43], Wilson again with ${}^2\mathrm{E}_6(2)$ in [75] and Rowley, Bates and Ballantyne with $\mathrm{E}_7(2)$ [4]. The classification for $\mathrm{E}_8(2)$ is very near completion thanks to the work of Rowley, Neuhaus [62], McGaw [61], Javed [41], Ballantyne, Ward, Aubad and myself. Recently, Craven has given a complete classification for all groups of the form $\mathrm{F}_4(q)$, $\mathrm{E}_6(q)$ and ${}^2\mathrm{E}_6(q)$ [23]. Moreover, Craven makes great strides in the classification for $\mathrm{E}_7(q)$ [25] where only a few minor cases remain. This leaves only the groups $\mathrm{E}_8(q)$ for $q \geq 2$. In this thesis, we explore some of the potential maximal subgroups for the case $q = 3$ with an excursion into the case $q = 2$.

Some information is already known about the maximal subgroups of $\mathrm{E}_8(3)$, mainly thanks to the extensive work of Liebeck and Seitz. Their work is not solely focused on $\mathrm{E}_8(3)$, but on exceptional groups of Lie type (both finite and algebraic) in a more general context. Their paper [48] enumerates the class of maximal subgroups known as maximal rank subgroups for all finite exceptional groups of Lie type. Moreover, they were able to further reduce the possibilities for the maximal subgroups of exceptional groups of Lie type in [52]. The 2003 survey [50] provides a concise resource detailing their joint work on the topic. It is from here that we obtain the theorem upon which this thesis, and [62], [41], [61], [4], are based.

Theorem 1.1 (Liebeck, Seitz, [50]). *Let G be a simple algebraic group defined over the algebraic closure $\overline{\mathrm{GF}(q)}$ where $\mathrm{GF}(q)$ denotes the field in prime characteristic p for $q = p^a$. Let σ be a Frobenius morphism* *of G and denote by G_σ the fixed point group of σ defined by $G_\sigma = \{g \in G \mid g\sigma = g\}$. Let H be a maximal subgroup of the finite exceptional group G_σ over the field* $\mathrm{GF}(q)$*. Then one of the following holds:*

i) *$H = M_\sigma$ where M is maximal closed σ-stable of positive dimension in G; the possibilities for M and H are:*

  (a) *Both M and H are parabolic subgroups;*

  (b) *M is reductive of maximal rank and the possibilities for H are determined in [48];*

  (c) $G = \mathrm{E}_7$*,* $p > 2$ *and* $H = (2^2 \times \Omega_8^+(q).2^2).\mathrm{Sym}(3)$ *or* ${}^3\mathrm{D}_4(q).3$*;*

  (d) $G = \mathrm{E}_8$*,* $p > 5$ *and* $H = \mathrm{PGL}_2(q) \times \mathrm{Sym}(5)$*;*

  (e) *M is as in ([50], Table 1) and $H = M_\sigma$ is as in ([50], Table 3).*

ii) *H is of the same type as G;*

iii) *H is an exotic local subgroup (see [52]);*

iv) $G = \mathrm{E}_8$*,* $p > 5$ *and* $H = (\mathrm{Alt}(5) \times \mathrm{Alt}(6)).2^2$*;*

v) *$F^*(H) = H_0$ is simple and not in* $\mathrm{Lie}(p)$*. The possibilities for H_0 are given upto isomorphism in [51].*

vi) *$F^*(H) = H(q_0)$ is simple and in* $\mathrm{Lie}(p)$*. Moreover, $rank(H(q_0)) \leq \frac{1}{2} rank(G)$ and one of the following holds:*

  (a) *Both M and H are parabolic subgroups;*

  (b) $q_0 \leq 9$*;*

  (c) $H(q_0) \cong \mathrm{A}_2(16)$ *or* $\mathrm{A}_2^\epsilon(16)$*;*

  (d) $q_0 \leq (2, p-1)u(G)$ *and* $H(q_0) = \mathrm{A}_1(q_0), {}^2\mathrm{B}_2(q_0)$ *or* ${}^2\mathrm{G}_2(q_0)$*. The values for $u(G)$ for each exceptional group are shown in the table below (see [47] for more details on how they were calculated):*

G	G_2	F_4	E_6	E_7	E_8
$u(G)$	*12*	*68*	*124*	*388*	*1312*

In cases (i) - (iv), H is determined upto G_σ-conjugacy.

For $G_\sigma \cong \mathrm{E}_8(3)$, the maximal subgroups corresponding to Theorem 1.1 (i) - (iv) are already known and determined. As such, completing a classification of the maximal subgroups is reduced to considering Theorem 1.1 (v) - (vi). From these points, we obtain a list of simple groups H_0 such that any almost simple group H with $H_0 \leq H$ is a potential maximal subgroup of $\mathrm{E}_8(3)$. We say a group H is almost simple if H contains a non-abelian simple group $H_0 = F^*(H)$ such that $H_0 \leq H \leq \mathrm{Aut}(H_0)$. The simple groups which correspond to Theorem 1.1 (v) and (vi) are given in Tables 1 and 2 respectively. We remark that (v) gives $\mathrm{Alt}(n)$ for $5 \leq n \leq 18$ as possibilities, however Craven [24] and Litterick [56] eliminate many of these cases, leaving only $\mathrm{Alt}(6)$ and $\mathrm{Alt}(7)$. As such, $\mathrm{Alt}(n)$ for $n \in \{5, 8, 9, ..., 18\}$ are not included in Tables 1 and 2. We remark that a more refined list for the case $F^*(H) \notin \mathrm{Lie}(p)$ can be found in [56].

Table 1: Possible maximal subgroups $H_0 < \mathrm{E}_8(3)$, $H_0 \in \mathrm{Lie}(3)$.

$\mathrm{Alt}(6) \cong \mathrm{L}_2(9)$	$\mathrm{L}_2(27)$	$\mathrm{L}_2(81)$	$\mathrm{L}_2(243)$	$\mathrm{L}_2(729)$
$\mathrm{L}_2(2187)$	$\mathrm{L}_3(3)$	$\mathrm{L}_3(9)$	$\mathrm{L}_4(3)$	$\mathrm{L}_4(9)$
$\mathrm{L}_5(3)$	$\mathrm{L}_5(9)$	$\mathrm{U}_3(3)$	$\mathrm{U}_3(9)$	$\mathrm{U}_4(3)$
$\mathrm{U}_4(9)$	$\mathrm{U}_5(3)$	$\mathrm{U}_5(9)$	$\mathrm{Sp}_4(3)$	$\mathrm{Sp}_4(9)$
$\mathrm{Sp}_6(3)$	$\mathrm{Sp}_6(9)$	$\mathrm{Sp}_8(3)$	$\mathrm{Sp}_8(9)$	$\Omega_7(3)$
$\Omega_7(9)$	$\Omega_9(3)$	$\Omega_9(9)$	$\Omega_4^+(9)$	$\Omega_8^+(3)$
$\Omega_8^+(9)$	$\Omega_8^-(3)$	$\Omega_8^-(9)$	$\mathrm{G}_2(3)$	$\mathrm{G}_2(9)$
${}^2\mathrm{G}_2(27)$	${}^2\mathrm{G}_2(243)$	${}^2\mathrm{G}_2(2187)$	$\mathrm{F}_4(3)$	$\mathrm{F}_4(9)$
${}^3\mathrm{D}_4(3)$	${}^3\mathrm{D}_4(9)$			

Table 2: Possible maximal subgroups $H_0 < \mathrm{E}_8(3)$, $H \notin \mathrm{Lie}(3)$.

$\mathrm{L}_2(7)$	$\mathrm{L}_2(8)$	$\mathrm{L}_2(11)$	$\mathrm{L}_2(13)$	$\mathrm{L}_2(16)$
$\mathrm{L}_2(17)$	$\mathrm{L}_2(19)$	$\mathrm{L}_2(25)$	$\mathrm{L}_2(29)$	$\mathrm{L}_2(31)$
$\mathrm{L}_2(32)$	$\mathrm{L}_2(41)$	$\mathrm{L}_2(49)$	$\mathrm{L}_2(61)$	$\mathrm{Alt}(7)$
$\mathrm{L}_3(5)$	$\mathrm{U}_3(8)$	$\mathrm{Sp}_6(2)$	$\Omega_8^+(2)$	${}^3\mathrm{D}_4(2)$
${}^2\mathrm{F}_4(2)'$	$\mathrm{Sz}(8)$	M_{11}	Th	

It remains to consider each of the groups in these tables to see whether they could embed as maximal subgroups of $\mathrm{E}_8(3)$. Analogous lists were made for the groups $\mathrm{E}_7(2)$ and $\mathrm{E}_8(2)$ in [4] and [41], [62], [61]. Determining the maximality of the groups in Tables 1 and 2 was split between myself, Parkin [64] and Rowley. As such, not all the groups listed are considered in this thesis. The structure of this thesis is as follows.

Section 2 outlines some theory surrounding algebraic groups and groups of Lie type before moving on to study $\mathrm{E}_8(3)$ more specifically. Details regarding the centralisers of semisimple elements, Sylow subgroup structures and the current known maximal subgroups are given here. Following this, the primary method of disproving the maximality of the groups in Tables 1 and 2 is given, including the definition of feasible decompositions. Subsequently, several groups are then shown not to be maximal in $\mathrm{E}_8(3)$.

Section 3 presents all work conducted on the groups $\mathrm{L}_2(3^n)$ for $n \in \{3,4,5,6,7\}$. We will determine that all groups mentioned here (and their automorphic extensions) are not maximal in $\mathrm{E}_8(3)$ apart from $\mathrm{L}_2(27)$, where this work is incomplete. Moreover, we detail an algorithm which explicitly constructs subgroups of the form $\mathrm{L}_2(3^n)$ inside $\mathrm{E}_8(3)$. In the $\mathrm{L}_2(81)$ case, all such subgroups are constructed in $\mathrm{E}_8(3)$ (upto conjugacy) followed by the automorphic extensions $\mathrm{PGL}_2(81)$ and $\mathrm{Aut}(\mathrm{L}_2(81))$. Overgroups of $\mathrm{Aut}(\mathrm{L}_2(81))$ are then found, thus disproving maximality. None of the work in this section, or any of the other sections in this thesis, would be possible without the use of Magma; the vast majority of calculations presented in this thesis are done inside the irreducible 248-dimensional matrix representation of $\mathrm{E}_8(3)$.

Section 4 gives the proof of perhaps the most interesting result in this thesis, namely that $\mathrm{L}_3(5)$ exists as a maximal subgroup of $\mathrm{E}_8(3)$ and that this subgroup is unique upto conjugacy. As with $\mathrm{L}_2(81)$, we explicitly construct $\mathrm{L}_3(5)$ inside $\mathrm{E}_8(3)$. Following this, we show that $\mathrm{E}_8(3)$ contains no subgroups isomorphic to $\mathrm{Aut}(\mathrm{L}_3(5))$. The algorithm presented here can also be used to construct the maximal Thompson group Th inside $\mathrm{E}_8(3)$.

In Section 5, we consider both $\mathrm{L}_2(11)$ and M_{11} simultaneously. As $\mathrm{L}_2(11)$ is a maximal subgroup of M_{11}, we will prove that we can reduce the problem of studying the maximality of M_{11} to studying only $\mathrm{L}_2(11)$. We shall find that neither group exists as a maximal subgroup of $\mathrm{E}_8(3)$.

Section 6 details work that is analogous to that in Section 3, but instead we study the potential maximality of $\mathrm{L}_2(8)$ inside $\mathrm{E}_8(2)$. This is a continuation of the work conducted by Javed [41] and Neuhaus [62] and we follow all the notation presented in Section 3. Despite my best efforts, this work remains incomplete.

Section 7 is my publication [42] alongside Rowley, Parkin and Javed which gives a comprehensive report on the maximal tori of finite exceptional groups of Lie type. Extra details and examples are given here which are not present in [42].

Section 8 gives the Brauer character tables and feasible decompositions for all groups considered in this thesis and more. Furthermore, for various groups from Tables 1 and 2, the structures of the projective covers of the irreducible GF(3)-modules are given here when they are calculable.

This thesis is accompanied by an appendix which is split into 2 parts. The first part gives all the tedious details regarding how the orders of the centralisers of semisimple elements in $\mathrm{E}_8(3)$ (which are given in Section 2) were calculated. The second half gives all the Magma code which supports the work throughout this thesis. Some of this code, particularly that involving groups of the form $\mathrm{L}_2(3^n)$, is adapted from the code shown in [41] and [62].

2 Background and Preliminary Results

2.1 Algebraic Groups and Groups of Lie Type

We shall briefly outline some background theory and definitions surrounding algebraic groups and groups of Lie type. Almost all work conducted in this thesis is done in the finite case, so many of the results given here are not explicitly used. Much more detail can be found in Malle & Testerman [59], Carter [16] and Litterick [56].

Let G be a linear algebraic group defined over the algebraically closed field $\overline{K} = \overline{\mathrm{GF}(q)}$, where $q = p^a$. It is a well known result that any linear algebraic group is isomorphic (as algebraic groups) to a closed subgroup of $\mathrm{GL}_n(\overline{K})$ for some n. The converse is also true, in that any closed subgroup of $\mathrm{GL}_n(\overline{K})$ is a linear algebraic group. We say that G is connected if it cannot be decomposed into a disjoint union of two non-empty closed subsets. For V an n-dimensional vector space over $\overline{K}$ and $\rho : G \to \mathrm{GL}(V)$, we say an element $\rho(g) \in \mathrm{GL}(V)$ is unipotent if $(\rho(g) - 1)^m = 0$ for some m and semisimple if it is diagonalisable endomorphism. We remark that $\rho(g)$ is unipotent if and only if it has p-power order. By the Jordan decomposition, every element $g \in G$ can be written uniquely in the form $g = g_u g_s = g_s g_u$ where $\rho(g_u)$ is unipotent and $\rho(g_s)$ is semisimple. We say that g is semisimple if $g = g_s$ and g is unipotent if $g = g_u$. We let G_u denote the set of all unipotent elements in G and we say that G is unipotent if $G = G_u$. If G is unipotent, then it can be embedded into $\mathrm{GL}_n(\overline{K})$ as a subgroup consisting entirely of upper triangular matrices. As the group of all upper triangular matrices is nilpotent, it follows that G is nilpotent and hence soluble. If G is connected and soluble, then G_u is a closed connected normal subgroup.

Let $R(G)$ denote the radical of G, i.e, the maximal closed connected soluble normal subgroup of G. We call the subgroup $R(G)_u \leq R(G)$ consisting of all unipotent elements the unipotent radical of G. As $R(G)$ is connected and soluble, it follows that $R(G)_u$ is a normal connected unipotent subgroup of $R(G)$. By the maximality of $R(G)$, any closed connected normal unipotent subgroup of G is contained in $R(G)_u$. As the unipotent elements of G are precisely the elements of p-power order, it follows that $R(G)_u$ is the largest connected normal subgroup of G which consists entirely of p-power elements. If $R(G)_u$ is trivial, then we say that G is reductive. If $R(G)$ is trivial and G is connected, then we say G is semisimple. Clearly, all semisimple groups are connected and reductive. A semisimple algebraic group is simple if it has no proper closed connected normal subgroups. This definition of $R(G)_u$ is analogous to that of the p-core $O_p(G)$ for finite groups G. We shall see in Section 2.3 that we use a slightly different notation for $O_p(G)$ in $E_8(3)$.

A linear algebraic group is called a torus if it is isomorphic to some direct product of multiplicative groups over $\overline{K}$. As such, each torus has an embedding into $GL_n(\overline{K})$ consisting entirely of diagonal matrices. Consequently, all tori are abelian groups. We say that $T \leq G$ is a maximal torus if it is maximal amongst all other torus subgroups of G. It is well known that, in the algebraic case, all maximal tori are conjugate. We remark that when reduced to the finite case, there can be multiple conjugacy classes of maximal tori in the finite group. Much more detail on this is given in [42] and Section 7.

Let $\overline{V}$ denote the Lie algebra of G. This is a vector space over $\overline{K}$ equipped with a binary operation known as the Lie bracket. The group G acts on $\overline{V}$ which gives a representation $\mathbf{ad} : G \to GL(\overline{V})$; we call this the adjoint representation of G.

A character of G is a morphism $\chi : G \to \overline{K}_\times$ where $\overline{K}_\times$ denotes the multiplicative group of $\overline{K}$; this is also an algebraic group. We denote by $X(G)$ the set of all characters of G. As there is one conjugacy class of maximal tori in G, it suffices to let $T < G$ be any maximal torus for the following definitions. Moreover, we assume that G is a connected reductive algebraic group. For $\chi \in X(G)$, define $\overline{V}_\chi = \{v \in \overline{V} \mid tv = \chi(t)v, \ \forall t \in T\}$ and $\Phi(G) = \{\chi \in X(T) \mid \chi \neq 0, \ \overline{V}_\chi \neq 0\}$. We call $\Phi(G)$ the set of roots of G with respect to T. Moreover, $\Phi(G)$ forms a root system for G. We may consider $\Phi(G)$ to be a subspace of some finite-dimensional Euclidean space E equipped with the standard Euclidean inner product $(\cdot, \cdot)$. Two roots $\alpha, \beta \in \Phi(G)$ are orthogonal if $(\alpha, \beta) = 0$. There exists a subset $\Pi(G) \subset \Phi(G)$ which forms a basis for $\Phi(G)$ such that for any $\alpha \in \Phi(G)$, there exist roots $\alpha_1, \dots, \alpha_n \in \Pi(G)$ and integers $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that $\alpha = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n$ where either $\lambda_i \leq 0$ for all $i = 1, \dots, n$ or $\lambda_i \geq 0$ for all $i = 1, \dots, n$. If $\lambda_i \geq 0$ for all $i = 1, \dots, n$, then we say α is a positive root. Otherwise, α is a negative root. We let $\Phi(G)^+$ denote the set of all positive roots of G and $\Phi(G)^-$ the set of all negative roots. As such an expression exists for all roots in $\Phi(G)$, we have that $\Phi(G) = \Phi(G)^+ \cup \Phi(G)^-$. We call $\Pi(G)$ the set of fundamental roots of G.

If $\Pi(G)$ cannot be decomposed into the union of two mutually orthogonal proper subsets $\Phi_1 \cup \Phi_2$, then we say that $\Phi(G)$ is an irreducible root system. Each irreducible root system is associated with a Dynkin diagram which consists of $|\Pi(G)|$ nodes. It is a well known result that all simple algebraic groups have irreducible root systems. The irreducible root systems have been classified into 9 different isomorphism types, namely A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8. The simple groups of classical type are those whose root system is of type A_n, B_n, C_n or D_n. For the root systems of type G_2, F_4, E_6, E_7, E_8, we say the associated simple group is of exceptional type. We remark that a root system is irreducible if and only if its associated Dynkin diagram is connected. Hence, classifying the irreducible root systems can be reduced to classifying the connected Dynkin diagrams. We refer the reader to [40] for an account of the classification and more details on root systems, but there are many other resources which also give these details. In Section 2.3, we give more detail on how we use root systems to learn more about $E_8(3)$. Moreover, we introduce notation there that will be used throughout this thesis.

Finally, we turn our attention to the finite case, where almost all the work in this thesis is conducted. Let $\sigma : \overline{K} \to \overline{K}, \ x \to x^q$, be the field automorphism of $\overline{K}$ which fixes $K = GF(q)$ pointwise. By considering the action of σ on the matrix entries, we obtain a group homomorphism $\sigma : \mathrm{GL}_n(\overline{K}) \rightarrow \mathrm{GL}_n(K)$ into the finite group $\mathrm{GL}_n(K) = \mathrm{GL}_n(q) = \{g \in \mathrm{GL}_n(\overline{K}) \mid \sigma(g) = g\}$; we call σ the Frobenius map of $\mathrm{GL}_n(\overline{K})$ with respect to K. For a given linear algebraic group G defined over $\overline{K} = \overline{\mathrm{GF}(q)}$ with Frobenius morphism σ, we let G_σ denote the fixed point group of G with respect to σ. If G is a semisimple algebraic group, then G_σ is called a finite group of Lie type. If G denotes the semisimple algebraic group of type E_8 defined over $\overline{\mathrm{GF}(3)}$ and σ is the Frobenius map of G which sends each matrix entry to its third power, then $G_\sigma \cong \mathrm{E}_8(3)$.

2.2 Fundamental Properties of $\mathrm{E}_8(3)$

In this section we cover some fundamental properties of $\mathrm{E}_8(3)$ that will be used frequently throughout this thesis. Unless stated otherwise, we use the notation given in the Atlas [20]. As remarked briefly in Section 2.1, $\mathrm{E}_8(3)$ exists as the fixed point group of the algebraic group of type E_8 defined over the algebraic closure $\overline{\mathrm{GF}(3)}$ under the standard Frobenius map. This linear algebraic group is simply connected and has only one isogeny type (this is true for all algebraic groups of type E_8 defined over $\overline{\mathrm{GF}(q)}$). Unless specified otherwise, we shall use G to denote $\mathrm{E}_8(3)$ for the remainder of this thesis.

The order of G is:

$$|\mathrm{E}_8(3)| = 2^{30} \cdot 3^{120} \cdot 5^5 \cdot 7^4 \cdot 11^4 \cdot 13^4 \cdot 19 \cdot 31 \cdot 37 \cdot 41^2 \cdot 61^2 \cdot 73^2 \cdot 271 \cdot 547 \cdot 757 \cdot 1093 \cdot 1181 \cdot 4561 \cdot 6481.$$

For $p = 7, 13, 41, 61, 73$, the Sylow p-subgroups are elementary abelian. The structures of the Sylow 5-subgroup and Sylow 11-subgroup are 5.5^4 and 121×121 respectively. From [48] and Table 3, the Sylow 2-subgroup has a center of order 2. The orders of all centralisers of semisimple elements of prime order and composite order ≤ 40 are given in Table 3. For a given class representative x, also given in Table 3 is the dimension of the fixed space $C_V(x)$ and the Brauer character value $\psi(x)$ of x on V where V denotes the minimal irreducible 248-dimensional adjoint GF(3)-module for G. A star $*$ in the $\psi(x)$ column indicates that $\psi(x)$ is not an integer and is instead located in Table 5 where it is expressed as a sum of the form $\omega_n^a + \omega_n^b + \cdots + z$ where ω_n denotes a primitive n^{th} root of unity. For all classes containing elements of order 41, 61, 73, 121, 757 and 1093, the Brauer character value is computable but omitted from Table 5 due to the length of the expressions. A ? in this column indicates that $\psi(x)$ is not known for the particular class. Additionally, the structures of the centralisers of prime order elements are given in Table 4. More detail on how these centraliser orders were obtained and the notation used in Table 3 is given in Section 9.1.

Table 3: Centralisers of semisimple elements in $E_8(3)$.

Class	$\|C_G(x)\|$	$\dim(C_V(x))$	$\psi(x)$	Powers
1A	$\|E_8(3)\|$	248	248	-
2A	$2^{27} \cdot 3^{64} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547 \cdot 757 \cdot 1093$	136	24	-
2B	$2^{30} \cdot 3^{56} \cdot 5^4 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41^2 \cdot 61 \cdot 73 \cdot 547 \cdot 1093$	120	-8	-
4A	$2^{26} \cdot 3^{63} \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 13^3 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73 \cdot 547 \cdot 757 \cdot 1093$	134	132	2A
4B	$2^{26} \cdot 3^{42} \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 61 \cdot 73 \cdot 547$	92	64	2B
4C	$2^{24} \cdot 3^{37} \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73$	82	28	2A
4D	$2^{26} \cdot 3^{31} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 61$	70	4	2A
4E	$2^{24} \cdot 3^{29} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 41 \cdot 61 \cdot 547$	66	-4	2A
4F	$2^{23} \cdot 3^{28} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 41 \cdot 61 \cdot 547$	64	8	2B
4G	$2^{26} \cdot 3^{26} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 41 \cdot 61$	60	0	2B
5A	$2^{20} \cdot 3^{30} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61 \cdot 73$	68	23	-
5B	$2^{11} \cdot 3^{20} \cdot 5^5 \cdot 41 \cdot 73 \cdot 1181$	48	-2	-
7A	$2^{19} \cdot 3^{36} \cdot 5^2 \cdot 7^4 \cdot 13^2 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73$	80	52	-
7B	$2^{11} \cdot 3^{15} \cdot 7^4 \cdot 13^2 \cdot 73$	38	3	-
8A	$2^{22} \cdot 3^{36} \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 19 \cdot 37 \cdot 41 \cdot 61 \cdot 73$	80	78	4C, 2A
8B	$2^{20} \cdot 3^{36} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^3 \cdot 41 \cdot 73 \cdot 757$	80	26	4A, 2A
8C	$2^{22} \cdot 3^{30} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61 \cdot 73$	68	44	4B, 2B
8D	$2^{20} \cdot 3^{28} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13^2 \cdot 41 \cdot 1093$	64	-6	4A, 2A
8E	$2^{22} \cdot 3^{22} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41$	52	12	4B, 2B
8F	$2^{19} \cdot 3^{21} \cdot 5 \cdot 7^2 \cdot 13 \cdot 61 \cdot 547$	50	34	4E, 2A
8G	$2^{22} \cdot 3^{20} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 61$	48	14	4C, 2A
8H	$2^{21} \cdot 3^{18} \cdot 5^2 \cdot 7 \cdot 13^2 \cdot 41$	44	-4	4B, 2B
8I	$2^{20} \cdot 3^{16} \cdot 5 \cdot 7^2 \cdot 13 \cdot 61$	40	-2	4C, 2A
8J	$2^{17} \cdot 3^{15} \cdot 5 \cdot 7 \cdot 11^2 \cdot 13^2$	38	6	4D, 2A
8K	$2^{19} \cdot 3^{15} \cdot 5 \cdot 7^2 \cdot 13 \cdot 41$	38	16	4G, 2B
8L	$2^{18} \cdot 3^{14} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	36	2	4D, 2A
8M	$2^{17} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	34	-2	4D, 2A
8N	$2^{19} \cdot 3^{13} \cdot 5 \cdot 7^2 \cdot 61$	34	2	4E, 2A
8O	$2^{15} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	32	0	4F, 2B
8P	$2^{18} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 61$	32	4	4G, 2B
8Q	$2^{19} \cdot 3^{11} \cdot 5^2 \cdot 7 \cdot 13$	30	0	4G, 2B

10A	$2^{20} \cdot 3^{30} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61 \cdot 73$	68	87	5A, 2B
10B	$2^{19} \cdot 3^{14} \cdot 5^4 \cdot 7 \cdot 13$	36	7	5A, 2B
10C	$2^{18} \cdot 3^{14} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	36	-1	5A, 2A
10D	$2^{11} \cdot 3^{12} \cdot 5^4 \cdot 41 \cdot 73$	32	-3	5A, 2B
10E	$2^{11} \cdot 3^{12} \cdot 5^4 \cdot 41 \cdot 73$	32	22	5B, 2B
10F	$2^{10} \cdot 3^8 \cdot 5^4 \cdot 73$	24	2	5B, 2B
11A	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^4 \cdot 13$	28	6	-
13AB	$2^{17} \cdot 3^{36} \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^4 \cdot 41 \cdot 73 \cdot 757$	80	*	-
13CD	$2^{10} \cdot 3^{15} \cdot 7^2 \cdot 13^4 \cdot 73$	38	*	-
13E	$2^6 \cdot 3^{12} \cdot 7^2 \cdot 13^4 \cdot 73$	32	14	-
13F	$2^8 \cdot 3^6 \cdot 13^4$	20	1	-
14A	$2^{19} \cdot 3^{20} \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 41 \cdot 61$	48	20	7A, 2B
14B	$2^{17} \cdot 3^{16} \cdot 5 \cdot 7^3 \cdot 13 \cdot 61$	40	-4	7A, 2A
14C	$2^{11} \cdot 3^{13} \cdot 7^3 \cdot 13^2 \cdot 73$	34	31	7B, 2A
14D	$2^{11} \cdot 3^7 \cdot 7^3 \cdot 13$	22	3	7B, 2A
14E	$2^{11} \cdot 3^5 \cdot 7^2 \cdot 13$	18	-1	7B, 2B
16A	$2^{13} \cdot 3^{12} \cdot 5^3 \cdot 41 \cdot 73$	32	-4	8C
16B	$2^{15} \cdot 3^{10} \cdot 5^2 \cdot 7 \cdot 61$	28	24	8P
16C	$2^{13} \cdot 3^{10} \cdot 5^2 \cdot 11^2 \cdot 13$	28	4	8E
16D	$2^{13} \cdot 3^7 \cdot 5 \cdot 7 \cdot 41$	22	6	8K
16E	$2^{12} \cdot 3^7 \cdot 5 \cdot 13 \cdot 41$	22	0	8H
16F	$2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$	20	8	8P
16G	$2^{13} \cdot 3^6 \cdot 5 \cdot 7 \cdot 13$	20	10	8Q
16H	$2^{13} \cdot 3^5 \cdot 5 \cdot 41$	18	-2	8K
16I	$2^{14} \cdot 3^4 \cdot 5 \cdot 7$	16	0	8P
16J	$2^{13} \cdot 3^4 \cdot 5 \cdot 13$	16	2	8Q
16K	$2^9 \cdot 3^4 \cdot 5^2 \cdot 41$	16	0	8O
19A	$2^5 \cdot 3^3 \cdot 7 \cdot 19 \cdot 37$	14	1	-
20A	$2^{19} \cdot 3^{20} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41$	48	19	10A, 4B
20B	$2^{19} \cdot 3^{20} \cdot 5^3 \cdot 7 \cdot 13 \cdot 41 \cdot 61$	48	35	10A, 4G
20C	$2^{17} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	34	7	10C, 4A
20D	$2^{17} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	34	39	10C, 4D
20E	$2^{18} \cdot 3^{12} \cdot 5^3 \cdot 7 \cdot 13$	32	-5	10A, 4G

20F	$2^{16} \cdot 3^{8} \cdot 5^{3} \cdot 7$	24	-1	10B, 4B
20G	$2^{16} \cdot 3^{8} \cdot 5^{3} \cdot 7$	24	15	10B, 4G
20H	$2^{16} \cdot 3^{7} \cdot 5^{2} \cdot 13$	22	3	10C, 4C
20I	$2^{16} \cdot 3^{7} \cdot 5^{2} \cdot 7$	22	11	10C, 4E
20J	$2^{15} \cdot 3^{6} \cdot 5^{2} \cdot 7$	20	3	10B, 4F
20K	$2^{9} \cdot 3^{6} \cdot 5^{3} \cdot 73$	20	4	10F, 4B
20L	$2^{11} \cdot 3^{6} \cdot 5^{3} \cdot 13$	20	5	10B, 4G
20M	$2^{16} \cdot 3^{5} \cdot 5^{2}$	18	-1	10C, 4D
20N	$2^{10} \cdot 3^{5} \cdot 5^{2} \cdot 41$	18	1	10C, 4E
20O	$2^{9} \cdot 3^{4} \cdot 5^{2} \cdot 41$	16	-2	10E, 4F
20P	$2^{9} \cdot 3^{2} \cdot 5^{3}$	12	0	10F, 4G
22A	$2^{9} \cdot 3^{6} \cdot 5 \cdot 11^{2} \cdot 13$	20	14	11A, 2A
22B	$2^{8} \cdot 3^{4} \cdot 11^{2} \cdot 13$	16	2	11A, 2B
26AB	$2^{17} \cdot 3^{20} \cdot 5^{2} \cdot 7 \cdot 11^{2} \cdot 13^{2} \cdot 41$	48	*	13AB, 2B
26CD	$2^{15} \cdot 3^{16} \cdot 5 \cdot 7 \cdot 11^{2} \cdot 13^{3}$	40	*	13AB, 2A
26EF	$2^{10} \cdot 3^{13} \cdot 7^{2} \cdot 13^{3} \cdot 73$	34	*	13CD, 2A
26GH	$2^{10} \cdot 3^{7} \cdot 7 \cdot 13^{3}$	22	*	13CD, 2A
26IJ	$2^{10} \cdot 3^{5} \cdot 7 \cdot 13^{2}$	18	*	13CD, 2B
26K	$2^{6} \cdot 3^{4} \cdot 7 \cdot 13^{3}$	16	-2	13E, 2A
26LM	$2^{8} \cdot 3^{4} \cdot 13^{3}$	16	*	13F, 2A
26N	$2^{8} \cdot 3^{2} \cdot 13^{2}$	12	5	13F, 2B
28AB	$2^{19} \cdot 3^{20} \cdot 5^{2} \cdot 7^{2} \cdot 13 \cdot 41 \cdot 61$	48	*	14A, 4G
28C	$2^{16} \cdot 3^{15} \cdot 5 \cdot 7^{3} \cdot 13 \cdot 61$	38	20	14B, 4A
28DE	$2^{11} \cdot 3^{13} \cdot 7^{3} \cdot 13^{2} \cdot 73$	34	*	14C, 4C
28F	$2^{10} \cdot 3^{12} \cdot 7^{3} \cdot 13^{2} \cdot 73$	32	13	14C, 4A
28G	$2^{18} \cdot 3^{12} \cdot 5^{2} \cdot 7^{2} \cdot 13$	32	8	14A, 4B
28HI	$2^{16} \cdot 3^{11} \cdot 5 \cdot 7^{2} \cdot 61$	30	*	14B, 4E
28JK	$2^{15} \cdot 3^{10} \cdot 5 \cdot 7^{2} \cdot 61$	28	*	14A, 4F
28L	$2^{17} \cdot 3^{8} \cdot 5 \cdot 7^{2}$	24	0	14A, 4G
28M	$2^{16} \cdot 3^{7} \cdot 5 \cdot 7^{2}$	22	4	14B, 4D
28N	$2^{15} \cdot 3^{7} \cdot 7^{3}$	22	0	14B, 4C
28O	$2^{10} \cdot 3^{6} \cdot 7^{3} \cdot 13$	20	-1	14D, 4A
28PQ	$2^{11} \cdot 3^{5} \cdot 7^{2} \cdot 13$	18	*	14C, 4E

28R	$2^{10} \cdot 3^4 \cdot 7^2 \cdot 13$	16	-3	14C, 4D
28S	$2^{10} \cdot 3^4 \cdot 7^2 \cdot 13$	16	11	14D, 4D
28TU	$2^{10} \cdot 3^4 \cdot 7^2 \cdot 13$	16	*	14E, 4F
28V	$2^{10} \cdot 3^4 \cdot 7^3$	16	7	14D, 4C
28W	$2^9 \cdot 3^3 \cdot 7^2 \cdot 13$	14	1	14E, 4B
28X	$2^{10} \cdot 3^2 \cdot 7^2$	12	3	14D, 4E
28YZ	$2^{10} \cdot 3^2 \cdot 7^2$	12	*	14E, 4G
28A′	$2^9 \cdot 3 \cdot 7^2$	10	1	14E, 4F
31A	$31 \cdot 271$	8	0	-
32A	$2^5 \cdot 5 \cdot 41$	8	0	16K
35A	$2^{11} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 13$	20	?	7A, 5A
37AB	$2^5 \cdot 3^3 \cdot 7 \cdot 19 \cdot 37$	14	?	-
38A	$2^5 \cdot 3 \cdot 19 \cdot 37$	10	?	19A, 2A
40AB	$2^{19} \cdot 3^{20} \cdot 5^3 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41$	48	*	20A, 8E
40C	$2^{16} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	32	21	20A, 8H
40DE	$2^{15} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$	32	*	20D, 8L
40F	$2^{18} \cdot 3^{12} \cdot 5^3 \cdot 7 \cdot 13$	32	9	20A, 8C
40GH	$2^{15} \cdot 3^{10} \cdot 5^2 \cdot 7 \cdot 61$	28	*	20B, 8P
40IJ	$2^{16} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 13$	26	*	20E, 8Q
40K	$2^{16} \cdot 3^8 \cdot 5^3 \cdot 7$	24	-3	20A, 8E
40L	$2^{17} \cdot 3^8 \cdot 5^2 \cdot 13$	24	1	20A, 8H
40MN	$2^{13} \cdot 3^6 \cdot 5^3 \cdot 7$	20	*	20F, 8E
40O	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 13$	20	9	20C, 8D
40P	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 13$	20	3	20H, 8A
40Q	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 13$	20	19	20H, 8G
40RS	$2^9 \cdot 3^6 \cdot 5^3 \cdot 73$	20	*	20K, 8C
40T	$2^{14} \cdot 3^6 \cdot 5^2 \cdot 7$	20	1	20C, 8B
40U	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	18	5	20E, 8Q
40VW	$2^{11} \cdot 3^5 \cdot 5^2 \cdot 41$	18	*	20D, 8M
40X	$2^{15} \cdot 3^5 \cdot 5 \cdot 7$	18	1	20E, 8K
40Y	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 41$	16	-1	20C, 8D
40ZA′	$2^9 \cdot 3^4 \cdot 5^2 \cdot 41$	16	*	20O, 8O
40B′	$2^{12} \cdot 3^4 \cdot 5^3$	16	-1	20F, 8C

40C′	$2^{12} \cdot 3^4 \cdot 5^3$	16	7	20F, 8E
40D′	$2^{14} \cdot 3^4 \cdot 5^2$	16	-3	20D, 8L
40E′ F′	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 7$	14	*	20G, 8Q
40G′ H′	$2^{13} \cdot 3^3 \cdot 5^2$	14	*	20M, 8M
40I′	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$	14	-1	20I, 8F
40J′	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$	14	7	20I, 8N
40K′	$2^{12} \cdot 3^2 \cdot 5^2$	12	1	20F, 8H
40L′	$2^{12} \cdot 3^2 \cdot 5^2$	12	-1	20H, 8G
40M′ N′	$2^9 \cdot 3^2 \cdot 5^3$	12	*	20K, 8E
40O′	$2^{11} \cdot 3^2 \cdot 5^2$	12	-1	20G, 8P
40P′ Q′	$2^{11} \cdot 3^2 \cdot 5^2$	12	*	20J, 8O
40R′ S′	$2^{11} \cdot 3^2 \cdot 5^2$	12	*	20M, 8L
40T′	$2^{13} \cdot 3^2 \cdot 5$	12	3	20H, 8I
40U′	$2^{10} \cdot 3 \cdot 5^2$	10	3	20M, 8M
40V′	$2^{12} \cdot 3 \cdot 5$	10	1	20M, 8J
40W′	$2^8 \cdot 5^2$	8	0	20O, 8O
40X′ Y′	$2^8 \cdot 5^2$	8	*	20P, 8P
41AE	$2^{12} \cdot 3^{12} \cdot 5 \cdot 7 \cdot 13 \cdot 41^2$	32	*	-
41FJ	$2^6 \cdot 3^4 \cdot 5 \cdot 41^2$	16	*	-
41KO	$2^2 \cdot 41^2$	8	*	-
44A	$2^7 \cdot 3^3 \cdot 11^2 \cdot 13$	14	?	22B, 4A
44B	$2^7 \cdot 3^2 \cdot 5 \cdot 11^2$	12	?	22A, 4B
44C	$2^7 \cdot 3 \cdot 11^2$	10	?	22B, 4D
61AF	$2^{11} \cdot 3^{10} \cdot 5 \cdot 7 \cdot 61^2$	28	*	-
61GK	61^2	8	*	-
73AF	$2^6 \cdot 3^{12} \cdot 7^2 \cdot 13^2 \cdot 73^2$	32	*	-
73GL	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73^2$	20	*	-
73M-(15)	73^2	8	*	-
121AK	$2^9 \cdot 3^{10} \cdot 5 \cdot 11^4 \cdot 13$	28	*	11A
121L-(22)	11^4	8	*	11A
271A-(9)	$31 \cdot 271$	8	?	-
547A-(39)	$2^5 \cdot 3 \cdot 547$	10	?	-
757A-(42)	$2^4 \cdot 3^3 \cdot 13 \cdot 757$	14	*	-

1093A-(78)	$2^4 \cdot 3 \cdot 1093$	10	*	-
1181A-(59)	$5 \cdot 1181$	8	?	-
4561A-(152)	4561	8	?	-
6481A-(270)	6481	8	?	-

Table 4: Centraliser structures for all semi-simple elements of prime order in $E_8(3)$.

Class	Shape of $C_G(x)$	Class	Shape of $C_G(x)$
2A	$SL_2(3) \times E_7(3)_{sc}$	41FJ	$82.(L_2(81).2)$
2B	$2.\Omega_{16}^{+}(3).2$	41KO	$2^2 \times 41^2$
5A	$10.(\Omega_{12}^{-}(3).2)$	61AF	$61 \times U_5(3)$
5B	$SU_5(9)$	61GK	61^2
7A	$7.{}^2E_6(3)$	73AF	$73.{}^3D_4(3)$
7B	$7.({}^3D_4(3).U_3(3))$	73GL	$73 \times U_3(9)$
11A	$121 \times L_5(3)$	73M-	73^2
13AB	$13.E_6(3)$	271A-	8401
13CD	$13.({}^3D_4(3).L_3(3))$	547A-	?
13E	$13^2 \times {}^3D_4(3)$	757A-	?
13F	$13^2 \times L_3(3)^2$	1093A-	?
19A	?	1181A-	5905
31A	8401	4561A-	4561
37AB	?	6481A-	6481
41AE	$82.(\Omega_8^{-}(3).2)$		

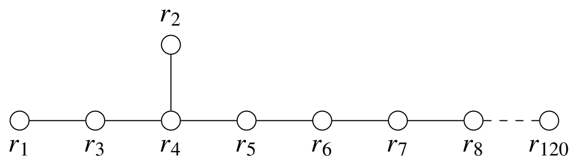
Table 5: The non-integer Brauer character values for semisimple conjugacy classes of elements of order ≤ 40 in $E_8(3)$.

Class	$\dim(C_V(g))$	Brauer Character Values	Powers
13AB	80	$26(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+79$ $-26(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+53$	-
13CD	38	$13(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+27$ $-13(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+14$	-
26AB	48	$6(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+21$ $-6(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+15$	13AB, 2B
26CD	40	$2(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})-1$ $-2(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})-3$	13AB, 2A
26EF	34	$15(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+38$ $-15(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+23$	13CD, 2A
26GH	22	$3(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+6$ $-3(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+3$	13CD, 2A
26IJ	18	$\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2}-1$ $-\omega_{13}^{11}-\omega_{13}^{8}-\omega_{13}^{7}-\omega_{13}^{6}-\omega_{13}^{5}-\omega_{13}^{2}-2$	13CD, 2B
26LM	16	$4(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+13$ $-4(\omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}+\omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2})+9$	13F, 2A
28AB	48	$16(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+56$ $-16(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+56$	14A, 4G
28DE	34	$14(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+35$ $-14(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+35$	14C, 4C

28HI	30	$8(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+24$ $-8(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+24$	14B, 4E
28JK	28	$4(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+8$ $-4(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+8$	14A, 4F
28PQ	18	$2(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+3$ $-2(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+3$	14C, 4E
28TU	16	$6(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+15$ $-6(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+15$	14E, 4F
28YZ	12	$2(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+7$ $-2(2\omega_{28}^{11}-2\omega_{28}^{9}+\omega_{28}^{7}-2\omega_{28})+7$	14E, 4G
40AB	48	$16(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+57$ $-16(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+57$	20A, 8E
40DE	32	$8(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+37$ $-8(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+37$	20D, 8L
40GH	28	$4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+9$ $-4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+9$	20B, 8P
40IJ	26	$8(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+25$ $-8(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+25$	20E, 8Q
40MN	20	$4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+17$ $-4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+17$	20F, 8E
40RS	20	$5(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+14$ $-5(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+14$	20K, 8C
40VW	18	$2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+3$ $-2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+3$	20D, 8M
40ZA′	16	$6(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+20$ $-6(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+20$	20O, 8O
40E′F′	14	$2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+5$ $-2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+5$	20G, 8Q
40G′H′	14	$4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+13$ $-4(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+13$	20M, 8M
40M′N′	12	$\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40}+2$ $-\omega_{40}^{15}+2\omega_{40}^{9}-2\omega_{40}^{7}-\omega_{40}^{5}+2\omega_{40}^{3}+2\omega_{40}+2$	20K, 8E
40P′Q′	12	$2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+5$ $-2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+5$	20J, 8O
40R′S′	12	$2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+7$ $-2(\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40})+7$	20M, 8L
40X′Y′	8	$\omega_{40}^{15}-2\omega_{40}^{9}+2\omega_{40}^{7}+\omega_{40}^{5}-2\omega_{40}^{3}-2\omega_{40}+4$ $-\omega_{40}^{15}+2\omega_{40}^{9}-2\omega_{40}^{7}-\omega_{40}^{5}+2\omega_{40}^{3}+2\omega_{40}+4$	20P, 8P

2.3 Root Systems

Root systems (and sub-systems) are utilised frequently throughout this thesis. We shall use the standard Bourbaki ordering described in [8]; this is consistent with the labelling used in Magma [6]. As such, the extended E_8 Dynkin diagram is given by the roots:



Let $\Phi(E_8)^+$ and $\Phi(E_8)^-$ denote the sets containing all the positive and negative roots inside the E_8 root system $\Phi(E_8)$. We have that $|\Phi(E_8)^+| = |\Phi(E_8)^-| = 120$. We shall use $\Pi(E_8)$ to denote the 8 fundamental roots which form a basis for $\Phi(E_8)$. We shall often create subsystems $\Phi \subset \Phi(E_8)^+$ which will contain roots of different labels. Each Φ is associated with a Dynkin type. For example, $\Phi = \{r_1, r_2, r_3, r_4\}$ has Dynkin type A_4 and we shall write $\Phi \cong A_4$; this can be seen as a sub-diagram of the E_8 Dynkin diagram. For $\Phi \subset \Phi(E_8)^+$, we consider the subgroup of $E_8(3)$ generated by the root subgroups corresponding to the roots in Φ. We denote this subgroup by $G(\Phi)$. We remark that $G(\Phi)$ is generated by the root subgroups corresponding to the roots in Φ and also the negatives of all these roots. For example, consider again $\Phi = \{r_1, r_2, r_3, r_4\} \cong A_4$. Then $G(\Phi)$ is generated by the root subgroups corresponding to the set $\Phi \cup \{-r_1, -r_2, -r_3, -r_4\}$. To avoid this becoming too notation heavy, the labels for the negative roots are omitted. Moreover, when defining these root subsystems, we represent the roots by only their label. For example, we shall write $\{1, 2, 3, 4\} \cong A_4$ instead of $\{r_1, r_2, r_3, r_4\} \cong A_4$.

We extend this notation to help us describe the parabolic subgroups of G. Given $\Phi \subset \Phi(E_8)^+$, we denote the corresponding parabolic subgroup by $P(\Phi)$. The following definition details the structure of such parabolic subgroups.

Definition 2.1. Let $\Phi \subset \Phi(E_8)^+$ and let $P = P(\Phi)$ be a parabolic subgroup of $E_8(3)$ (i.e. Φ corresponds to a sub-graph of the E_8 Dynkin diagram).

1. The unipotent radical $O_3(P) = Q(\Phi) \leq P$ is the largest normal subgroup of P consisting of entirely unipotent elements. That is, elements x such that $(x-1)^n = 0$ for some n.

2. A Levi decomposition of P is $P = Q(\Phi) : L(\Phi)$. The group $L(\Phi) \cong P/Q(\Phi)$ is called the Levi complement of P.

From our earlier notation, we have that $G(\Phi) = L(\Phi)$. If we are working with a parabolic subgroup (so $\Phi \subset \Pi(E_8)$) like in Sections 3.1 - 3.4, we shall use $L(\Phi)$. If $\Phi \not\subset \Pi(E_8)$, then the corresponding subgroup $G(\Phi)$ is not a Levi-complement of a parabolic subgroup, so we always use $G(\Phi)$ in these cases. By definition, if we have two root subsystems Φ_1, Φ_2 such that $\Phi_2 \subset \Phi_1$, then $G(\Phi_2) \leq G(\Phi_1)$. For a given Φ, the Magma code given in Section 9.2.2 constructs $Q(\Phi)$ and $G(\Phi)$ inside $GL_{248}(3)$. Table 6 contains the Dynkin types of Φ that we consider in this thesis and the isomorphism type of the corresponding group $G(\Phi)$. We remark that $G(\Phi)$ is being used here as not all Dynkin types given in Table 6 exist as subgraphs of the E_8 Dynkin diagram.

Table 6: Root subsystems $\Phi \subset \Phi(\mathrm{E}_8)^+$ and the corresponding isomorphism types of $G(\Phi)$.

Dynkin type of Φ	Isomorphism type of $G(\Phi)$
$A_n, n \in \{1,2,3,4,5,6,7,8\}$	$\mathrm{SL}_{n+1}(3)$
D_4	$2^{2\cdot}\Omega_8^+(3)$
D_5	$2^{\cdot}\Omega_{10}^+(3)$
D_6	$2^2.\Omega_{12}^+(3)$
D_7	$2.\Omega_{14}^+(3)$
D_8	$2.\Omega_{16}^+(3)$
E_6	$\mathrm{E}_6(3)$
E_7	$\mathrm{E}_7(3)_{\mathrm{sc}} \sim 2.\mathrm{E}_7(3)$
E_8	$\mathrm{E}_8(3)$

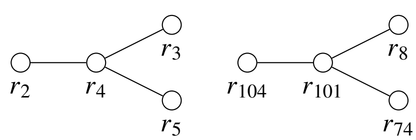
For $\alpha, \beta \in \Phi(\mathrm{E}_8)^+$, define

$$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)},\ n_{\beta\alpha} = \frac{2(\beta,\alpha)}{(\beta,\beta)}$$

where $(.,.)$ denotes the standard Euclidean inner product. We can use $n_{\alpha\beta}$ to define a graph which has α and β as nodes. Indeed, the roots α and β are joined by an edge of weight $w_{\alpha\beta} = n_{\alpha\beta} \cdot n_{\beta\alpha}$. As all roots in $\Phi(\mathrm{E}_8)$ have the same length, $w_{\alpha\beta}$ is either 0 or 1. We remark that the Dynkin diagram of E_8 (and any other Dynkin diagram) can be constructed in this way by considering the fundamental roots. More details surrounding this theory can be found in [15] and Section 7. The Magma code used to calculate $w_{\alpha\beta}$ can be found in Section 9.2.1.

We say that α and β are orthogonal if $w_{\alpha\beta} = 0$ and that two sets $\Phi_1, \Phi_2 \subset \Phi(\mathrm{E}_8)^+$ are mutually orthogonal if every root in Φ_1 is orthogonal to every root in Φ_2. If we can partition $\Phi \subset \Phi(\mathrm{E}_8)^+$ into a disjoint union of mutually orthogonal subsystems $\Phi = \Phi_1 \dot\cup \Phi_2 \dot\cup \ldots \dot\cup \Phi_n$, then it is often the case that $G(\Phi) \cong G(\Phi_1) \times G(\Phi_2) \times \ldots \times G(\Phi_n)$. We remark that this is not always true, as sometimes $G(\Phi_i)$ and $G(\Phi_j)$ (for $i \neq j$) share a non-trivial center.

For example, let $\Phi = \{2,3,4,5,8,74,101,104\}$. Then $\Phi = \Phi_1 \dot\cup \Phi_2$ where $\Phi_1 = \{2,3,4,5\} \cong \mathrm{D}_4$ and $\Phi_2 = \{8,74,101,104\} \cong \mathrm{D}_4$ are sets of mutually orthogonal roots. The Dynkin diagram of Φ is



and we follow the notation in [42] and [19] by writing $\Phi \cong 2\mathrm{D}_4$. In this case, we find that $Z(G(\Phi_1)) = Z(G(\Phi_2)) \cong 2^2$. Hence, $G(\Phi) \cong 2^2.(\Omega_8^+(3) \times \Omega_8^+(3))$.

We illustrate this notation with a further example. Suppose $\Phi = \{2,3,4,5,7,8\}$. Then $\Phi = \Phi_1 \dot\cup \Phi_2$ where $\Phi_1 = \{2,3,4,5\} \cong \mathrm{D}_4$ and $\Phi_2 = \{7,8\} \cong \mathrm{A}_2$. In this case, we write $\Phi \cong \mathrm{D}_4 + \mathrm{A}_2$. From Table 6, we see that $L(\Phi_2) \cong \mathrm{SL}_3(3) \cong \mathrm{L}_3(3)$, hence $L(\Phi_2)$ has a trivial center. Therefore, we have $L(\Phi) \sim 2^{2\cdot}\Omega_8^+(3) \times \mathrm{L}_3(3)$. Resources such as [49] and [14] give more background information on root subgroups.

2.4 Known Maximals

Thanks to the work of Liebeck, Seitz and many others, several maximal subgroups of G are already known. These are described in Theorem 1.1 (i) - (iv). In this section, we list the maximal subgroups corresponding to Theorem 1.1 $(i)(a)$ and $(i)(b)$, namely the maximal parabolic subgroups and the subgroups of maximal rank.

The maximal parabolic subgroups of G are the parabolic subgroups $P(\Phi)$ where $|\Phi| = 7$. If $\Phi \subset \Pi(\mathrm{E}_8)$ and $|\Phi| = 7$, we say that $P(\Phi)$ is a standard maximal parabolic subgroup. As such, there is a 1-1 correspondence between the standard maximal parabolic subgroups of $\mathrm{E}_8(q)$ and the sub-graphs of the E_8 Dynkin diagram which consist of exactly 7 nodes. It is a well known result that any parabolic subgroup is conjugate to some standard parabolic subgroup. Table 7 details the structures of all standard maximal parabolic subgroups of G and the Dynkin types of the associated sets Φ. The first column indicates which node of the E_8 Dynkin diagram is removed to obtain Φ.

Table 7: The structures of the standard maximal parabolic subgroups of $\mathrm{E}_8(3)$.

Node removed	Dynkin type of Φ	Structure of $P(\Phi)$
1	D_7	$[3^{78}] : 2.\Omega^+_{14}(3)$
2	A_7	$[3^{92}] : \mathrm{SL}_8(3)$
3	$\mathrm{A}_1 + \mathrm{A}_6$	$[3^{98}] : \mathrm{SL}_2(3) \times \mathrm{L}_7(3)$
4	$\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_4$	$[3^{106}] : \mathrm{SL}_2(3) \times \mathrm{L}_3(3) \times \mathrm{L}_5(3)$
5	$\mathrm{A}_3 + \mathrm{A}_4$	$[3^{104}] : \mathrm{SL}_4(3) \times \mathrm{L}_5(3)$
6	$\mathrm{A}_2 + \mathrm{D}_5$	$[3^{97}] : \mathrm{L}_3(3) \times 2^{\cdot}\Omega^+_{10}(3)$
7	$\mathrm{A}_1 + \mathrm{E}_6$	$[3^{83}] : \mathrm{SL}_2(3) \times \mathrm{E}_6(3)$
8	E_7	$[3^{92}] : \mathrm{E}_7(3)_{sc} \sim [3^{92}] : 2.\mathrm{E}_7(3)$

We conclude this section by listing the maximal subgroups of maximal rank inside G; these are found in their general form (for all q) in [48].

Table 8: The structures of the subgroups of maximal rank in $\mathrm{E}_8(3)$.

$2.\Omega^+_{16}(3).2$	$2.(\mathrm{L}_2(3) \times \mathrm{E}_7(3)).2$	$\mathrm{L}_9(3).2$
$\mathrm{U}_9(3).2$	$(\mathrm{L}_3(3) \times \mathrm{E}_6(3)).2$	$(\mathrm{U}_3(3) \times^2 \mathrm{E}_6(3)).2$
$(\mathrm{L}_5(3))^2.4$	$(\mathrm{U}_5(3))^2.4$	$\mathrm{SU}_5(9).4$
$\mathrm{PGU}_5(9).4$	$2^2.(\Omega^+_8(3))^2.2^2.(\mathrm{Sym}(3) \times 2)$	$2^2.\Omega^+_8(9).(\mathrm{Sym}(3) \times 2)$
$({}^3\mathrm{D}_4(3))^2.6$	${}^3\mathrm{D}_4(9).6$	$(\mathrm{L}_3(3))^4.\mathrm{GL}_2(3)$
$(\mathrm{U}_3(3))^4.\mathrm{GL}_2(3)$	$(\mathrm{U}_3(9))^2.8$	$\mathrm{U}_3(81).8$
$2^4.(\mathrm{L}_2(3))^8.2^4.\mathrm{AGL}_3(2)$	$3^8.(2.\Omega^+_8(2).2)$	$11^4.(5 \times \mathrm{SL}_2(5))$
$13^4.(2.(3 \times \mathrm{U}_4(2)))$	$[10^4].((4 * 2^{1+4}).\mathrm{Alt}(6).2)$	$[8401].30$
$73^2.(12 * \mathrm{GL}_2(3))$	4561.30	$61^2.(5 \times \mathrm{SL}_2(5))$
$7^4.(2.(3 \times \mathrm{U}_4(2)))$		

2.5 The 248-dimensional Module

As briefly mentioned in Section 2.2, the minimal irreducible representation of $\mathrm{E}_8(q)$ is the 248-dimensional adjoint representation. This gives G a natural embedding inside $\mathrm{GL}_{248}(3)$. Moreover, this representation equips G with an action on the 248-dimensional E_8 Lie algebra, which we shall label V. The vast majority of the computational work conducted in this thesis is done inside this 248-dimensional representation. In this section, we present some results which will allow us to determine the non-maximality of the groups in Tables 1 and 2 by only studying their action on V. Unless specified otherwise, we shall assume throughout the rest of this thesis that $K = \mathrm{GF}(3)$ and that V denotes the 248-dimensional irreducible adjoint module for G.

2.5.1 Potential Embeddings Into $\mathrm{E}_8(3)$

Let H be any group from either Table 1 or 2. Should H exist as a maximal subgroup of G, then H must have some action on V and with it there must exist some 248-dimensional KH-submodule of V. This submodule, which we often label $V|_H$, is composed of irreducible KH-modules. Using Magma, we can often calculate these irreducible modules and with it, study the possible embeddings of H into G. To do this, we use the Brauer character information given in Tables 3 and 5. The majority of these values were calculated using Magma; the code used to do this is shown in [62] and Section 9.2.11. Using these Brauer characters, we are able to obtain lists of what we call feasible decompositions for groups in Tables 1 and 2 (providing the group is sufficiently small). These describe possible composition series for $V|_H$ by sums of irreducible KH-modules; many examples are shown in Section 8.

There are two ways we can obtain these feasible decompositions, both of which hinge on solving a system of simultaneous equations involving known Brauer character values of semisimple elements in G. We remark that both methods require us to be able to calculate the irreducible modules and Brauer character table of H over K. The first method involves using the code in Section 9.2.12; this follows the code shown in [62] where it was used in $\mathrm{E}_8(2)$. This involves forming all possible composition series consisting of irreducible KH-modules such that the dimension of each irreducible module is ≤ 248. The process starts by considering 248 trivial modules. After forming each possibility, the code then uses Brauer character information from G to determine whether the newly formed decomposition is feasible. Typically, this is a quick and efficient way of obtaining the feasible decompositions of H. However, for some groups, it is more efficient to consider the system of equations more directly. This is best illustrated with an example.

Suppose $H \cong \mathrm{Sz}(8)$. Shown below is the Brauer character table of H over K.

Table 9: Brauer character table for Sz(8).

Sz(8)	1A	2A	4AB	5A	7AC	13AC
φ_1	1	1	1	1	1	1
φ_2	28	-4	0	-2	0	2
φ_3	64	0	0	-1	1	-1
φ_4	91	-5	-1	1	0	0
φ_5	105	9	-3	0	0	1
φ_6	195	3	3	0	-1	0

The column headed 1A gives the dimensions of the irreducible modules φ_i. We require the sum of the dimensions of the modules in the composition series for $V|_H$ to be equal to 248. As such, we search for all non-negative integer solutions to the equation:

$$x_1 + 28x_2 + 64x_3 + 91x_4 + 105x_5 + 195x_6 = 248.$$

Following this, we consider the column headed 2A. It must be that the class 2A from H fuses into either 2A or 2B in G. From Table 3, we find that the Brauer character values for these classes are 24 and −8 respectively. Hence, we take the solutions to the previous equation and check if they satisfy either of the following:

$$x_1 - 4x_2 + 0x_3 - 5x_4 + 9x_5 + 3x_6 = 24 \quad (1)$$

$$x_1 - 4x_2 + 0x_3 - 5x_4 + 9x_5 + 3x_6 = -8 \quad (2).$$

If equation (1) is satisfied, then we write 2A → 2A to indicate that the class 2A from H fuses into the class 2A in G (as 24 is the Brauer character value for the class 2A in G on V). Similarly, if instead (2) is satisfied, then we write 2A → 2B. This notation forms part of what we call a fusion pattern; each feasible decomposition is associated to a fusion pattern. This process is then repeated for all other columns in the Brauer character table until the fusion of the conjugacy classes of H into the conjugacy classes of G is completely determined. For full clarity, the complete system of equations for Sz(8) is:

$$x_1 + 28x_2 + 64x_3 + 91x_4 + 105x_5 + 195x_6 = 248$$
$$x_1 - 4x_2 + 0x_3 - 5x_4 + 9x_5 + 3x_6 = 24, -8$$
$$x_1 + 0x_2 + 0x_3 - x_4 - 3x_5 + 3x_6 = 134, 92, 82, 70, 66, 64, 60$$
$$x_1 - 2x_2 - x_3 + x_4 + 0x_5 + 0x_6 = 23, -2$$
$$x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 - x_6 = 52, 3$$
$$x_1 + 2x_2 - x_3 + 0x_4 + 1x_5 + 0x_6 = 14, 1.$$

Note that the final equation (which corresponds to the column 13AC in Table 9) needs only be checked against the integer Brauer character values for the elements of order 13 in G as we have only integers in Table 9. Should we have a group H with a Brauer character table containing non-integer values, then we must also consider the non-integer Brauer character values in G. Solving these systems of equations yields only one possible solution, given by $x_1 = 1, x_2 = 1, x_3 = 2, x_4 = 1, x_5 = 0, x_6 = 0$. As such, the sole feasible decomposition for Sz(8) is given by $\varphi_1 + \varphi_2 + 2\varphi_3 + \varphi_4 + 0\varphi_5 + 0\varphi_6$. This indicates which irreducible KH-modules are present in the composition series of $V|_H$ and the multiplicity in which they occur. Furthermore, the associated fusion pattern for this feasible decomposition is 2A → 2B, 4AB → 4G, 5A → 5B, 7AC → 7B, 13AC → 13F. In particular, this shows that if $H \cong$ Sz(8) exists as a subgroup of G, then the elements of H must lie inside the classes 2B, 4G, 5B, 7B and 13F of G. Another detailed example of this method is given in Section 3.1. The feasible decompositions cannot be calculated for all groups shown in Tables 1 and 2. In the following section, we shall see how we can use feasible decompositions to determine non-maximality. We remark that this notion of feasible decomposition is also seen in [55] where they are used to study subgroup embeddings in exceptional algebraic groups over fields of different characteristics (whereas they are used exclusively with $E_8(3)$ here).

2.5.2 Determining Maximality

We now turn our attention to investigating the maximality of the groups in our lists. Given the general nature of Theorem 1.1, many of the groups listed in Tables 1 and 2 will not even exist as subgroups of G. By using some elementary properties, some of these groups can be easily identified.

Lemma 2.2. The following groups do not embed as subgroups of $G \cong \mathrm{E}_8(3)$:

$\mathrm{L}_5(9)$	$\mathrm{Sp}_8(9)$	$\Omega_9(9)$
$\Omega_8^-(9)$	${}^2\mathrm{G}_2(2187)$	$\mathrm{F}_4(9)$
$\mathrm{L}_2(16)$	$\mathrm{L}_2(17)$	$\mathrm{L}_2(29)$
$\mathrm{L}_2(49)$	${}^2\mathrm{G}_2(243)$	

Proof. The groups $\mathrm{Sp}_8(9)$, $\Omega_9(9)$, $\Omega_8^-(9)$, ${}^2\mathrm{G}_2(2187)$, $\mathrm{F}_4(9)$, $\mathrm{L}_2(16)$, $\mathrm{L}_2(17)$, $\mathrm{L}_2(29)$ all have order with a prime divisor that does not divide $|G|$. The group $\mathrm{L}_5(9)$ contains an element of order $11^2 \cdot 61$, implying that an element of order 61 commutes with an element of order 11. Consulting Table 3, we see that G has one conjugacy class of order 11 elements and the centraliser order is not divisible by 61. From Table 34 of Section 7.5, we see that ${}^2\mathrm{G}_2(243)$ contains a cyclic maximal torus of order $217 = 7 \cdot 31$, thus implying an element of order 31 is centralised by an element of order 7. Upon consulting Table 3, we see that G has two conjugacy classes of order 7 elements and neither admits a centraliser of order divisible by 31. Finally, $\mathrm{L}_2(49)$ has a Sylow 5-subgroup of exponent 25, whereas the Sylow 5-subgroup of G has exponent 5 (details of how a Sylow 5-subgroup of G was constructed are given in Section 9.1). □

Disproving the maximality of the remaining groups in our tables is a more laborious task. As mentioned in Section 2.5.1, if H_0 is a group from Table 1 or 2 and G contains a subgroup isomorphic to H_0, then H_0 must have some compatible action on the 248-dimensional adjoint module V. It is of great interest to us to try and determine whether H_0 fixes a vector on V.

In general, let G be an algebraic group over an algebraically closed field $\overline{\mathrm{GF}(q)}$ ($q = p^a$) with Lie algebra $\overline{V}$ and let $\mathscr{X}$ denote the collection of maximal subgroups of positive dimension in G; these are detailed in Theorem 1.1 and [53]. Moreover, as in ([22], Section 3) and ([14], Section 12), $\mathrm{Aut}^+(G)$ is used to denote the group generated by the inner, diagonal, graph and p-power field automorphisms of G. If σ denotes a Frobenius endomorphism of G, then every automorphism of G_σ extends to an element of $\mathrm{Aut}^+(G)$. We now recall the definition of a strongly imprimitive subgroup from [22].

Definition 2.3 ([22], Definition 3.2)**.** If σ is a Frobenius endomorphism on G and a subgroup $H \leq G$ is contained in the fixed point group G_σ, then H is called strongly imprimitive if H is contained in a σ-stable, $N_{\mathrm{Aut}^+(G)}(H)$-stable member of $\mathscr{X}$.

Using the following result, again from [22], we can determine whether a subgroup H is strongly imprimitive by studying it's action on $\overline{V}$. We use $\overline{V}^\circ$ to denote the connected component of $\overline{V}$ containing the identity.

Theorem 2.4 ([22], Proposition 4.5)**.** *If H is a finite subgroup of G that centralizes a line on $\overline{V}^\circ$, then H is strongly imprimitive.*

Moving this back to our particular situation, if we assume G is of type E_8, then we have $\overline{V}^\circ = \overline{V}$ as G is simply connected. Consequently, if H_0 is a simple group from Table 1 or 2 such that $H_0 < G_\sigma \cong \mathrm{E}_8(q)$ and H_0 fixes some non-zero vector of V, then H_0 will fix the same vector inside $\overline{V}$. Then, by Theorem 2.4, H_0 will be contained in a σ-stable member, X, of $\mathscr{X}$. Furthermore, if $H \leq G_\sigma$ is any automorphic extension of H_0, then since $H \leq \mathrm{Aut}^+(G)$ and X is $N_{\mathrm{Aut}^+(G)}(H_0)$-stable and maximal in G, we have that $H \leq X$. In particular, if $H < \mathrm{E}_8(3)$ such that $H_0 = F^*(H)$ is a group shown in Table 1 or 2 with $\dim(C_V(F^*(H))) \neq 0$, then H is not maximal in $\mathrm{E}_8(3)$. These results are summarised in the following lemma.

Lemma 2.5. If $H < G_\sigma$ with $\dim(C_V(F^*(H))) \neq 0$, then H is not a maximal subgroup of G_σ.

By using results of Litterick [55], in many cases, we are able to show $\dim(C_V(F^*(H))) \neq 0$ by studying only the feasible decompositions. Firstly, should we be able to calculate the irreducible KH-modules for a given group H, and then use the method shown in Section 2.5.1 to find that no feasible decompositions exist, then we have shown that H cannot embed as a subgroup of G. The following result lists all groups H from Tables 1 and 2 with this property.

Lemma 2.6. Let $H \cong \mathrm{L}_2(41)$, $\mathrm{L}_2(32)$, $\mathrm{U}_4(3)$ or $\mathrm{Sp}_6(3)$. Then $G \cong \mathrm{E}_8(3)$ contains no subgroups isomorphic to H.

Proof. None of the groups listed here have any feasible decompositions, hence none of them yield an embedding into G. □

Now suppose that H has at least one feasible decomposition. Using group cohomology, we may be able to quickly determine if H fixes a vector $v \in V$. Indeed, for each irreducible KH-module φ, there is an associated cohomological dimension. This is defined to be $\dim(H^1(H, \varphi))$. For our purposes, we need not delve into group cohomology; the cohomological dimension is simply an integer which we can obtain in Magma. By using the cohomological dimension and the following result, we can immediately deduce whether certain feasible decompositions of H will lead to $C_V(H) \neq \{0\}$.

Lemma 2.7 ([55], Lemma 3.6)**.** Let H be a finite group and M a finite-dimensional KH-module, with composition factors $W_1, \ldots, W_r$ of which m are trivial. Set $n = \sum \dim(H^1(H, W_i))$ and assume that $H^1(H, K) = \{0\}$. Then

1. If $n < m$ then M contains a trivial submodule of dimension at least $m - n$,
2. If $m = n$ and M contains no nonzero trivial submodule, then $H^1(H, M) = \{0\}$,
3. Suppose that $m = n > 0$, and that for each i we have
$$H^1(H, W_i) = \{0\} \iff H^1(H, W_i^*) = \{0\}.$$
Then M has a non-zero trivial submodule or quotient.

How we use Lemma 2.7 in practice is best illustrated with an example. Let us consider $H \cong \mathrm{Sz}(8)$ again. Previously, we found that H has one feasible decomposition given by $\varphi_1 + \varphi_2 + 2\varphi_3 + \varphi_4 + 0\varphi_5 + 0\varphi_6$. Using Magma, we calculate the cohomological dimensions of the irreducible modules $\varphi_1, \ldots, \varphi_6$ to be $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, $\varphi_5 = 0$, $\varphi_6 = 0$. Hence, by Lemma 2.7 (with $M = V|_H$, $m = 1$, $n = 0$), we have that $V|_H$ contains a trivial submodule and thus $\dim(C_V(H)) \neq 0$. As there is only one feasible decomposition, we can conclude by Lemma 2.5 that any subgroup $H < G$ with $F^*(H) \cong \mathrm{Sz}(8)$ is not maximal in G. Using the same method, we can eliminate the following groups.

Lemma 2.8. Suppose $H < \mathrm{E}_8(3) \cong G$ such that $F^*(H) \cong \Omega_8^+(2)$, $\mathrm{L}_2(13)$, $\mathrm{Sp}_6(2)$. Then H is not a maximal subgroup of G.

Proof. The Brauer character tables and feasible decompositions for $\Omega_8^+(2)$, $\mathrm{L}_2(13)$, $\mathrm{Sp}_6(2)$ are given in Section 8. By Lemma 2.7, all feasible decompositions associated to these groups imply the existence of a fixed non-zero vector. Hence, by Lemma 2.5, these groups cannot exist as maximal subgroups of G. □

To conclude this section, we introduce one final technique for determining non-maximality using feasible decompositions. This method hinges on studying the projective covers of the irreducible KH-modules. Firstly, we have two preliminary definitions, which can be generalised to $K = \mathrm{GF}(q)$.

Definition 2.9. Let M be a finite-dimensional KH-module.

1. The socle of M, denoted $\mathrm{Soc}(M)$, is the sum of all irreducible submodules of M.
2. The radical of M, denoted $\mathrm{Rad}(M)$, is the intersection of all maximal submodules of M.
3. M is said to be completely reducible if $M = \mathrm{Soc}(M)$.
4. The quotient $M/\mathrm{Rad}(M)$ is called the head of M.
5. The head and socle are dual concepts, meaning that $\mathrm{Soc}(M^*)^* \cong M/\mathrm{Rad}(M)$.

Lemma 2.10 ([55], Lemma 3.7)**.** Let M be a finite-dimensional KH-module. Then

1. There exists a finite-dimensional projective KH-module P such that $M/\mathrm{Rad}(M) \cong P/\mathrm{Rad}(P)$ and this P is unique upto isomorphism.
2. M is a homomorphic image of P.
3. $P/\mathrm{Rad}(P) \cong \mathrm{Soc}(P)$.
4. $\dim(P)$ is divisible by the order of a Sylow p-subgroup of H.
5. If $M = W \oplus U$ then there exists P_W and P_U such that P_i is projective and W is a homomorphic image of P_W and equivalently U of P_U.

Using the notation from Lemma 2.10, P is called the projective cover of M and the dual notion is called the injective hull. A consequence of Lemma 2.10 (2.) is that any finite-dimensional KH-module exists as a quotient of some finite-dimensional projective module. In particular, any module $V|_H$ will exist as a quotient of some projective module whose composition series consists of projective indecomposable modules corresponding to irreducible KH-modules. Lemma 2.11 details the possible multiplicities of these projective indecomposables in the composition series.

Lemma 2.11 ([55], Lemma 3.9)**.** Let $\varphi_1, \dots, \varphi_r$ be irreducible KH-modules and $P_1, \dots, P_r$ the corresponding projective indecomposables. Let M be a self-dual KH-module having composition factors φ_i with multiplicity r_i, such that M has no irreducible direct summands, and let $P = \bigoplus P_i^{n(Pi)}$ be the projective cover of M. Then

$$n(P_i) + n(P_i^*) \leq r_i$$

for all i. In particular, $n(P_i) \leq r_i/2$ when φ_i is self dual.

The following result gives us the tools to determine whether a given H fixes a vector by only studying the feasible decompositions and the projective indecomposable modules corresponding to the irreducible KH-modules.

Lemma 2.12 ([62], Lemma 3.6)**.** Let M be a sum of projective indecomposables $\sum P_i$ whose trivial composition factors all lie inside the second socle layer, $\mathrm{Soc}(M/\mathrm{Soc}(M))$. Then for any proper quotient Q of M, if Q has no trivial submodules then there exists a projective quotient S of Q such that S contains all trivial composition factors of Q.

Several examples of projective indecomposable modules and their socle layer structures are given in Section 8; the code used to produce these structures can be found in Section 9.2.14. More detail about the notation used here is given in the opening paragraphs of Section 8. For the larger groups in Tables 1 and 2, the projective indecomposables were either not computable or their structures were too complex for this method to be viable. Lemma 2.12 can be used to identify trivial submodules providing that the trivial composition factors of all projective indecomposables lie in the second socle layer. This can be made clear with an example.

Let $H \cong \mathrm{L}_2(7)$. The Brauer character table and feasible decompositions for H can be found in Section 8.14. Feasible decompositions 2, 5, 6, 7, 9, 12 and 14 exhibit fusion patterns which contradict the information shown in Table 3 and so are not valid. Furthermore, by Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $\mathrm{L}_2(7) \cong F^*(H)$ that are associated to feasible decompositions 1, 3, 4, 8 and 15 are not maximal. This leaves only decompositions 10, 11, 13 remaining, which are given by:

10. $17\varphi_1 + 0\varphi_2 + 14\varphi_3 + 21\varphi_4$ $(2\mathrm{A} \to 2\mathrm{A}, 4A \to 4E, 7AB \to 7B)$,

11. $16\varphi_1 + 7\varphi_2 + 6\varphi_3 + 22\varphi_4$ $(2\mathrm{A} \to 2\mathrm{B}, 4A \to 4F, 7AB \to 7B)$,

13. $14\varphi_1 + 5\varphi_2 + 6\varphi_3 + 24\varphi_4$ $(2\mathrm{A} \to 2\mathrm{B}, 4A \to 4G, 7AB \to 7B)$.

Also given in Section 8.14 are the projective covers corresponding to the 3 non-trivial irreducible KH-modules. Their socle layer structures are :

$$P(\varphi_2) = \varphi_2$$
$$P(\varphi_3) = \varphi_3, \qquad P(\varphi_4) = \varphi_4|\varphi_1|\varphi_4.$$

We see that all trivial composition factors lie inside the second socle layer, hence Lemma 2.12 applies. Suppose for contradiction that there are no trivial submodules. Then, there exists a projective quotient containing all the trivial composition factors. As $P(\varphi_4) = \varphi_4|\varphi_1|\varphi_4$, for every trivial composition factor, there are at least 2 composition factors of dimension 7 (as $\dim(\varphi_4) = 7$). However, we see that in the remaining feasible decompositions, this does not happen. Hence, we must have that there does exist a trivial submodule and so all subgroups $H < G$ such that $F^*(H) \cong \mathrm{L}_2(7)$ is associated to feasible decomposition 10, 11 or 13 are not maximal. All 15 feasible decompositions have now been eliminated, meaning we can conclude our work on $\mathrm{L}_2(7)$ with the following result. We remark that the case $F^*(H) \cong \mathrm{L}_2(8)$ can be dealt with in a similar way, details of this are given in [64].

Lemma 2.13. Let $H < \mathrm{E}_8(3) \cong G$ such that $F^*(H) \cong \mathrm{L}_2(7)$. Then H is not maximal in G.

3 $\mathrm{L}_2(3^n)$, $n = 3, 4, 5, 6, 7$

In this section we shall investigate the potential maximality of $\mathrm{L}_2(3^n) < G \cong \mathrm{E}_8(3)$ for $n \in \{3, 4, 5, 6, 7\}$. The work involving $\mathrm{L}_2(27)$ is incomplete. However, Lemma 3.1 summarises our findings for the other groups.

Lemma 3.1. Let $H < G$ such that $F^*(H) \cong \mathrm{L}_2(3^n)$ for any $n \in \{4, 5, 6, 7\}$. Then H is not a maximal subgroup of G. In the cases of $F^*(H) \cong \mathrm{L}_2(3^5), \mathrm{L}_2(3^6)$, G contains no subgroups isomorphic to H.

We maintain the notation used previously throughout this section, namely $G \cong E_8(3)$ and $K = \mathrm{GF}(3)$.

3.1 $L_2(243)$ and $L_2(729)$

Firstly, we prove that G contains no subgroups isomorphic to either $L_2(3^5)$ or $L_2(3^6)$. Both groups contain semisimple elements for which the Brauer character values are incalculable, thus meaning we cannot calculate the feasible decompositions. However, we shall see that it suffices to consider only the semisimple elements of order ≤ 7. We use the method of solving simultaneous equations described in Section 2.5.1. The Brauer character tables for $L_2(3^5)$ and $L_2(3^6)$ are omitted due to the length of the expressions in each column. However, these can be easily obtained using MAGMA.

Suppose $H \cong L_2(3^5)$. Of the 26 irreducible KH-modules, 23 have dimension ≤ 248. We now consider the Brauer character fusion. Consider the equation

$$\begin{aligned} &x_1 + 15x_2 + 20(x_3 + x_4) + 45(x_5 + x_6) \\ &+ 60(x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}) + 80x_{13} \\ &+ 135(x_{14} + x_{15}) + 180(x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21}) \\ &+ 240x_{22} + 243x_{23} = 248 \end{aligned}$$

corresponding to the first column of the Brauer character table. Using MATHEMATICA, we find there are 15083 non-negative integer sets of solutions to this equation. Of these solutions, none of them satisfy the equation $x_1 - 5x_2 + 5x_5 + 5x_6 - 5x_{14} - 5x_{15} - x_{23} = 24,\text{-}8$ relating to the fusion of involutions. Hence there are no possible embeddings of H into G.

Now suppose $H \cong L_2(3^6)$. We find that H has 68 irreducible modules over K, of which 43 have dimension ≤ 248. We use an identical approach to the previous case, although this case is more involved. The first equation

$$\begin{aligned} &x_1 + 12x_2 + 18x_3 + 24(x_4 + x_5) + 27x_6 + 48x_7 + 54(x_8 + x_9 + x_{10}) \\ &+ 64x_{11} + 72(x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21}) \\ &+ 96(x_{22} + x_{23}) + 108(x_{24} + x_{25}) + 162(x_{26} + x_{27} + x_{28}) \\ &+ 216(x_{29} + x_{30} + x_{31} + x_{32} + x_{33} + x_{34} + x_{35} + x_{36} + x_{37} + x_{38} + x_{39} + x_{40} + x_{41} + x_{42}) \\ &+ 243x_{43} = 248 \end{aligned}$$

has 103723 non-negative integer solutions. Of these solutions, 3886 satisfy the equation

$$x_1 - 6x_3 + 3x_6 + 6x_8 - 2x_9 + 6x_{10} - 6x_{26} - 6x_{27} - 6x_{28} + 3x_{43} = 24, -8$$

corresponding to the fusion of the involutions of H into G. Of these 3886 solutions, 1185 satisfy the equation for the fusion of elements of order 4, which is

$$\begin{aligned} &x_1 - 6x_2 + 6(x_3 + x_8 + x_{10} + x_{26} + x_{27} + x_{28}) - 6(x_{24} + x_{25}) \\ &+ 12(x_4 + x_7 + x_{14} + x_{16} + x_{19} + x_{20} + x_{32} + x_{33} + x_{38} + x_{39} + x_{40} + x_{41}) \\ &- 12(x_5 + x_{12} + x_{13} + x_{15} + x_{17} + x_{18} + x_{21} + x_{29} + x_{30} + x_{31} + x_{34} + x_{35} + x_{36} + x_{37} + x_{42}) \\ &+ 2x_9 - 8x_{11} + 3(x_6 + x_{43}) + 24x_{22} - 24x_{23} = -4,\ 0,\ 4,\ 8,\ 28,\ 64,\ 132. \end{aligned}$$

Continuing on, we now look at the fusion of elements order 5. Of the 1185 remaining possibilities, only 25 satisfy the equation

$$\begin{aligned}&x_1 + 4x_9 - x_{11} - 3(x_2 - x_3 + 2x_5 + x_6 - x_7 + 2x_8 + x_{12} + x_{13} + x_{14} + x_{15}\\&\quad - 4x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} - x_{24} - x_{25} + x_{26} + x_{27} + x_{28} - x_{43})\\&+ 6(x_{22} + x_{29} + x_{30} + x_{31} + x_{33} + x_{37} + x_{38} + x_{40} + x_{42})\\&- 9(x_{23} + x_{32} + x_{34} + x_{35} + x_{36} + x_{39} + x_{41})\\&+ 9(x_4 + x_{10}) + 12x_{16} = 23,\ -2.\end{aligned}$$

None of these satisfy the equation corresponding to the fusion of elements of order 7, which is

$$\begin{aligned}&x_1 + 5(x_2 + x_{43}) + 4x_3 - 4(x_4 + x_5)\\&+ 6(x_6 + x_7 + x_{31} + x_{32} + x_{33} + x_{35} + x_{36} + x_{38} + x_{39} + x_{42})\\&- 2(x_8 + x_9 + x_{10} + x_{22} + x_{23}) + (x_{11}) + 2(x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{20} + x_{21})\\&- 12(x_{18} + x_{19}) + 3x_{24} + 10x_{25} - 6(x_{26} + x_{27})\\&+ 8x_{28} - 8(x_{29} + x_{30} + x_{34} + x_{37} + x_{40} + x_{41}) = 52,\ 3.\end{aligned}$$

Hence there are no possible embeddings of H into G, thus concluding our work in this section.

3.2 $\mathrm{L}_2(2187)$

We now look to establish whether groups of the form $\mathrm{L}_2(3^n)$, for $n \in \{3, 4, 7\}$, can be maximal in G. To do this, we resort to constructing all copies of such groups upto G-conjugacy. We follow the method described in [62] and [41], although we must make several adaptations as we are working over a field of different characteristic. We go through this method below, including all preliminary results that we need. Following this, we implement the method in MAGMA to disprove the maximality of $\mathrm{L}_2(2187)$ in G.

3.2.1 Methodology

All results and methods given in this section can be applied to all $\mathrm{L}_2(3^n)$ ($n \in \{3, 4, 7\}$). As such, we use H to denote $\mathrm{L}_2(3^n)$ throughout this section. Firstly, we have a theorem detailing some key properties regarding the structure of H. This result is a cornerstone in our construction of these groups.

Theorem 3.2 ([37], 15.1.1)**.** *Let $G \cong \mathrm{L}_2(q)$ with $q = p^a$, p a prime. Then*

1. *Let $P \in \mathrm{Syl}_p(G)$. Then $P \cong p^a$, P is disjoint from its conjugates, and $N_G(P)$ is a Frobenius group with a cyclic complement which acts irreducibly on P.*
2. *If t is a prime distinct from p and 2, then $T \in \mathrm{Syl}_t(G)$ is cyclic.*
3. *If p is odd, then $S \in \mathrm{Syl}_2(G)$ is dihedral and has order 4 if and only if $q \equiv 3, 5 \pmod 8$.*
4. *If p is odd and R is a 4-subgroup of G, then $C_G(R) = R$.*
5. *If Q is a non-trivial subgroup of G such that $|Q| = m^d$ for m an odd prime, then $N_G(Q)$ does not contain a subgroup isomorphic to* Alt(4)*. Moreover, if $N_G(Q)$ contains a 4-subgroup, then Q is cyclic.*

Using Theorem 3.2 (1), we are able to devise an algorithm for constructing these groups. The following Lemma details a method for generating groups of the form $\mathrm{L}_2(3^n)$.

Lemma 3.3. Let $H \cong L_2(3^n)$ and $S \in \mathrm{Syl}_3(H)$. Then $S \cong 3^n$ and there exists an element $x \in N_H(S)$ of order $(3^n - 1)/2$, such that $N_H(S) = \langle S, x\rangle$, and x acts irreducibly on S. Moreover, there exists an involution $t \in H$ inverting x such that $H \cong \langle N_H(S), t\rangle$.

Proof. By 3.2 (1), we have that $S \cong 3^n$ and $N_H(S) = \langle S, x\rangle$ with x acting irreducibly on S. Additionally, from the proof of 3.2 (found in [37]), we get that $N_H(S)$ is maximal in H. The normaliser $N_H(\langle x\rangle)$ is a dihedral group of order $3^n - 1$ (see [[31], 1.3]) and hence has a presentation of the form $\langle x, t : x^m = t^2 = 1, txt = x^{-1}\rangle$ where $m = (3^n - 1)/2$. Therefore, we see that there exists an involution $t \in H$ inverting x. By [[37], 2.7.7], we have that $N_{N_H(S)}(\langle x\rangle) = \langle x\rangle$. In particular, $t \notin N_H(S)$. By maximality of $N_H(S)$ in H, we have $H \cong \langle N_H(S), t\rangle$. □

Unless specified otherwise, we now assume $S \in \mathrm{Syl}_3(H)$ and $x \in N_H(S)$ such that x has order $(3^n-1)/2$ and $N_H(S) = \langle S, x\rangle = S : \langle x\rangle$. By Lemma 3.3, in order to construct H, we must first find all copies of $S : \langle x\rangle$. As we are working upto G-conjugacy, it suffices to search for $S : \langle x\rangle$ upto G-conjugacy also. Due to the large order of G, a direct search is impossible. Hence, we wish to restrict our search to smaller subgroups. Given $\Phi \subset \Phi(E_8)^+$ and the corresponding parabolic subgroup $P = P(\Phi)$, we denote the unipotent radical of P by $Q(\Phi)$ and the Levi complement by $L(\Phi)$ (see Definition 2.1). Then, by the Levi decomposition, we have $P = Q(\Phi) : L(\Phi)$. Using the following Theorem, we are able to narrow down our search for groups of the form $S : \langle x\rangle$ to parabolic subgroups of G.

Theorem 3.4 (Borel-Tits)**.** *Let G be a simple linear algebraic group defined over the algebraic closure $\overline{\mathrm{GF}(q)}$ where $q = p^a$, p prime. Let σ be a Frobenius morphism of G and let G_σ denote the fixed point group of σ. Let Q be a non-identity p-subgroup of G_σ. Then there exists a parabolic subgroup $P_\sigma \leq G_\sigma$ such that $N_{G_\sigma}(Q) \leq P_\sigma$ and $Q \leq O_p(P_\sigma)$.*

By Theorem 3.4, we know that there exists some parabolic subgroup of G, say P, such that $S : \langle x\rangle \leq P$ and $S \leq O_3(P)$. As we are working upto G-conjugacy, it suffices to consider only the standard parabolic subgroups of G, that is, the parabolic subgroups $P(\Phi)$ where Φ is a subset of the fundamental roots $\Pi(E_8)$. We can actually restrict our search even further. Let $P_1 < P_2$ be standard parabolic subgroups of G such that $S : \langle x\rangle \leq P_1$ and $S \leq O_3(P_1)$. If there exists some conjugate of x inside P_2, say x^g for $g \in P_1$, then we can conjugate $S : \langle x\rangle$ into P_2 and S into $O_3(P_2)$. Indeed, a Sylow 3-subgroup of any parabolic subgroup P of G is also a Sylow 3-subgroup of G itself. Hence, by definition we have that

$$O_3(P_1) = \bigcap \mathrm{Syl}_3(P_1) \leq \bigcap \mathrm{Syl}_3(P_2) = O_3(P_2),$$

hence $S^g \leq O_3(P_1)^g = O_3(P_1) \leq O_3(P_2)$. Therefore it suffices to search for $S : \langle x\rangle$ inside those standard parabolic subgroups such that x cannot be conjugated into any smaller parabolic subgroup. We have a definition for groups with this property.

Definition 3.5. Let $\Phi_1 \subset \Phi(E_8)^+$ and $g \in L(\Phi_1)$. We say that $\langle g\rangle$ (or g) is $L(\Phi_1)$-cuspidal if $\langle g\rangle$ is not $L(\Phi_1)$-conjugate to a subgroup in any $L(\Phi_2)$ for $\Phi_2 \subset \Phi_1$. Given an element $g \in G$ that is $L(\Phi)$-cuspidal for some $\Phi \subset \Phi(E_8)^+$, we say that $\langle g\rangle$ (or g) is a Levi-cuspidal subgroup (or element) of G.

Using Definition 3.5 and the prior remarks, we know it is sufficient to search for $S : \langle x\rangle$ inside the standard parabolic subgroups of G such that $\langle x\rangle$ is a Levi-cuspidal subgroup of order $(3^n - 1)/2$. We will see later that even with these restrictions, we still will encounter a vast number of possibilities for S. Constructing copies of H from each possible S will be a very lengthy process, so if possible we would like to discard some of these subgroups before proceeding to constructing H. To do this, we consider the irreducible KH-module known as the Steinberg module.

Definition 3.6. Let G be a finite group of Lie type over a field $K = \mathrm{GF}(q)$. The Steinberg module is a projective and injective KG-module of dimension q^m, the order of a Sylow p-subgroup of G.

It turns out that by considering the multiplicity of the Steinberg module in a given feasible decomposition of H, we are able to rule out many of the subgroups S that we come across.

Lemma 3.7 ([41], Lemma 3.1.4)**.** Let $H \cong \mathrm{L}_2(3^n)$, $S \in \mathrm{Syl}_3(H)$, $N = N_H(S)$ and assume H is a subgroup of $G \cong \mathrm{E}_8(3)$. Let V_r be the composition factor of $V|_H$ with multiplicity n_r isomorphic to the Steinberg module. Then $\dim(C_V(N)) \geq n_r$ and if $\dim(C_V(N)) > n_r$, it follows that $C_V(H) \neq \{0\}$.

Using Lemma 3.7, we can discard any elementary abelian group S such that $\dim(C_V(\langle S, x\rangle)) > n_r$, where n_r is the multiplicity of the Steinberg module in the given feasible decomposition of H. We remark that we may use Lemma 3.7 to discount any 3-group O such that $\dim(C_V(\langle O, x\rangle)) > n_r$. For example, suppose $S \leq O \leq G$ for some 3-group O such that $\dim(C_V(\langle O, x\rangle)) > n_r$. Then we have that $\dim(C_V(\langle S, x\rangle)) > n_r$, so by Lemma 3.7 any H constructed from S fixes a vector on V. Hence by Lemma 2.5, we have H is not maximal in G.

Having now established where we should start looking for S, we now describe the process of how we do this in practice. Suppose that P is a standard parabolic subgroup containing a Levi-cuspidal element x of order $(3^n - 1)/2$. In practice, we know some properties that x must have. For example, we will (in most cases) be working with a specific feasible decomposition of H with a given fusion pattern. Hence, we can calculate which G-conjugacy class x fuses into. By Theorem 3.2, we know x acts irreducibly on S. Therefore, we search inside $O_3(P)$ for all elementary abelian subgroups S of order 3^n such that x acts irreducibly on S.

It is well known that for any p-group b and its Frattini subgroup $\Phi(b)$, the quotient group $b/\Phi(b)$ is elementary abelian. Hence, we wish to search inside $O_3(P)/\Phi(O_3(P))$ (which crucially is a smaller 3-group than $O_3(P)$) for subgroups such that their pre-image under the quotient map is an elementary abelian group of order 3^n. Suppose there exists an elementary abelian subgroup $S \leq O_3(P)$ of the desired properties and let $\psi : O_3(P) \to O_3(P)/\Phi(O_3(P))$ be the quotient map. Moreover, we consider $O_3(P)/\Phi(O_3(P))$ to be a $\langle x\rangle$-module. Indeed, x acts on $O_3(P)$ and so we can consider x acting on the quotient $O_3(P)/\Phi(O_3(P))$. As x acts irreducibly on S, the subgroup $\psi(S)$ is an irreducible $\langle x\rangle$-submodule of $O_3(P)/\Phi(O_3(P))$. Moreover, we have

$$\Phi(O_3(P)) \cap S = \{s \in S \mid \psi(s) = \Phi(O_3(P))\} \subseteq S$$

and this intersection is stabilised by x. Hence, due to the irreducible action of x on S, we must have that $\Phi(O_3(P)) \cap S = S$ or $\{1\}$. In particular, $\psi(S)$ is either a n or 0-dimensional irreducible $\langle x\rangle$-submodule of $O_3(P)/\Phi(O_3(P))$. As such, we search for S inside either the preimages of all n-dimensional irreducible $\langle x\rangle$-submodules of $O_3(P)/\Phi(O_3(P))$ or $\Phi(O_3(P))$ itself.

Suppose $\psi(S)$ is a n-dimensional irreducible $\langle x\rangle$-submodule of $O_3(P)/\Phi(O_3(P))$. As S has no $\langle x\rangle$-stable proper subgroups, we have that x acts faithfully on $\psi(S)$. If $\langle x\rangle$ doesn't act faithfully on $O_3(P)/\Phi(O_3(P))$, then it won't act faithfully on any of its submodules. In particular, $\langle x\rangle$ will not act faithfully on $\psi(S)$, so $\psi(S)$ must be 0-dimensional, meaning we search for S inside $\Phi(O_3(P))$.

In practice we consider $O_3(P)/\Phi(O_3(P))$ to be acted on by $\langle O_3(P), x\rangle$ instead of just $\langle x\rangle$. We do this so that our method remains practical to implement in Magma. Indeed, we wish to use the function `GModule(G,A,B)` which returns the module (over the field that `G` is defined) of the action of `G` on the quotient `A / B`. However, this requires three arguments `G,A,B` such that `A` and `B` are normal subgroups of `G` and the quotient `A / B` is an elementary abelian p-group. We wish to consider the action of $\langle x\rangle$ on $O_3(P)/\Phi(O_3(P))$, and setting `A`

= $O_3(P)$, B = $\Phi(O_3(P))$ satisfies MAGMA's requirements. However, as neither $O_3(P)$ nor $\Phi(O_3(P))$ are normal subgroups of $\langle x \rangle$, we cannot set G = $\langle x \rangle$. Setting G = $\langle O_3(P), x \rangle$ solves these issues. We note that studying the action of $\langle O_3(P), x \rangle$ on $O_3(P)/\Phi(O_3(P))$ instead of $\langle x \rangle$ does not make a difference. Indeed, every element of $\langle O_3(P), x \rangle$ is of the form ox^i for some $i \in \{1, ..., (3^n - 1)/2\}$. Moreover, we know that $O_3(P)/\Phi(O_3(P))$ is abelian, so any $o \in O_3(P)$ will act trivially on $O_3(P)/\Phi(O_3(P))$.

Let k be the dimension of $O_3(P)/\Phi(O_3(P))$ and $\rho : \langle O_3(P), x \rangle \to GL_k(3)$ be the representation of the action on $O_3(P)/\Phi(O_3(P))$. Let A denote $\mathrm{Im}(\rho)$. We have that $\langle x \rangle$ acts faithfully on A. By our previous remarks, any $o \in O_3(P)$ acts trivially on the quotient $O_3(P)/\Phi(O_3(P))$, hence $\rho|_{\langle x \rangle} : \langle x \rangle \to A$ is surjective. We now have a proposition.

Proposition 3.8. Let A be as above. Then $|A| = \frac{3^n - 1}{2}$.

Proof. We have $A = \mathrm{Im}(\rho)$. By the first isomorphism theorem we have that

$$\frac{\langle x \rangle}{\mathrm{Ker}(\rho|_{\langle x \rangle})} \cong A.$$

Hence $|A|$ divides $|\langle x \rangle| = (3^n - 1)/2$. Now suppose for contradiction $|A| < (3^n - 1)/2$. Then $\rho|_{\langle x \rangle}$ is not injective, so there exists some non-identity element in $\langle x \rangle$ that acts trivially on $O_3(P)/\Phi(O_3(P))$. However, this contradicts our assumption that $\langle x \rangle$ acts faithfully on $O_3(P)/\Phi(O_3(P))$. □

We see from the proof above that if $|A| < (3^n - 1)/2$, then it must be that $\langle x \rangle$ does not act faithfully on $O_3(P)/\Phi(O_3(P))$. Hence, by our previous remarks, it must be that any elementary abelian groups of interest are contained in $\Phi(O_3(P))$. This is true not only for P, but all 3-groups that we shall encounter as part of this process. Should we find a 3-group b such that $\langle x \rangle$ does not act faithfully on $b/\Phi(b)$ (which is checked by calculating $|A|$), then we store b in a set called `ActnGpDiff`. For all groups in $b \in$ `ActnGpDiff`, we search inside the Frattini subgroup $\Phi(b)$ for our elementary abelian subgroups S. Hence, we repeat this entire process but with $\Phi(b)$ (or $\Phi(O_3(P))$) in place of b (or $O_3(P)$).

When considered as a $\langle x \rangle$-module, $O_3(P)/\Phi(O_3(P))$ decomposes into a direct sum of the form

$$U \oplus V_1^1 \oplus \cdots \oplus V_{n_1}^1 \oplus \cdots \oplus V_1^m \oplus \cdots \oplus V_{n_m}^m$$

where U is a direct sum of irreducible submodules whose dimension is not n and $V_1^i \oplus \cdots \oplus V_{n_i}^i$ is a direct sum of n_i isomorphic n-dimensional irreducible submodules. Let V_i denote $V_1^i \oplus \cdots \oplus V_{n_i}^i$. If V is a n-dimensional irreducible submodule of $O_3(P)/\Phi(O_3(P))$, then V is isomorphic to V_1^i for some $i \in \{1, ..., m\}$. Consequently, V is a submodule of V^i, so the pre-image of V under ψ is contained in the pre-image of V^i. Hence, we consider the pre-images of all such V^i. These pre-images are 3-groups, smaller than $O_3(P)$, in which our subgroups S are contained. If $\Phi(O_3(P)) = \{1\}$, then $\psi^{-1}(V^i)$ is an elementary abelian group for all i. We wish to save these groups for further investigation, so we store them in a set named `FinSub`. If $\Phi(O_3(P)) \neq \{1\}$, then we gather the pre-images $\psi^{-1}(V^i)$ in a set called `SetSub2` and repeat the above process on every group we have added to this set.

Let $b \leq O_3(P)$ be one of the 3-groups we have collected and suppose that $b/\Phi(b) \cong V_1^1 \oplus \cdots \oplus V_{n_1}^1 = V^1$ where $V_i^1 \cong V_j^1$ for $i, j \in \{1, ..., n_1\}$, $|A| = (3^n - 1)/2$ and $\Phi(b)$ is not trivial. In this case, the algorithm never terminates as the pre-image of V^i will be continually added to `SetSub2` without ever being broken down. Whenever we find a b with such properties, we add it to a set called `BadSub`.

This concludes the first stage of the algorithm. We start with $O_3(P)$, where P is some standard parabolic subgroup of G and slowly break this down into smaller 3-groups which are added to `SetSub2`. This process is repeated until `SetSub2` is empty, by which point we will have a collection of 3-groups in each set `FinSub`, `ActnGpDiff` and `BadSub`. For $b \in$ `ActnGpDiff`, we know that $S \leq \Phi(b)$ and so we repeat the first stage of the algorithm but consider $\Phi(b)$ instead. Throughout this process, more groups will be added to `FinSub` and `BadSub` and this is repeated until `ActnGpDiff` is empty. It remains to consider the groups in `BadSub`.

Let $b \in$ `BadSub` and suppose that $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_m$ where $V_i \cong V_j$ for $i, j \in \{1, \ldots, m\}$. As before, we let $\psi : b \to b/\Phi(b)$ be the quotient map. Let V be any irreducible submodule of $b/\Phi(b)$. Then, for any non-zero $v \in V$, we have that $V = \langle v \rangle$. Moreover, we have that

$$V = \{x^i v : 1 \leq i \leq (3^n - 1)/2\} \cup \{0\}$$

as $|\{x^i v : 1 \leq i \leq (3^n - 1)/2\}| = (3^n - 1)/2$. This equality is checked for every group b and submodule V by checking that the fixed space in $b/\Phi(b)$ of every non-identity element in $\langle x \rangle$ is 0.

Our aim here is to generate all irreducible submodules of $b/\Phi(b)$. By the above remarks, if we can collect one non-zero vector from every irreducible submodule then we can generate them all. We now illustrate a method that allows us to collect such vectors. Let $v \in b/\Phi(b)$ be a non-zero vector. By the direct sum decomposition of $b/\Phi(b)$, we can write $v = v_1 + \ldots + v_m$ where $v_i \in V_i$ for $i \in \{1, \ldots, m\}$. Consider the vector $v_1 \in V_1$, we say that v_1 is the projection of v onto V_1. Suppose $v_1 \neq 0$, we fix $v_1 \in V_1$ and start collecting all vectors of the form $v_1 + w$ where $w \in V_2 \oplus \ldots \oplus V_m$. It suffices to have v_1 fixed and only consider vectors of the form $v_1 + w$. Indeed, we have already established that $V_1 = \{x^i v_1 : 1 \leq i \leq (3^n - 1)/2\} \cup \{0\}$. We shall show that by considering $v_1 + w$, we need not consider any vectors of the form $x^i v_1 + w$ for $1 \leq i \leq (3^n - 1)/2$. Suppose that $V = \langle v_1 + w \rangle$ is an irreducible module of $b/\Phi(b)$. Then $V = \langle x^i v_1 + w \rangle = \langle x^i (v_1 + x^{-i} w) \rangle = \langle v_1 + x^{-i} w \rangle$. As we are already collecting all vectors of the form $v_1 + x^{-i} w$, we need not consider $x^i v_1 + w$ as they both generate the same irreducible module V.

Now suppose that the projection of v onto V_1 is the zero vector. In this case, we consider $v_2 \in V_2$, the projection of v onto V_2. If $v_2 \neq 0$, then we repeat the above process and gather all vectors of the form $v_2 + w$ for $w \in V_3 \oplus \ldots \oplus V_m$. If $v_2 = 0$, then we consider the projection of v onto V_3 instead and repeat the process for all V_i. By the end, we will have gathered all vectors of the form $v_i + w$ for $w \in V_{i+1} \oplus \ldots \oplus V_m$. We store all such vectors in a set called `SetKeep`. Every irreducible submodule of $b/\Phi(b)$ can be generated by a vector inside `SetKeep`. Recall that we are searching for our elementary abelian subgroups S in the pre-images of irreducible submodules of $b/\Phi(b)$. By considering $\psi^{-1}(v) \in b$ for $v \in$ `SetKeep`, we are able to generate the pre-images of all irreducible submodules of $b/\Phi(b)$. However, in practice, the size of `SetKeep` can become very large when calculated this way. Hence, we wish to cut down the number of vectors in `SetKeep` that we need to consider, and we shall see shortly that there is an efficient way of doing this.

Let H be one of the pre-images generated by a vector from `SetKeep`. We remark that H itself is too large to directly search inside for S. Hence, we wish to employ the previous tactic of searching inside a quotient group of H. We must find some subgroup $A \leq \Phi(b)$ such that H/A is elementary abelian. An obvious choice would be $\Phi(H)$, however, there is better choice that improves the efficiency of the algorithm. Let C denote the commutator subgroup $[b, \Phi(b)]$. Note that C is normal in both b and $\Phi(b)$. We set A to be the pre-image of $\Phi(\Phi(b)/C)$ under the quotient map $\Phi(b) \to \Phi(b)/C$. As $C \leq A$, we have $\Phi(b)/A \leq Z(b/A)$ and $\Phi(\Phi(b)) \leq A$. Hence $\Phi(b)/A$ is elementary abelian as it is the homomorphic image of an elementary abelian group $\Phi(b)/\Phi(\Phi(b))$.

Now suppose t is the pre-image of some vector $v \in$ `SetKeep` and let $H = \langle \Phi(b), t^{\langle x \rangle} \rangle$ be the pre-image of the irreducible module generated by v. Then H has a decomposition

$$H = \Phi(b) \dot\cup t^x \Phi(b) \dot\cup t^{x^2} \Phi(b) \dot\cup \ldots \dot\cup t^{x^{\frac{3^n-1}{2}}} \Phi(b).$$

Suppose that $S \le H$ intersects $\Phi(b)$ trivially. Then by the above decomposition of H, we can write

$$S = \{1, t^x f_1, t^{x^2} f_2, \ldots, t f_{\frac{3^n-1}{2}}\}$$

for some $f_i \in \Phi(b)$. Recall that both S and $\Phi(b)/A$ are elementary abelian and $\Phi(b)/A \le Z(b/A)$. Hence, the images under the quotient map $\psi : \Phi(b) \to \Phi(b)/A$ of elements $t^{x^i} f_i$, $t^{x^j} f_j$ commute and are order 3. In particular, we have

$$\begin{aligned}
\psi(tf_1)^3 &= \psi(tf_1)\psi(tf_1)\psi(tf_1) \\
&= \psi(t)\psi(f_i)\psi(t)\psi(f_i)\psi(t)\psi(f_i) \\
&= \psi(t)\psi(t)\psi(t)\psi(f_i)\psi(f_i)\psi(f_i) \\
&= \psi(t)^3\psi(f_1)^3 \\
&= \psi(t)^3 = \psi(1).
\end{aligned}$$

Hence $\psi(t) \in \Phi(b)/A$ has order 3, so we have that $t^3 \in A$. Consequently, we disregard any vector $v \in$ `SetKeep` whose pre-image does not cube into A as otherwise we have $S \le \Phi(b)$. Using this makes working with H/A more practical than the more natural choice $H/\Phi(H)$ as we are often able to disregard a large number of vectors in `SetKeep`. Should we ever come across a group b such that none of the pre-images of the vectors in `SetKeep` cube into A, it must be that all elementary abelian groups of interest lie inside $\Phi(b)$. Hence, we store b in a set labelled `SetKeepZero` and deal with these cases in the same way `ActnGpDiff` is considered.

Having now trimmed down our `SetKeep`, we construct all groups of the form $\langle \Phi(b), t^{\langle x \rangle} \rangle$ where t is the pre-image of some vector in `SetKeep` and add them to a now empty `SetSub2`. We now repeat the first part of the algorithm, breaking down every group in `SetSub2` while adding new groups to `FinSub` and `ActnGpDiff`. It is highly likely that throughout this process we encounter more groups that would typically be added to `BadSub`. As we are currently working through the groups in `BadSub`, we add any new groups which would previously be added here into a new set called `BadSetNew`. We emphasise that this process of building up `SetKeep` and considering the groups $\langle \Phi(b), t^{\langle x \rangle} \rangle$ must happen for every group in $b \in$ `BadSub`. After the first pass of this part of the algorithm, we will have broken down all groups in `BadSub` and added new groups to `FinSub`, `ActnGpDiff` and `BadSetNew`. At this point, we set `BadSub := BadSetNew`, empty `BadSetNew` and repeat exactly what we have just done to start breaking down the groups in our new `BadSub`. This entire process is repeated until both `BadSub` and `BadSetNew` are empty.

It now remains to deal with the 3-groups contained in `FinSub` and `ActnGpDiff`. Firstly consider $b \in$ `ActnGpDiff`. By our previous remarks, we have that any suitable elementary abelian groups S will be found inside $\Phi(b)$. Hence we set `SetSub2` to be the set of all $\Phi(b)$ for $b \in$ `ActnGpDiff` and empty `ActnGpDiff`. We then repeat the entire algorithm, breaking down these groups and adding new groups to `FinSub` and the now empty `ActnGpDiff` and `BadSub`. This process is repeated until `ActnGpDiff` is empty and no new groups are added to `SetSub2`. We remark that if any groups are added to `BadSub`, then these must be dealt with in the same way as before.

Finally, all we have left to consider are the elementary abelian 3-groups in `FinSub` which contain the subgroups S we have been hunting for. Let $F \in$ `FinSub` and suppose $S \leq F$ has order 3^n with x acting irreducibly on S. By Lemma 3.3, we have that $H = \langle S, x, t \rangle \cong \mathrm{L}_2(3^n)$ for some involution $t \in G$ inverting x. Moreover, by Lemma 3.7, we can assume that $\dim(C_V(\langle F, x \rangle)) \leq n_r$ where n_r is the multiplicity of the Steinberg module in the feasible decomposition of $\mathrm{L}_2(3^n)$ we are considering. Indeed, suppose $\dim(C_V(\langle F, x \rangle)) > n_r$, then, as $\langle S, x \rangle \leq \langle F, x \rangle$, we have that $\dim(C_V(\langle S, x \rangle)) \geq \dim(C_V(\langle F, x \rangle)) > n_r$. Hence by Lemma 3.7 and Lemma 2.5, we would have that $H = \langle S, x, t \rangle$ is not maximal in G.

Considering F to be a $\langle x \rangle$-module over K, we have that S is an irreducible submodule of F of dimension n. We remark that every irreducible submodule of F is n-dimensional as F is the pre-image of a direct sum of n-dimensional isomorphic irreducible $\langle x \rangle$-modules under a quotient map $b \to b/\{1\}$ (recall that $\Phi(b) = \{1\}$ in order for $F \leq b$ to have been added to `FinSub`). Using Magma, we realise F as a $\langle x \rangle$-module using `GModule` with the arguments $\langle F, x \rangle$, F and the identity group $\{1\}$. This returns $F/\{1\} \cong F$ as a module over K being acted on by $\langle F, x \rangle$ and also a map $F/\{1\} \cong F \to \bar{F}$ where $\bar{F}$ denotes F when considered as a $\langle x \rangle$-module. As was the case when we considered $O_3(P)/\Phi(O_3(P))$ as a $\langle x \rangle$-module, F is abelian so any $f \in F$ acts trivially on F, so having $\langle F, x \rangle$ act on F here is not going to cause any problems. The command `MinimalSubmodules` on $\bar{F}$ returns a list of all irreducible n-dimensional submodules of $\bar{F}$ from which we calculate their pre-images under the map $F \to \bar{F}$. This set of pre-images contains all our subgroups S. As a final check, we use Lemma 3.7 and run through these pre-images, disregarding any such that $\dim(C_V(\langle S, x \rangle)) > n_r$. The code implementing this method into Magma is given in Section 9.2.7.

Having finally got our hands on S, we need to find all involutions $t \in G$ inverting x so that we can start generating all subgroups of the form $H = \langle S, x, t \rangle \cong \mathrm{L}_2(3^n) \leq G$. All such involutions are contained in the extended centraliser of x in G, which is defined to be

$$C_G^*(x) = \{g \in G \mid x^g = x \text{ or } x^g = x^{-1}\}.$$

We note that $C_G^*(x) = \langle C_G(g), t \rangle$ where t is an involution inverting x. Hence, if we can find one inverting involution, we can find them all. We shall see our work in Section 9.1 will be invaluable to us in obtaining $C_G^*(x)$.

3.2.2 Non-Maximality

We now implement the above methodology in Magma in order to disprove the maximality of $\mathrm{L}_2(2187)$ in G; the code used here can be found in Sections 9.2.3 and 9.2.4. In order to do this, we shall use Lemma 3.7 and Lemma 2.5. The dimension of the Steinberg module of $\mathrm{L}_2(2187)$ is 2187, hence this cannot be a composition factor in any composition series for $V|_H$. So by Lemma 3.7 and Lemma 2.5, in order to show any $H \cong \mathrm{L}_2(2187)$ is not maximal in G, it suffices to show that $C_V(N) \neq \{0\}$ where $N = N_H(S)$ is as in Lemma 3.3.

By Lemma 3.3, we have that $H \cong \mathrm{L}_2(2187) \cong \langle S, x, t \rangle$ where $x \in N_H(S)$ has order 1093. From Section 3.2.1, we know to search for H in the standard parabolic subgroups $P(\Phi)$ of G such that $\langle x \rangle$ is $L(\Phi)$-cuspidal. We identify such subgroups in the following Lemma.

Lemma 3.9. Let $g \in G$ have order 1093 and suppose $\langle g \rangle$ is a Levi-cuspidal subgroup of G. Set $\mathfrak{J} = \{\{1,3,4,5,6,7\}, \{2,4,5,6,7,8\}, \{3,4,5,6,7,8\}\}$. Then $\langle g \rangle$ is $L(\Phi)$-cuspidal for some $\Phi \in \mathfrak{J}$. Furthermore, in each $L(\Phi) \cong \mathrm{L}_7(3)$, there is one $L(\Phi)$-conjugacy class of $L(\Phi)$-cuspidal subgroups $\langle g \rangle$.

Proof. Using Magma, it can be easily verified that 1093 divides only the orders of $L(\Phi)$ for $\Phi \in \mathfrak{J} \cup \mathfrak{J}'$ where $\mathfrak{J}' = \{\{1,2,3,4,5,6,7\}, \{1,2,4,5,6,7,8\}, \{1,3,4,5,6,7,8\}, \{2,3,4,5,6,7,8\}\}$. Moreover, for $P \in$

$\mathrm{Syl}_{1093}(G)$, we have $P \cong 1093$ and so by Sylows theorem, there is exactly 1 $\langle g \rangle$-conjugacy class in each $L(\Phi)$. Let $\Phi \in \mathfrak{J}$ and $\Phi' \in \mathfrak{J}'$. We remark that the groups $L(\Phi')$ are the Levi-complements of maximal parabolic subgroups of G. Moreover, we have that each group $L(\Phi)$ is contained in some $L(\Phi')$. Consequently, we can conjugate $\langle g \rangle \leq L(\Phi')$ into some $L(\Phi)$. Hence $\langle g \rangle$ is not $L(\Phi')$-cuspidal. The result now follows from the fact that the groups $L(\Phi)$ are the smallest Levi-complements of parabolic subgroups that have order divisible by 1093. □

We are now ready to start using the code in Section 9.2.3. For each $\Phi \in \mathfrak{J}$, we construct $P = P(\Phi)$ using functions `makesub` and `makeQ` from Section 9.2.2. We remark that $Q(\Phi) = O_3(P)$ can be generated in a similar way to $L(\Phi)$ using root subgroups. Indeed, $Q(\Phi)$ is generated by all root subgroups corresponding to all roots $r \in \Phi(\mathrm{E}_8)^+ \backslash \Phi$ (and their negatives) such that the i^{th} component of r is positive for all $r_i \in \Pi(\mathrm{E}_8) \backslash \Phi$. We illuminate this with an example. Set $\Phi = \{1, 2, 3, 4\} \cong \mathrm{A}_4$. Then $P(\Phi) \cong Q(\Phi) : L(\Phi) = [3^{110}] : \mathrm{L}_5(3)$. To generate $Q(\Phi)$, we search for all roots $r \in \Phi(\mathrm{E}_8)^+$ such that $r \notin \Phi$ and the 5^{th}, 6^{th}, 7^{th} and 8^{th} components of r are positive. After collecting all such roots and their negatives, we are able to generate $Q(\Phi)$ using their corresponding root subgroups.

After setting `O := makeQ(Phi), S := makesub(Phi)` for some `Phi` $\in \mathfrak{J}$ and defining `x1093 := Element(S,1093)` to be an element of order 1093 inside the Levi-complement, we are ready to run stage 1 of the algorithm in order to start the search for elementary abelian subgroups of order 3^7. Table 10 shows the sizes of sets `FinSub`, `BadSub` and `ActnGpDiff` after running Code 9.2.3 on each $\Phi \in \mathfrak{J}$. We remark that if `SetKeepZero` or `ActnGpDiff` are not mentioned, in any of the work on $\mathrm{L}_2(27)$, $\mathrm{L}_2(81)$ or $\mathrm{L}_2(2187)$, then they remained empty.

Table 10: The sizes of sets FinSub, BadSub and ActnGpDiff for each $\Phi \in \mathfrak{J}$ after running Code 9.2.3.

	Φ		
	{1,3,4,5,6,7}	{2,4,5,6,7,8}	{3,4,5,6,7,8}
FinSub	10	8	11
BadSub	4	6	3
ActnGpDiff	0	0	0

We now consider each $\Phi \in \mathfrak{J}$ separately. Firstly, suppose $\Phi = \{1, 3, 4, 5, 6, 7\}$ and let b be any group in `FinSub`. Then b is order 3^7 and $\dim(C_V(\langle b, x_J \rangle)) > 2$. Now let b be any group contained in `BadSub`. Then b is either order 3^{14} and $\dim(C_V(\langle b, x \rangle)) = 3$, or b is order 3^{35} and $\dim(C_V(\langle b, x \rangle)) = 1$. Hence, we can conclude that $Q(\Phi)$ contains no suitable elementary abelian subgroups of order 3^7 in this case.

Now let $\Phi = \{3, 4, 5, 6, 7, 8\}$ and $b \in$ `FinSub`. Then b is either order 3^7 and $\dim(C_V(\langle b, x \rangle)) > 2$ (this is the case for ten of the groups in `FinSub`), or b is order 3^{14} and $\dim(C_V(\langle b, x \rangle)) = 1$. Now let $b \in$ `BadSub`. Then b is order 3^{14} and $\dim(C_V(\langle b, x \rangle)) = 3$. Hence, as in the first case, $Q(\Phi)$ contains no suitable elementary abelian subgroups of order 3^7.

Finally, let $\Phi = \{2, 4, 5, 6, 7, 8\}$ and $b \in$ `FinSub`. Then b is either order 3^7 and $\dim(C_V(\langle b, x \rangle)) > 2$ or b is order 3^{14} and $\dim(C_V(\langle b, x \rangle)) = 1$. Now let $b \in$ `BadSub`. Then b is either order 3^{14} and $\dim(C_V(\langle b, x \rangle)) = 3$ or b is order 3^{49} and $\dim(C_V(\langle b, x \rangle)) = 0$. In the latter case, we must delve further into the algorithm, employing Code 9.2.4 which deals with `BadSub`. Let b be any of the groups just mentioned with $|b| = 3^{49}$ and $\dim(C_V(\langle b, x \rangle)) = 0$. Then b is isomorphic to the direct sum of two 7-dimensional modules. Moreover, each group has a `SetKeep` of size 2188. Going through a `SetKeep` of this size takes less than a day when split across 12 separate MAGMA sessions. For each group b, 6 groups get added to `FinSub` and 2 to `BadSetNew`. So in total, we now have 18 groups in `FinSub` and 6 groups in `BadSetNew` to consider. Let b be any of these newly found 3-groups, then $\dim(C_V(\langle b,x\rangle)) \geq 1$, thus concluding this case. Having now dealt with all 3 of our cases, we have the final result of this section.

Lemma 3.10. Let $H \leq G$ such that $F^*(H) \cong L_2(3^7)$. Then H is not a maximal subgroup of G.

Proof. Let $S \cong 3^7$ be such that $S \leq b$ where b is any 3-group obtained by the process described above. Then $\dim(C_V(\langle S,x\rangle)) \geq 1$. By Lemma 3.7, any group $H \cong \mathrm{L}_2(2187) = \langle S,x,t\rangle$ constructed from $\langle S,x\rangle$ would be such that $C_V(H) \neq \{0\}$. Hence, by Lemma 2.5, H is not maximal in G. □

3.3 $\mathrm{L}_2(81)$

Having now shown $\mathrm{L}_2(2187)$ cannot be maximal in G, we employ the same methods to show the same result for $H \cong \mathrm{L}_2(81)$. Unlike with $\mathrm{L}_2(2187)$, we are able to calculate the irreducible KH-modules and consequently, we can obtain all possible feasible decompositions for H; these are given in Section 8.3. By Lemma 2.7, any H associated to feasible decomposition 2, 3 or 6 will be such that $C_V(H) \neq \{0\}$. Hence in these cases, by Lemma 2.5, we have that H and any automorphic extension of H are not maximal in G. Furthermore, from Table 3, we see that 10B squares into 5A, contradicting the fusion possibility for feasible decomposition 5, allowing us to rule out this case also. This leaves us with only two remaining decompositions, namely 1 and 4. In this section, we establish the non-maximality of H in these two cases.

3.3.1 Methodology

Let $H \cong \mathrm{L}_2(81)$, $S \in \mathrm{Syl}_3(H)$ and $N = N_H(S)$. By Lemma 3.3, we have that $S \cong 3^4$ and $H = \langle S,x,t\rangle$ where $x \in N_H(S) = S : x$ is order 40 acting irreducibly on S and t is an involution inverting x. The Steinberg module of H over K is denoted by φ_{12} in Section 8.3 and does not appear as a composition factor in either feasible decomposition 1 or 4. Therefore, by Lemma 3.7 and Lemma 2.5, disproving the maximality of a given H is reduced to showing $C_V(N) \neq \{0\}$.

To show this, we use the code in Sections 9.2.3 and 9.2.6 to start constructing all groups of the form $\langle S,x\rangle$. However, some adaptations have to be made to this code in order to make it compatible with our new smaller field. With $\mathrm{L}_2(2187)$, we were searching for elementary abelian subgroups of order 3^7 that were acted on irreducibly by an element of order 1093. Hence any occurrence of 1093 in the code must be replaced with 40. Similarly, if b is a 3-group we are working with, then $b/\Phi(b)$ is now a direct sum of the form $U \oplus V^1_{n_1} \oplus V^2_{n_2} \oplus \ldots \oplus V^m_{n_m}$ where each $V^i_{n_i}$ is direct sum of n_i irreducible 4-dimensional submodules (where they were 7-dimensional previously). So any occurrence of 7 in Section 9.2.3 must be replaced with 4. Similarly, the code in Section 9.2.4 must be changed.

We shall see shortly that working with $\mathrm{L}_2(81)$ is much more involved than with $\mathrm{L}_2(2187)$ and we have to be more careful when using MAGMA. In particular, dealing with `BadSub` will now be a much more laborious task. Previously, we had only two sets `SetKeep` to work with, both containing 2188 vectors. This was manageable, however using the same approach with $\mathrm{L}_2(81)$ could lead to `SetKeep` containing over 500000 vectors, which is far too many to consider. To avoid this, we use the following procedure.

Procedure 3.11. Let b be such that $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_m$ where V_i is an irreducible $\langle x\rangle$-submodule, $V_i \cong V_j$ and $\dim(V_i) = 4$ for all $i,j \in \{1,\ldots,m\}$. In particular, b represents a group that would be found in BadSub. Let F denote the pre-image of $V_1 \oplus \ldots \oplus V_r$ for some $1 \le r < m$. Then $\Phi(b) < F$, b/F is elementary abelian and $b/F \cong V_1 \oplus \ldots \oplus V_{m-r}$. As b/F is elementary abelian and $b/\Phi(b) < b/F$, we may use it in place of $b/\Phi(b)$ when dealing with BadSub. We remark that $b/\Phi(b)$ must first be calculated in order to find F.

Recall that SetKeep contains all vectors $v_i + w$ where $v_i \in V_i$ and $w \in V_{i+1} \oplus \ldots \oplus V_m$ whose pre-image under the quotient map cubes into A. After considering b/F in place of $b/\Phi(b)$, we are now gathering these vectors into SetKeep from a direct sum of length $m - r$, as opposed to m. In particular, SetKeep will now contain all vectors $v_i + w$ where $v_i \in V_i$ and $w \in V_{i+1} \oplus \ldots \oplus V_{m-r}$ whose pre-image under the quotient map cubes into A. Consequently, fewer vectors are added to SetKeep, making it more manageable.

As r (and consequently $|F|$) increases, the size of SetKeep decreases. However, the caveat is that the pre-images we obtain $\langle F, t^{\langle x \rangle} \rangle$ (labelled H in Section 3.2.1 or Sub4aa in Section 9.2.6) are larger, meaning more iterations of the algorithm must be performed. The value for r was chosen on a case by case basis; we need r to be large enough to ensure |SetKeep| is kept to a manageable size but not too large to the point where our pre-images are barely smaller than the group b we started with. The code showing how this was implemented in practice is shown in Section 9.2.6.

To assist in future explanations (particularly in Section 3.4), we introduce some notation. With $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_m$ and $F \cong V_1 \oplus \ldots \oplus V_r$ as described above, we define CN=m-r; this is a variable used in Code 9.2.6 which controls the size of SetKeep. When utilising this procedure, instead of giving all the details again, we simply reference this procedure (or Code 9.2.6) and state the value of CN being used. For example, suppose we are considering $\mathrm{L}_2(27)$ and that we have a group b such that $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_9$. In this case, calculating SetKeep using $b/\Phi(b)$ would be impractical as we would accumulate hundreds of thousands of vectors, hence we wish to utilise the process stated above. So we set CN=3 to instead consider $b/F \cong V_1 \oplus V_2 \oplus V_3$ to obtain a more manageable SetKeep of size 757. Many detailed examples showing this procedure in use are given in Section 3.4 on our work with $\mathrm{L}_2(27)$. As such, we often omit all the minute details for the $\mathrm{L}_2(81)$ case.

Before we start the construction of the groups of the form $3^4 : \langle x \rangle$, we need to identify the standard parabolic subgroups $P(\Phi)$ such that $\langle x \rangle$ is $L(\Phi)$-cuspidal. These parabolic subgroups will be different for each feasible decomposition, so we consider them separately. In order to find the relevant $P(\Phi)$, we use the Brauer character of x on V and the information in Table 3. From Section 8.3, we know that either $x \in$ 40DE (if H corresponds to feasible decomposition 1) or $x \in$ 40X′Y′ (if x corresponds to feasible decomposition 4). The respective Brauer characters of both these classes can be found in Table 5.

Using this information and our work in Section 3.2.1, we know to search for all standard parabolic subgroups $P(\Phi)$ such that $\langle x \rangle$ is $L(\Phi)$-cuspidal where $x \in$ 40DE $\cup$ 40X′Y′. Upon finding such groups, we try to find representatives for all $\langle x \rangle$-conjugacy classes and proceed to search inside $O_3(P(\Phi))$ for elementary abelian subgroups of order 3^4 using Code 9.2.3 and 9.2.6. We remark that we are not able to obtain representatives for all $\langle x \rangle$-conjugacy classes in some cases due to the order and structure of $L(\Phi)$; more detail on how we work around this is given in the following sections. Table 11 lists all isomorphism types of $L(\Phi)$ that contain elements of order 40. It now remains to consider each of these individually to search for groups in which $\langle x \rangle$ is Levi-cuspidal. Before we consider each feasible decomposition separately, we list some fusion properties for elements of order 40 in G. This information is used frequently in the proofs for both Lemma 3.14 and 3.13.

Lemma 3.12. Let $\Phi \subseteq \Pi(\mathrm{E}_8)$ and let $P = P(\Phi)$ be the standard parabolic subgroup associated to Φ.

1. If $|\Phi| = 1$ and $g \in P$ has order 4, then $g \in$ 4A and $g^2 \in$ 2A.

2. If $|\Phi| = 2$ and $g \in P$ has order 8, then $g \in 8\text{B}$, $g^2 \in 4\text{A}$ and $g^4 \in 2\text{A}$.

3. If $|\Phi| = 3$ and $g \in P$ has order 40, then $\dim(C_V(g)) = 48$. Furthermore, if $g \in P$ has order 5, then $g \in 5\text{A}$.

Proof. Using Magma, it can easily be shown that if $|\Phi| = 1$ and $g \in P(\Phi)$ is order 4, then $\dim(C_V(g)) = 134$. Similarly, if $|\Phi| = 2$ and $g \in P(\Phi)$ has order 8 then $\dim(C_V(g)) = 80$. We are able to distinguish the two G-classes of order 8 elements with fixed space dimension 80 using the fact that $\dim(C_V(g^2)) = 134$. Finally, suppose $|\Phi| = 3$. If $\Phi \not\cong \text{A}_3$, then Φ can be decomposed into the sum of at least two orthogonal root subsystems, in which case we are reduced to the cases when $|\Phi| = 1, 2$. So suppose $\Phi \cong \text{A}_3$ and $g \in P(\Phi) \cong \text{L}_3(3)$ has order 40 (resp. 5). Then $\dim(C_V(g)) = 48$ (resp. 68). All relevant class labelling and fusion information may be found in Table 3. □

Table 11: Levi-complements of standard parabolic subgroups containing elements of order 40.

$\|\Phi\|$	Isomorphism type of $L(\Phi)$
3	$\text{SL}_4(3)$
4	$\text{L}_5(3)$, $\text{SL}_2(3) \times \text{SL}_4(3)$, $2^{2\cdot}\Omega_8^+(3)$
5	$2^{\cdot}\Omega_{10}^+(3)$, $\text{SL}_2(3) \times \text{L}_5(3)$, $\text{L}_3(3) \times \text{SL}_4(3)$, $\text{SL}_6(3)$, $\text{SL}_2(3) \times 2^{2\cdot}\Omega_8^+(3)$, $\text{SL}_2(3)^2 \times \text{SL}_4(3)$
6	$2^{\cdot}\Omega_{10}^+(3) \times \text{SL}_2(3)$, $\text{L}_3(3) \times \text{SL}_2(3) \times \text{SL}_4(3)$, $\text{SL}_4(3)^2$, $\text{E}_6(3)$, $2^2.\Omega_{12}^+(3)$, $\text{L}_3(3) \times 2^{2\cdot}\Omega_8^+(3)$ $\text{SL}_2(3) \times \text{SL}_6(3)$, $\text{SL}_2(3)^2 \times \text{L}_5(3)$, $\text{L}_7(3)$, $\text{L}_3(3) \times \text{L}_5(3)$
7	$2.\Omega_{14}^+(3)$, $\text{SL}_8(3)$, $\text{SL}_2(3) \times \text{L}_7(3)$, $\text{L}_3(3) \times \text{SL}_2(3) \times \text{L}_5(3)$, $\text{L}_5(3) \times \text{SL}_4(3)$, $2^{\cdot}\Omega_{10}^+(3) \times \text{L}_3(3)$, $\text{E}_6(3) \times \text{SL}_2(3)$, $2.\text{E}_7(3)$

3.3.2 Feasible Decomposition 4

Lemma 3.13. Suppose that $\langle g \rangle$ is a Levi-cuspidal subgroup of G such that $x \in 40\text{X}'\text{Y}'$. Set $\mathfrak{I} = \{\{1,3,4,6,7,8\},\{2,3,4,6,7,8\}\}$. Then $\langle g \rangle$ is $L(\Phi)$-cuspidal for some $\Phi \in \mathfrak{I}$. Moreover, in each $L(\Phi)$ for $\Phi \in \mathfrak{I}$, there are 4 $L(\Phi)$-classes of cuspidal subgroups $\langle g \rangle$.

Proof. We have that $\langle g \rangle \cong 40$, $\dim(C_V(g)) = 8$ and the Brauer character of g on V is either $\frac{\alpha-5}{8}$ or $\frac{\tau(\alpha)-5}{8}$ where α and τ are defined before Table 5. Moreover, from Table 3, we find that $g^8 \in 5\text{B}$. Hence by Lemma 3.12, we have that $g, g^8 \notin P(\Phi)$ when $|\Phi| = 3$. We also remark that when $|\Phi| = 1, 2$, the parabolic subgroups $P(\Phi)$ contain no elements of order 5. In particular, $g^8 \notin P(\Phi)$ in these cases. We shall use this frequently in the following proof. Table 11 lists all isomorphism types of Levi-subgroups that contain elements of order 40, and we consider each of these cases.

Firstly consider $L(\Phi) \cong \text{SL}_4(3)$. From Lemma 3.12 we see that if $g \in L(\Phi)$, then $\dim(C_V(g)) = 48$, contradicting $\dim(C_V(g)) = 8$.

Now suppose $g \in L(\Phi)$ where $|\Phi| = 4$. If $L(\Phi) \cong \text{L}_5(3)$, then there is one $L(\Phi)$-conjugacy class of C_{40}-subgroups. If g is the representative generator for this class, we would have $\dim(C_V(g)) = 48$, again contradicting $\dim(C_V(g)) = 8$. Now we consider $L(\Phi) \cong \text{SL}_2(3) \times \text{SL}_4(3)$. There are three $L(\Phi)$-classes of C_{40}-subgroups and if h denotes a generator representing one of these classes, we have $\dim(C_V(h)) \in$

$\{24, 26, 48\}$. Now suppose $L(\Phi) \sim 2^{2\cdot}\Omega_8^+(3)$. Using Magma, we find that a Sylow 5-subgroup of $L(\Phi)$ has order 5^2 and contains only non-identity elements from the class 5A. This contradicts our requirement that $g^8 \in 5B$, so no suitable elements are to be found here.

Next assume $g \in L(\Phi)$ where $|\Phi| = 5$. First consider $L(\Phi) \sim 2^{\cdot}\Omega_{10}^+(3)$. We have that $2^{2\cdot}\Omega_8^+(3) \sim L(\{2, 3, 4, 5\}) \leq L(\Phi)$ and a Sylow 5-subgroup of $L(\Phi)$ has order 5^2. Hence by Sylow's theorem, any element of order 5 from $L(\Phi)$ can be conjugated into $L(\{2, 3, 4, 5\}) \sim 2^{2\cdot}\Omega_8^+(3)$. We have previously shown $2^{2\cdot}\Omega_8^+(3)$ only contains 5A elements. This contradicts our requirement that $g^8 \in 5B$, so no suitable elements are to be found here. An identical argument works for both $L(\Phi) \cong E_6(3)$ and $L(\Phi) \cong 2^2.\Omega_{12}^+(3)$, hence no suitable elements are to be found here either.

Now suppose that $L(\Phi) \cong SL_2(3) \times L_5(3)$. We have shown that g cannot lie inside the $L_5(3)$ factor, hence we must have that $g^8 \in L_5(3)$. However, $L_5(3)$ has one class of elements of order 5 and this fuses to 5A, a contradiction. Next we assume that $L(\Phi) \cong L_3(3) \times SL_4(3)$. We know that g cannot be contained in $SL_4(3)$, hence as $L_3(3)$ contains no elements of order 5, we have $g^8 \in SL_4(3)$, which is not possible. If $L(\Phi) \cong SL_6(3)$, a quick Magma calculation shows this forces $\dim(C_V(g)) \in \{24, 26, 48\}$, hence no suitable elements are to be found here. Consider $L(\Phi) \sim SL_2(3) \times 2^{2\cdot}\Omega_8^+(3)$. We know that $g \notin 2^{2\cdot}\Omega_8^+(3)$, hence we must have $g^8 \in 2^{2\cdot}\Omega_8^+(3)$. However, we have already showed that all elements of order 5 in $2^{2\cdot}\Omega_8^+(3)$ belong to 5A, thus contradicting $g^8 \in 5B$. Now assume $L(\Phi) \cong SL_2(3)^2 \times SL_4(3)$. Following a similar argument to before, we must have that $g^8 \in SL_4(3)$, ruling out this case.

Now suppose $g \in L(\Phi)$ where $|\Phi| = 6$. If $L(\Phi) \sim 2^{\cdot}\Omega_{10}^+(3) \times SL_2(3)$, we must have $g^8 \in 2^{\cdot}\Omega_{10}^+(3)$ as we have shown g cannot live inside either factor completely. However, we know that $2^{\cdot}\Omega_{10}^+(3)$ does not contain any 5B elements, so no suitable elements are to be found here. Similarly, if $L(\Phi) \cong L_3(3) \times SL_2(3) \times SL_4(3)$, then we get $g^8 \in SL_4(3)$, another contradiction.

Now we consider $L(\Phi) \cong SL_4(3)^2$. There are 52 $L(\Phi)$-classes of C_{40}-subgroups. Of these 52 classes, 4 contain generators that satisfy our restrictions on g. By our previous work, we must have that $\langle g \rangle$ is $L(\Phi)$-cuspidal for $J \in \{\{1, 3, 4, 6, 7, 8\}, \{2, 3, 4, 6, 7, 8\}\}$.

Now assume $L(\Phi) \sim L_3(3) \times 2^{2\cdot}\Omega_8^+(3)$. We know $g \notin 2^{2\cdot}\Omega_8^+(3)$. Hence, by following an identical argument to the case $L(\Phi) \sim SL_2(3) \times 2^{2\cdot}\Omega_8^+(3)$, we see there are no suitable elements here. Suppose $L(\Phi) \cong SL_2(3) \times SL_6(3)$. As we have $g \notin SL_6(3)$, it must be that $g^8 \in SL_6(3)$. However, $SL_6(3)$ contains only one class of elements of order 5 and this class fuses to 5A, contradicting $g^8 \in 5B$. Assume $L(\Phi) \cong SL_2(3)^2 \times L_5(3)$ or $L(\Phi) \cong L_3(3) \times L_5(3)$. Following an identical proof to $L(\Phi) \cong SL_2(3) \times L_5(3)$, we see no suitable elements are to be found here due to the fusion of elements of order 5 in $L_5(3)$. For the final case of $|\Phi| = 6$, assume $L(\Phi) \cong L_7(3)$. We find that $L(\Phi)$ has one conjugacy class of elements of order 5 and this fuses into 5A, contradicting $g^8 \in 5B$.

Finally we consider the maximal parabolic subgroups of G. Suppose $g \in L(\Phi)$ where $|\Phi| = 7$. First we consider $L(\Phi) \cong SL_8(3)$. Here we find two $L(\Phi)$-classes of C_{40}-subgroups that satisfy our conditions. We have already seen that $SL_4(3)^2 \leq L(\Phi)$ contains four suitable classes. In the 248-dimensional representation, we generate two representatives of these four classes. These remain non-conjugate inside $L(\Phi)$. Hence we are able to conjugate both classes of C_{40}-subgroups in $L(\Phi)$ into $SL_4(3)^2$. Consequently, $\langle g \rangle$ is not $L(\Phi)$-cuspidal. Now suppose $L(\Phi) \cong SL_2(3) \times L_7(3)$ or $L(\Phi) \cong L_3(3) \times SL_2(3) \times L_5(3)$. Following a very similar argument to many of the previous cases, we find that either $g^8 \in L_7(3)$ or $g^8 \in L_5(3)$, which we have shown cannot happen.

Let $L(\Phi) \cong L_5(3) \times SL_4(3)$. We find that there are four $L(\Phi)$-classes of C_{40}-subgroups that satisfy our conditions. We have already shown that $\mathrm{SL}_4(3)^2 \leq L(\Phi)$ contains four suitable classes of C_{40}-subgroups. A quick Magma calculation shows that these four classes remain non-conjugate in the larger parabolic $L(\Phi)$. Hence $\langle g\rangle$ is not $L(\Phi)$-cuspidal here. Now suppose $L(\Phi) \sim 2^{\cdot}\Omega_{10}^{+}(3) \times \mathrm{L}_3(3)$. As in the case $L(\Phi) \sim 2^{\cdot}\Omega_{10}^{+}(3) \times \mathrm{SL}_2(3)$, we get that $g^8 \in 2^{\cdot}\Omega_{10}^{+}(3)$ which is not possible. Similarly, if $L(\Phi) \cong \mathrm{E}_6(3) \times \mathrm{SL}_2(3)$ then we must have $g^8 \in \mathrm{E}_6(3)$ as $\mathrm{E}_6(3)$ contains no suitable C_{40}-subgroups. However, we have already shown this contradicts our assumptions.

Finally, suppose $L(\Phi) \sim 2.\mathrm{E}_7(3)$ or $L(\Phi) \sim 2.\Omega_{14}^{+}(3)$. Let $P \in Syl_5(L(\Phi))$, then $P \cong 5^2$. As $2^2.\Omega_{12}^{+}(3) \leq L(\Phi)$, by Sylow's theorem we can follow an identical argument to this case in order to rule out these final two subgroups. □

We have now established the parabolic subgroups $P(\Phi)$ in which we must search for groups of the form $N = S : x$. To do this, we follow the method described in Section 3.2.1 and search for all $S \cong 3^4$ inside $O_3(P(\Phi))$ that are acted on irreducibly by x. We use the code in Sections 9.2.3 and 9.2.6; the outcome of running this code is shown in the following section.

Code Outcomes

From Lemma 3.13, we need only consider the two standard parabolic subgroups $P(\Phi)$ such that $\Phi \cong 2\mathrm{A}_3$, namely $\{1,3,4,6,7,8\}, \{2,3,4,6,7,8\}$. Moreover, we found that in each $L(\Phi)$, there are four $L(\Phi)$-classes of cuspidal subgroups $\langle g\rangle$ where $g \in 40\mathrm{X}'\mathrm{Y}'$.

Firstly, suppose $\Phi = \{1,3,4,6,7,8\}$ and let x_1, x_2, x_3, x_4 denote representatives from each of the four conjugacy classes of cuspidal elements. Table 12 shows the initial sizes of sets `FinSub`, `BadSub` and `ActnGpDiff` after running Code 9.2.3.

Table 12: The outcome of running the code in Section 9.2.3 for $\Phi = \{1,3,4,6,7,8\}$.

	Representative			
	x_1	x_2	x_3	x_4
FinSub	10	8	8	8
BadSub	5	6	7	8
ActnGpDiff	12	12	12	13

Firstly, let b be any 3-group from any of the sets `ActnGpDiff` from Table 12. Then either $\dim(C_V(\langle b, x_i\rangle)) \geq 1$ for $i \in \{1,2,3,4\}$ or $\Phi(b) = \{1\}$. If $\dim(C_V(\langle b, x_i\rangle)) \geq 1$, then by Lemma 3.7 and Lemma 2.5 we have that any group H constructed from $N_H(S) = S : x \leq \langle b, x_i\rangle$ is not maximal in G. In the remaining cases we have $\dim(C_V(\langle b, x_i\rangle)) \leq 1$ and $\Phi(b) = \{1\}$. Here we must search for $S : x$ inside $\Phi(b)$ as detailed in Section 3.2.1. However $\Phi(b)$ is trivial, so no suitable groups are to be found here. We now consider each representative x_i individually.

Firstly, we consider x_1. Of the 10 groups in `FinSub`, we find that 9 are such that $\dim(C_V(\langle b, x_1\rangle)) \geq 1$, so we can discount these. Let b denote the remaining case. We have $|b| = 3^{12}$ and $\dim(C_V(\langle b, x_1\rangle)) = 0$. We must search inside b for all elementary abelian groups of order 3^4 that are acted on irreducibly by x. Such elementary abelian subgroups correspond to the minimal $\langle x\rangle$-submodules of b when we consider b to be a x-module over K; we let $\bar{b}$ denote b when considered as a x-module. Indeed, we know the image of S in $\bar{b}$ will be an irreducible module of dimension 4. However, we do not want to ask for all irreducible submodules of $\bar{b}$ as such submodules may not have dimension 4, hence we use `MinimalSubmodules` instead. After obtaining these minimal submodules, we then take their pre-image from $\bar{b}$ back into b. From Lemma 3.7, it suffices to keep only those pre-images S such that $\dim(C_V(\langle S, x_1\rangle)) = 0$. For the case we are considering here, we find 6643 minimal $\langle x\rangle$-submodules and hence 6643 possibilities for S. Of these 6643 groups, 6560 are such that $\dim(C_V(\langle S, x_1\rangle)) = 0$. We save these groups for later consideration.

Next, we consider the groups in `BadSub`. Of these 5 groups, 4 are such that $\dim(C_V(\langle b, x_1\rangle)) = 0$ and thus must be further examined. In 1 of these 4 cases, we must consider b/F where F is some overgroup of $\Phi(b)$. In the other cases, it suffices to run the code as a whole without alterations. After these groups have been sufficiently broken down, we find only one elementary abelian 3-group b of interest. We have that $|b| = 3^8$ and $\dim(C_V(\langle b, x_1\rangle)) = 0$. After finding the minimal $\langle x\rangle$-submodules of $\bar{b}$ (as described previously) and taking pre-images, we find 80 elementary abelian groups S such that $|S| = 3^4$ and $\dim(C_V(\langle S, x_1\rangle)) = 0$. Again, we save these groups for later work.

Next, we consider representative x_2. In `BadSub`, we need to consider 5 of the 6 groups. After defining overgroups of $\Phi(b)$ when appropriate and working through the algorithm to break these groups down, we are left with a `FinSub` of size 93 (including our original `FinSub` of size 8). Of these 93 groups, 83 are such that $\dim(C_V(\langle b, x_1\rangle)) = 0$. After going through these 83 groups and taking pre-images of minimal $\langle x_2\rangle$-submodules when necessary, we are left with 6800 elementary abelian groups of order 3^4 that we must consider.

Moving on to x_3, of the 7 groups in `BadSub`, 6 are such that $\dim(C_V(\langle b, x_3\rangle)) = 0$. After these have been broken down, we are left with a `FinSub` of size 891. Of these 891 groups, 800 contain possibilities for S. After calculating minimal submodules and taking pre-images, we find 800 elementary abelian groups of order 3^4 from which we must look to construct copies of H.

Finally, we consider representative x_4. Of the 8 groups in `BadSub`, 6 are such that $\dim(C_V(\langle b, x_4\rangle)) = 0$. After defining overgroups when necessary and running through the algorithm, we end up with a `FinSub` of size 892. After finding minimal $\langle x_4\rangle$-submodules of these groups and discarding all groups that would lead to a non-maximal H, we are left with 962 elementary abelian groups of order 3^4 that we must consider.

Now we consider $\Phi = \{2, 3, 4, 6, 7, 8\}$. As before, let x_1, x_2, x_3, x_4 denote representatives from each of the four conjugacy classes of cuspidal elements. Table 13 shows the initial sizes of sets `FinSub`, `BadSub` and `ActnGpDiff` after running the code in Section 9.2.3.

Table 13: The outcome of running the code in Section 9.2.3 for $\Phi = \{2, 3, 4, 6, 7, 8\}$.

	Representative			
	x_1	x_2	x_3	x_4
FinSub	10	12	8	8
BadSub	3	3	10	10
ActnGpDiff	11	11	17	17

Firstly consider representative x_1. We must consider all 3 groups in `BadSub` here, these have orders 3^{64}, 3^{60} and 3^{48}. After working through the algorithm and defining overgroups F of $\Phi(b)$ when necessary, we are left with a `FinSub` of size 12. After calculating minimal submodules and taking pre-images, we find 6716 elementary abelian groups of order 3^4 from which we must look to construct copies of H.

Next consider x_2. As before, we must consider all the groups in `BadSub`. After breaking these 3 groups down, we get a `FinSub` of size 12. From these 12 groups, we find 6640 possibilities for S from which we shall look to construct copies of H.

For x_3, we must consider 6 of the 10 groups in `BadSub`. The largest of these is 3^{26}, meaning that breaking them down is a quick process. We end up with a `FinSub` of size 2592 and after taking minimal submodules, we are left with 2560 elementary abelian groups of order 3^4 that we must consider.

Finally, we consider x_4. We need only consider 7 of the 10 groups in `BadSub`. As with x_3, the largest of these is 3^{26}. After running through the algorithm, we are left with a `FinSub` of size 2591. Of these groups S in `FinSub`, 2480 are such that $\dim(C_V(\langle S, x_4\rangle)) = 0$

Constructing Copies of $\mathrm{L}_2(81)$

We now look to construct all groups of the form $H = \langle S, x, t\rangle \cong \mathrm{L}_2(81)$ inside G upto conjugacy. In the previous section, we found all possibilities for both x and S, it now remains to find all inverting involutions t of x in G and then check whether the group $\langle S, x, t\rangle$ is isomorphic to $\mathrm{L}_2(81)$. As remarked previously, we shall find all inverting involutions t inside the extended centraliser $C_G^*(x)$. Moreover, we have that $C_G^*(x) = \langle C_G(x), t\rangle$ where t is any inverting involution of x.

Firstly we consider the case $\Phi = \{2, 3, 4, 6, 7, 8\}$. Let x be any of the x_i mentioned in the previous section. The centraliser $C_G(x)$ can be easily constructed inside a much smaller subgroup. We find that $C_G(x) = C_{G(\Phi')}(x)$ where $\Phi' = \{2, 3, 4, 6, 7, 8, 24, 106\} \cong \mathrm{A}_8$. We remark that $C_G(x) \cong 5^2 \times 16^2$. It now remains to find an inverting involution of x. To begin, we construct the centraliser $C_1 = C_G(x^8)$; the process of doing this is described in Section 9.1.2. We now wish to construct the extended centraliser of $x^4 \in 10\mathrm{F}$, as we have that $C_G^*(x) \leq C_G^*(x^4)$. Using `FindCent`, we construct $C_{C_1}(x^4) = C_G(x^4)$. An inverting involution t' of x^4 can be easily found inside $N_{L(\Phi)}(\langle x^4\rangle)$. Then we have $C_2 = C_G^*(x^4) = \langle C_{C_1}(x^4), t'\rangle$. Finally, an involution t inverting x can be found inside $N_{C_2}(\langle x\rangle)$ which can be constructed using `LMGNormaliser`.

A similar process can be followed to find inverting involutions for all four representatives from the case $\Phi = \{1, 3, 4, 6, 7, 8\}$. Set $\Phi = \{1, 3, 4, 6, 7, 8\}$, $\Phi_1 = \Phi \cup \{82, 100\} \cong 2\mathrm{A}_4$, $\Phi_2 = \Phi \cup \{5, 69\} \cong \mathrm{D}_8$ and let x denote any of our four representatives x_i for $i \in \{1, 2, 3, 4\}$. Then let $C_1 = \langle C_{G(\Phi_1)}(x^8), C_{G(\Phi_2)}(x^8)\rangle \leq C_G(x^8)$. As before, we construct $C_{C_1}(x^4)$ and find an inverting involution t' of x^4 inside $N_{L(\Phi)}(\langle x^4\rangle)$. Having obtained the subgroup $C_2 = \langle C_{C_1}(x^4), t'\rangle \leq C_G^*(x^4)$, we find an inverting involution t of x inside $N_{C_2}(\langle x\rangle)$. Then $C_G^*(x) = \langle C_{G(\Phi_1)}(x), t\rangle$.

Let x be any of our representatives from either $\Phi = \{1, 3, 4, 6, 7, 8\}$ or $\{2, 3, 4, 6, 7, 8\}$. Then $C_G^*(x)$ contains four classes of involutions inverting x, which gives 6400 inverting involutions in total. By the fusion pattern for this feasible decomposition, we have that all involutions in H must be contained in 2B, and all 6400 of these involutions are. Before we start generating groups of the form $\langle S, x, t\rangle$ (where $S \cong 3^4$ was found in the previous section), we wish to cut down the number of these involutions that we need to consider. Many of these inverting involutions, when paired with x, will generate the same group $\langle x, t\rangle$. After keeping only those that generate a unique subgroup $\langle x, t\rangle \cong \mathrm{Dih}(80)$, we have only 160 inverting involutions left. We can now proceed to generate all groups of the form $\langle S, x, t\rangle$.

Not all groups we generate will be isomorphic to $\mathrm{L}_2(81)$; we must be very careful when checking whether $\langle S, x, t\rangle$ is indeed a copy of $\mathrm{L}_2(81)$. It is possible for the group we generate, $H = \langle S, x, t\rangle$, to have a very large order, meaning that checking the orders of these groups and comparing them to $\mathrm{L}_2(81)$ is not a good strategy. Firstly, we look at the number of composition factors in $V|_H$. We expect there to be 11 composition factors, as we have this many in the feasible decomposition. Even after keeping only those H such that $V|_H$ has 11 composition factors, we still may have groups not isomorphic to $\mathrm{L}_2(81)$. A quick way of removing these non-isomorphic groups is by checking element orders. Looping through the groups we have generated, we find 20 random elements from each and check their orders. Should we find an element of order not contained in $\{1, 2, 3, 4, 5, 8, 10, 20, 40, 41\}$ (the orders of all elements in $\mathrm{L}_2(81)$), then we know $H \ncong \mathrm{L}_2(81)$ and so we discard it. Following this check, we can be reasonably happy that all the groups we have left are isomorphic to $\mathrm{L}_2(81)$ and we can directly check this isomorphism.

After performing these checks on all groups of the form $\langle S, x, t\rangle$ (for all S, t we have found), we find a total of 320 groups isomorphic to $\mathrm{L}_2(81)$ across the two cases $\{1, 3, 4, 6, 7, 8\}$ and $\{2, 3, 4, 6, 7, 8\}$, with each contributing 160 groups towards this total. We remark that no groups were found for representatives x_3 and x_4, while 80 groups were found for each x_1 and x_2 in the two cases.

Non-Maximality

It now remains to find overgroups for each of the 320 groups we have generated and their automorphic extensions. Firstly, for one of our subgroups $H \cong \mathrm{L}_2(81)$, we search for elements of G in some sufficiently small subgroup that along with H generate a group isomorphic to $\mathrm{PGL}_2(81)$. From this, we then construct the entire automorphism group $\mathrm{Aut}(H)$ and then an overgroup containing it. We now explain how we do this in practice.

Recall each H is of the form $\langle S, x, t\rangle$ where x is order 40. We wish to search inside $C_G(x)$ for elements k such that $\langle S, x, t, k\rangle \cong \mathrm{PGL}_2(81)$. From Table 3, we see that $|C_G(x)| = 2^8 \cdot 5^2$. Moreover, we find that $C_G(x) = C_{G(\Phi)}(x)$ where $\Phi = \{1, 2, 3, 4, 6, 7, 8, 97\} \cong 2\mathrm{A}_4$ (this is true for all 320 groups H) and this can be easily calculated using `LMGCentraliser`. Now for each of our 320 groups H, we search randomly in $C_{G(\Phi)}(x)$ for elements k such that $\langle H, k\rangle$ is a suitably small subgroup of G. By suitably small, we mean that the restriction $V|_{\langle H,k\rangle}$ consists of between 2 and 11 composition factors. If $V|_{\langle H,k\rangle}$ has only one composition factor, then it is likely that $\langle H, k\rangle$ is all of G, meaning any subsequent calculations would run indefinitely. After having found some suitable element k, we can quickly check the order of $\langle H, k\rangle$ and compare it to $|\mathrm{PGL}_2(81)| = 2^5 \cdot 3^4 \cdot 5 \cdot 41$. We remark that having equality here does not guarantee an isomorphism, in fact, most of the time there is no isomorphism. Through the generation of lots of groups of this order, we see that $V|_{\langle H,k\rangle}$ typically consists of 9 composition factors. Using this information, we now search randomly for elements $k \in C_{G(\Phi)}(x)$ such that $V|_{\langle H,k\rangle}$ consists of 9 composition factors. This is much preferred to checking the order of our subgroup as calculating composition factors is far quicker. After having found such k, we then check whether $\langle H, k\rangle$ is isomorphic to $\mathrm{PGL}_2(81)$. Using this method, we are able to find a group isomorphic to $\mathrm{PGL}_2(81)$ containing each of our subgroups H very quickly.

From each subgroup isomorphic to $\langle H, k\rangle \cong \mathrm{PGL}_2(81)$, we now wish to construct the full automorphism group $\mathrm{Aut}(H)$. We have that $\mathrm{Out}(H) \cong 2 \times 4$, which implies that $\mathrm{PGL}_2(81)$ can be extended to $\mathrm{Aut}(H)$ by some element of order 4, which we shall label τ. Moreover, as $H \trianglelefteq \mathrm{Aut}(H)$, the cyclic subgroup $\langle x\rangle$ is normalised by all elements of the outer automorphism group. In particular, should $\mathrm{Aut}(H)$ embed as a subgroup of G, then τ can be found inside $N_G(\langle x\rangle)$. We remark that we need not construct all of $N_G(\langle x\rangle)$, we just need a suitably large subgroup containing τ. Instead, we construct $N_{C_G(x^{20})}(\langle x\rangle)$, which may or may not be the entire normaliser $N_G(\langle x\rangle)$. As x^{20} is an involution, the centraliser $C_G(x^{20})$ can be easily constructed using the Bray method [13] and `CentraliserOfInvolution`. Moreover, we have that $x^{20} \in 2\mathrm{B}$, meaning that $C_G(x^{20}) \sim 2.\Omega^+_{16}(3).2$ (see Table 4). This is particularly important as despite the large order of this centraliser, functions such as `LMGNormaliser` work, allowing us to easily find $N_{C_G(x^{20})}(\langle x\rangle)$. Although, this still takes approximately 11 hours of computation time. By the remarks at the end of Section 3.3.2, the normaliser $N_{C_G(x^{20})}(\langle x\rangle)$ only has to be calculated 4 times in order to extend all copies of $\mathrm{PGL}_2(81)$ to $\mathrm{Aut}(H)$. We find that $|N_{C_G(x^{20})}(\langle x\rangle)| = 2^{11}\cdot 5^2$ and upon searching randomly inside this normaliser, we quickly find elements τ that generate the full automorphism group $\mathrm{Aut}(H)$ when paired with the relevant copies of $\mathrm{PGL}_2(81)$. In particular, we now have 320 groups of the form $\langle H, k, \tau\rangle \cong \mathrm{Aut}(H)$ which are overgroups for both the 320 copies of $\mathrm{L}_2(81)$ and the 320 copies of $\mathrm{PGL}_2(81)$ previously constructed.

It now remains to find overgroups for each of these automorphism groups. Thankfully, this is easily and quickly accomplished by taking random elements $h \in N_{C_G(x^{20})}(\langle x\rangle)$ until we find a group $O = \langle H, k, \tau, h\rangle$ such that $V|_O$ has more than 1 composition factor. In all 320 cases, we quickly find overgroups O such that $V|_O$ has 6 composition factors and $|O| = 2^{12}\cdot 3^8\cdot 5^2\cdot 41^2$. Hence we can conclude that any group $H \leq G$ such that $F^*(H) \cong \mathrm{L}_2(81)$ and $F^*(H)$ is associated to feasible decomposition 4, is not maximal in G. The code used to generate all subgroups mentioned here is given in Section 9.2.8.

3.3.3 Feasible Decomposition 1

Now suppose that $V|_H$ has composition series consisting of the factors shown in feasible decomposition 1. We proceed as in the previous section, firstly by identifying which $\Phi \subset \Pi(\mathrm{E}_8)$ are such that $\langle x\rangle$ is $L(\Phi)$-cuspidal.

Lemma 3.14. Suppose that $\langle g\rangle$ is a Levi-cuspidal subgroup of G such that $g \in$ 40DE. Set $\mathfrak{J} = \{\{1,2,3,4,5\},\{2,3,4,5,6\},\{1,2,4,5,7,8\},\{1,2,4,6,7,8\},\{1,2,3,5,6,7\},\{1,2,3,6,7,8\}\}$. Then $\langle g\rangle$ is $L(\Phi)$-cuspidal for some $\Phi \in \mathfrak{J}$. Moreover, in each $L(\Phi)$, there is 1 conjugacy class of cuspidal $\langle g\rangle$-subgroups.

Proof. Let $g \in$ 40DE, then from Table 3, we have $\dim(C_V(g)) = 32$, $g^2 \in$ 20D and $g^5 \in$ 8L. Moreover, from Table 5, we have that the Brauer character of g on V is either α or $\tau(\alpha)$. Hence by Lemma 3.12, we have that $g \notin P(\Phi)$ when $|\Phi| = 3$ and $g^5 \notin P(\Phi)$ when $|\Phi| = 2$. We also remark that if $|\Phi| = 1$, the parabolic subgroups $P(\Phi) \cong \mathrm{SL}_2(3)$ contain no elements of order 8. In particular, $g^5 \notin P(\Phi)$. We proceed as in the proof of Lemma 3.13.

Firstly, suppose $g \in L(\Phi)$ where $L(\Phi)$ is isomorphic to one of $\mathrm{SL}_4(3)$, $\mathrm{L}_5(3)$, $\mathrm{SL}_2(3)\times\mathrm{SL}_4(3)$ or $\mathrm{SL}_6(3)$. From the proof of Lemma 3.13, we know that this forces $\dim(C_V(g)) \in \{24, 26, 48\}$, allowing us to rule out these cases. If $L(\Phi) \sim 2^{2\cdot}\Omega_8^+(3)$, then we can show using Magma that $\dim(C_V(g)) = 32$ or $\dim(C_V(g)) = 48$. When $\dim(C_V(g)) = 32$, we find that the Brauer character of g on V is $9 \neq \alpha, \tau(\alpha)$, allowing us to discount this case.

Now assume $g \in L(\Phi)$ where $|\Phi| = 5$. Consider $L(\Phi) \sim 2^{\cdot}\Omega_{10}^+(3)$. Using Magma, we find there are 11 $L(\Phi)$-classes of C_{40}-subgroups. Of these 11 classes, only 1 has a representative subgroup generated by an element g that satisfies our conditions. As g cannot be conjugated into a smaller Levi-complement, we have that $\langle g\rangle$ is $L(\Phi)$-cuspidal for $\Phi \in \mathfrak{J} = \{\{1,2,3,4,5\},\{2,3,4,5,6\}\}$.

Suppose $L(\Phi) \cong \mathrm{SL}_2(3)\times\mathrm{L}_5(3)$. Then by Lemma 3.12, we must have that $g^5 \in \mathrm{L}_5(3)$ as we have shown g itself cannot live inside $\mathrm{L}_5(3)$. However, $\mathrm{L}_5(3)$ contains nine classes of elements of order 8 and none of them fuse into 8L. Now let $L(\Phi) \cong \mathrm{L}_3(3)\times\mathrm{SL}_4(3)$ or $L(\Phi) \cong \mathrm{SL}_2(3)^2\times\mathrm{SL}_4(3)$. As $g \notin \mathrm{SL}_4(3)$ and both $\mathrm{SL}_2(3)$, $\mathrm{L}_3(3)$ contain no elements of order 20, we have $g^2 \in \mathrm{SL}_4(3)$. However, this gives $\dim(C_V(g^2)) = 48$, a contradiction.

For the final case of $|\Phi| = 5$, assume $L(\Phi) \sim \mathrm{SL}_2(3)\times 2^{2\cdot}\Omega_8^+(3)$. We know that $g \notin 2^{2\cdot}\Omega_8^+(3)$, hence it must be that $g^5 \in 2^{2\cdot}\Omega_8^+(3)$. Consequently, this gives $\dim(C_V(g^5)) \in \{48, 52, 68, 80\}$, giving us a contradiction.

Now suppose $|\Phi| = 6$. We remark that, due to computing limitations, we cannot apply the previous methods to certain parabolic subgroups for $|\Phi| \geq 6$. This is because we are unable to calculate conjugacy class representatives inside the 248-dimension representation using `LMGClasses`. Consequently, we consider the groups $E_6(3)$, $2^2.\Omega_{12}^+(3)$, $2.\Omega_{14}^+(3)$, $E_6(3) \times SL_2(3)$, $2.E_7(3)$ separately. Note that this does not change the statement of this Lemma. Indeed, showing these groups intersect 40DE non-trivially will not change the fact that $\langle g \rangle$ is $L(\Phi)$-cuspidal for $\Phi \in \mathfrak{J}$.

Suppose $L(\Phi) \sim 2^{\cdot}\Omega_{10}^+(3) \times SL_2(3)$. If $g \in 2^{\cdot}\Omega_{10}^+(3)$ then $\langle g \rangle$ is not $L(\Phi)$ cuspidal, hence $g \notin 2^{\cdot}\Omega_{10}^+(3)$. Through direct calculation in Magma, we obtain all $L(\Phi)$-conjugacy class representatives for C_{40}-subgroups, of which there are 33. Of these, only one has a representative generator that satisfies our conditions. Therefore, this class must intersect the smaller Levi-complement $2^{\cdot}\Omega_{10}^+(3)$, meaning $\langle g \rangle$ is not $L(\Phi)$-cuspidal here.

Let $L(\Phi) \cong SL_2(3) \times SL_4(3) \times L_3(3)$. We find that there are 24 $L(\Phi)$-classes of C_{40}-subgroups, of which 2 have a representative generator with $\dim(C_V(g)) = 32$. One of these has Brauer character on V equal to 9, so we may discount this class. However, the remaining class does satisfy our conditions on $\langle g \rangle$. As we cannot conjugate $\langle g \rangle$ into a smaller Levi-compliment, we have that $\langle g \rangle$ is $L(\Phi)$-cuspidal for $\Phi \in \mathfrak{J} = \{\{1,2,4,5,7,8\},\{1,2,4,6,7,8\},\{1,2,3,5,6,7\},\{1,2,3,6,7,8\}\}$. Moreover, g is of the form $(2A_{SL_2(3)}, 20A_{SL_4(3)}, 8A_{L_3(3)})$ where $2A_{SL_2(3)}$ denotes an element from the 2A class of the $SL_2(3)$ component of $L(\Phi)$.

If $L(\Phi) \cong SL_4(3)^2$, then by following the same argument for the case $L(\Phi) \cong L_3(3) \times SL_4(3)$, we get $g^2 \in SL_4(3)$ which we have shown cannot happen. If $L(\Phi) \sim SL_2(3)^2 \times 2^{2\cdot}\Omega_8^+(3)$, then as in the case $L(\Phi) \sim SL_2(3) \times 2^{2\cdot}\Omega_8^+(3)$, we get $\dim(C_V(g^5)) \in \{48, 52, 68, 80\}$. Similarly, if $L(\Phi) \sim L_3(3) \times 2^{2\cdot}\Omega_8^+(3)$, from Lemma 3.13 we know that $g^5 \notin L_3(3)$. Therefore $g^5 \in 2^{2\cdot}\Omega_8^+(3)$, which we have shown is not possible.

Let $L(\Phi) \cong SL_2(3) \times SL_6(3)$. Through direct calculation, we find one $L(\Phi)$-conjugacy class of C_{40}-subgroups such that $\dim(C_V(g)) = 32$. However, the Brauer character of g on V is $9 \neq \alpha, \tau(\alpha)$. So no suitable elements are found here. Assume $L(\Phi) \cong SL_2(3)^2 \times L_5(3)$. Following the argument used in the case $L(\Phi) \cong SL_2(3) \times L_5(3)$, we see that no suitable elements are to be found here either.

Suppose $L(\Phi) \cong L_7(3)$. Through direct calculation, we find that there are 8 C_{40}-classes and none of these contain generators such that $\dim(C_V(g)) = 32$. For the final case of $|\Phi| = 6$, suppose $L(\Phi) \cong L_3(3) \times L_5(3)$. From Lemma 3.13, we see that the $L_3(3)$ component does not contain any 8L elements, meaning that we must have $g^5 \in L_5(3)$, which we have already shown cannot happen.

We now consider some of the maximal parabolic subgroups of G. Assume $L(\Phi) \cong SL_8(3)$. Through direct calculation, we find that there are 90 $L(\Phi)$-conjugacy classes of elements of order 40. However, none of these classes intersect 40DE, so no suitable elements are found here. We remark that here it was easier to work with conjugacy classes of elements rather than cyclic subgroups due to the computation time associated with calculating subgroup classes.

Suppose $L(\Phi)$ is isomorphic to one of $SL_2(3) \times L_7(3)$, $L_3(3) \times SL_2(3) \times L_5(3)$, $L_5(3) \times SL_4(3)$. Respectively, we find 2, 2, 3 $L(\Phi)$-classes of C_{40}-subgroups and in each $L(\Phi)$, only 1 of these classes intersects 40DE non-trivially. As $L_3(3) \times SL_2(3) \times SL_4(3) \leq L(\Phi)$, it must be that $\langle g \rangle$ intersects $L(\Phi')$ for some $\Phi' \in \{\{1,2,4,5,7,8\},\{1,2,4,6,7,8\},\{1,2,3,5,6,7\},\{1,2,3,6,7,8\}\}$. Hence $\langle g \rangle$ is not $L(\Phi)$ cuspidal here.

Assume $L(\Phi) \sim 2^{\cdot}\Omega_{10}^+(3) \times L_3(3)$. We find that there are 6 $L(\Phi)$-classes of C_{40}-subgroups, of which 2 lie inside 40DE. Recall that both $L(\{1,2,3,4,5\})$ and $L(\{1,2,4,5,7,8\})$ contain suitable C_{40}-subgroups. Calculation inside $L(\Phi)$ shows that these two subgroups are not $L(\Phi)$-conjugate. Hence the 2 suitable classes we have found in $L(\Phi)$ must non-trivially intersect the smaller Levi-complements $L(\{1,2,3,4,5\})$ and $L(\{1,2,4,5,7,8\})$, allowing us to discount this case. □

We have now established some of the standard parabolic subgroups $P(\Phi)$ such that $\langle x\rangle$ is $L(\Phi)$-cuspidal. It is in these parabolic subgroups that we wish to search for groups of the form $S : x = N_H(S) = N$. However, as remarked in the proof of Lemma 3.14, due to computational limits we were not able to consider some of the parabolic subgroups $P(\Phi)$ where $|\Phi| \geq 6$. We cannot apply our previous methods to these cases, and so we consider them separately in the following result.

Lemma 3.15. Let $P(\Phi)$ be a parabolic subgroup of G such that $L(\Phi)$ is isomorphic to one of $\mathrm{E}_6(3)$, $2^2.\Omega^+_{12}(3)$, $2.\Omega^+_{14}(3)$, $\mathrm{E}_6(3)\times\mathrm{SL}_2(3)$, $2.\mathrm{E}_7(3)$. Suppose that $\langle S,x\rangle = N \leq P(\Phi)$. Then $\dim(C_V(N)) > 0$.

Proof. We use the following method for all groups considered in this Lemma. Firstly, we wish to find representatives for all $L(\Phi)$-conjugacy classes of cyclic subgroups of order 40 that intersect 40DE non-trivially. As remarked earlier, using `LMGClasses` is out of the question due to the orders of these groups. Hence, we must resort to other methods in order to obtain these representatives. After having found such representatives, we then use Code 9.2.3 and 9.2.6 to start the process of constructing groups of the form $\langle S,x\rangle$ where x is one of our representatives and $S \cong 3^4$. Due to the large number of trivial factors in the feasible decomposition, it is highly likely that we need only run the first part of the algorithm in order to find fixed vectors and thus finish the proof.

Firstly suppose $L(\Phi) \cong \mathrm{E}_6(3)$. Using Magma and Frank Lübeck's database [58], we find that $L(\Phi)$ has 2 conjugacy classes of involutions. We wish to find representatives for all conjugacy classes intersecting 40DE non-trivially inside the centralisers corresponding to these 2 classes of involutions. As every element of order 40 must fuse into one of these 2 classes of involutions, finding all conjugacy classes of elements order 40 in each centraliser of involution will give us at least 1 representative from each conjugacy class of elements order 40 in the larger group $L(\Phi)$. Both classes of involutions in G intersect $L(\Phi)$ non-trivially, meaning it is easy to find non-conjugate involutions inside $L(\Phi)$. Suppose that $t_1 \in L(\Phi)\cap 2\mathrm{A}$, $t_2 \in L(\Phi)\cap 2\mathrm{B}$. Using `CentraliserOfInvolution` and then `LMGClasses`, we obtain both their centralisers and a complete list of conjugacy class representatives. We find that $C_{L(\Phi)}(t_1) \sim \mathrm{SL}_2(3).(\mathrm{L}_6(3):2)$ and $C_{L(\Phi)}(t_2) \sim 2.(\Omega^+_{10}(3):2)$. Inside $C_{L(\Phi)}(t_1)$ and $C_{L(\Phi)}(t_2)$ we find 4 and 2 conjugacy classes respectively that intersect 40DE. We are interested in the classes of cyclic subgroups, so we check to see whether any of these representatives generate the same cyclic group. After this check, we have 5 cases to deal with. We now perform the first part of the algorithm using Code 9.2.3 on each of these 5 representatives (so we set `x40` to be one of these representatives and then run the code 5 times). The results of this are discussed at the end of this proof.

Now let $L(\Phi) \sim 2^2.\Omega^+_{12}(3)$. This time, we shall search inside centralisers of elements order 5 for our 40DE elements. Let $P \in \mathrm{Syl}_5(L(\Phi))$. We know that $P \cong 5^2$ and so the fusion of elements order 5 inside $L(\Phi)$ is controlled by $N_{L(\Phi)}(P)$. By this, we mean that if $N_{L(\Phi)}(P)$ has n conjugacy classes of order 5 elements, then $L(\Phi)$ also has n conjugacy classes of order 5 elements. Using `LMGRadicalQuotient` on $L(\Phi)$, we obtain $L(\Phi)/Z(L(\Phi)) \cong \Omega^+_{12}(3)$ as a permutation group and a corresponding quotient map $\psi : L(\Phi) \mapsto L(\Phi)/Z(L(\Phi))$. Through direct calculation inside this permutation representation, we find that $N_{L(\Phi)/Z(L(\Phi))}(P)$ has 2 conjugacy classes of elements order 5. Consequently, the same holds for $L(\Phi)$. Finding non-conjugate elements of order 5 in $L(\Phi)$ inside the 248-dimensional representation is more tricky than the situation we had with $\mathrm{E}_6(3)$ as $5\mathrm{B}\cap L(\Phi) = \emptyset$. However, we are able to obtain these representatives by taking the pre-images under ψ of class representatives from $N_{L(\Phi)/Z(L(\Phi))}(P)$. Now working inside the

248 dimensional representation again, we find the centralisers of both these elements and then the conjugacy classes of both these centralisers. Overall, we find 28 representatives of classes from these centralisers that live inside 40DE. From these 28 elements, we find 20 unique cyclic subgroups of order 40. As before, we now run Code 9.2.3 on each of these.

Let $L(\Phi) \sim 2.\Omega_{14}^{+}(3)$. We follow a very similar approach to $2^2.\Omega_{12}^{+}(3)$. This time, we have that $P \in \mathrm{Syl}_5(L(\Phi))$ has structure $P \cong 5^3$ and as before $5B \cap L(\Phi) = \emptyset$. Using `LMGRadicalQuotient` on $L(\Phi)$, we obtain $L(\Phi)/Z(L(\Phi)) \cong \Omega_{14}^{+}(3)$ as a permutation group. We find that both $N_{L(\Phi)/Z(L(\Phi))}(P)$ and $L(\Phi)$ have 3 conjugacy classes of elements order 5. We obtain representatives for these by taking pre-images of representatives from $N_{L(\Phi)/Z(L(\Phi))}(P)$ under the quotient map. After finding the centralisers of all these elements and the conjugacy classes of each centraliser, we find 18 elements of 40DE that we must consider. After generating cyclic subgroups, we are left with 8 cases.

Next suppose $L(\Phi) \cong \mathrm{E}_6(3) \times \mathrm{SL}_2(3)$. As $\mathrm{SL}_2(3)$ does not contain any elements of order 5, it must be that the 8^{th} power of any element of order 40 from $L(\Phi)$ lives inside the $\mathrm{E}_6(3)$ component. It is well known that $\mathrm{E}_6(3)$ contains $3^6 = 729$ conjugacy classes of semi-simple elements. Moreover, conjugate matrices share the same characteristic polynomial. By working inside the 27-dimensional representation for $\mathrm{E}_6(3)$, we are able to find representatives for all 729 conjugacy classes of semisimple elements simply by considering characteristic polynomials. Indeed, using Magma, we generate random semisimple elements and calculate their characteristic polynomial. If we find 2 elements of the same order with distinct characteristic polynomials, we know they are non-conjugate and so we collect them in a set. By repeatedly using this method to build up this set, we are able to obtain all 729 representatives. We find that there is only 1 conjugacy class of elements of order 5 in $\mathrm{E}_6(3)$; we shall let g denote a representative of this class. We now wish to construct $C_{L(\Phi)}(g)$. We are not able to use `LMGCentraliser` here, however, we may use the structure of $L(\Phi)$ to help us find $C_{L(\Phi)}(g)$. As $L(\Phi)$ is a direct product, we have that g is of the form (h_1, h_2) where $h_1 \in \mathrm{E}_6(3)$ and $h_2 \in \mathrm{SL}_2(3)$. As $\mathrm{SL}_2(3)$ contains no elements of order 5, we have that $h_2 = 1$ and $h_1^5 = 1$. Also, we have that $C_{L(\Phi)}(g) \cong C_{\mathrm{E}_6(3)}(h_1) \times C_{\mathrm{SL}_2(3)}(h_2)$. As h_2 is the identity, we must have that g commutes with all elements in the $\mathrm{SL}_2(3)$ component of $L(\Phi)$, and this can be quickly verified on Magma. So now it remains to find $C_{\mathrm{E}_6(3)}(h_1)$. We remark that h_1 is the projection of g onto $\mathrm{E}_6(3)$, and so we identify g with h_1 from now on. Moreover, as we are now working entirely inside $L(\{1,2,3,4,5,6\}) \cong \mathrm{E}_6(3)$, we let $\Phi = \{1,2,3,4,5,6\} \cong E_6$. Using Frank Lübeck's database [58], we are able to obtain a complete list of all $3^6 = 729$ centraliser orders of semisimple elements inside $\mathrm{E}_6(3)$. As $L(\Phi)$ has only 1 class of elements of order 5, we may assume without loss that $g \in L(\Phi_1) \leq L(\Phi)$ where $\Phi_1 = \{1,3,4\} \cong \mathrm{A}_3$. Now set $\Phi_2 = \Phi_1 \cup \{2,5\} \cong \mathrm{D}_5$ and $\Phi_3 = \Phi_1 \cup \{5,6\} \cong \mathrm{A}_5$. Then $C = \langle C_{L(\Phi_2)}(g), C_{L(\Phi_3)}(g) \rangle \leq C_{L(\Phi)}(g)$. We now consider our list of known centraliser orders, removing the orders for the centralisers of involutions and also the centraliser of the identity. Searching through the remaining 726 orders, we find only 1 that is divisible by $|C|$, which is $|C|$ itself. Hence $C = C_{L(\Phi)}(g) \sim 20.(\mathrm{U}_4(3):4)$. Moving back to $L(\Phi) \cong \mathrm{E}_6(3) \times \mathrm{SL}_2(3)$, we have that $C_{L(\Phi)}(g) \cong C \times \mathrm{SL}_2(3)$. Having finally obtained the centraliser corresponding to the single class of order 5 elements inside $L(\Phi)$, we now find all the conjugacy classes of elements order 40 that intersect 40DE. We find 8 such classes and from these representatives, there are only 2 unique cyclic subgroups. We are now ready to use the code in Section 9.2.3.

Finally, suppose $L(\Phi) \cong \mathrm{E}_7(3)_{sc} \sim 2.\mathrm{E}_7(3)$. In a perfect world, we could proceed exactly as we did with $\mathrm{E}_6(3)$ and consider the 3 classes of involutions in order to find our elements of order 40. This is ideal as computing centralisers of involutions is extremely quick and does not require any effort. Unfortunately, $L(\Phi)$ has a cyclic center of order 2, meaning that this approach is not possible. Instead, we follow a similar approach to $\mathrm{E}_6(3) \times \mathrm{SL}_2(3)$ and consider elements of order 5. Luckily, as was the case with $\mathrm{E}_6(3) \times \mathrm{SL}_2(3)$,

5^2 is the largest power of 5 dividing $|L(\Phi)|$, meaning that we can conjugate any element of order 5 from $L(\Phi)$ into $E_6(3)$. Consequently, $L(\Phi)$ has only one conjugacy class of elements order 5. We wish to find the centraliser corresponding to this class, which means we must consult the database [58] again. This time, we have $3^7 = 2187$ semisimple classes to contend with and obtaining centraliser orders for all of these classes takes hours of looking through the files on [58]. Eventually, we are able to obtain a complete list of all 2187 centraliser orders. We remove the orders of centralisers for involutions and the identity as we know that these cannot correspond to our class of order 5 elements. This leaves 2183 orders to consider. Let $t \in L(\Phi) \cap 2B$ and let $C = C_{L(\Phi)}(t)$. Now let $g \in C$ be order 5. Then $C_C(g) \sim (10 \times SL_2(3)).(\Omega_8^-(3).2)$. As remarked earlier, for $P \in Syl_5(L(\Phi))$, we have $P \cap 5B = \emptyset$, hence $g \in 5A$. Moreover, we have that $|C_{L(\Phi)}(g)|$ divides $|C_G(g)|$, which is given in Table 3. Upon searching the list of 2183 centraliser orders for orders that divide $|C_G(g)|$ but are also divisible by $|C_C(g)|$, we find only 1 possibility. This is $|C_C(g)|$ itself, hence $C_C(g) \cong C_{L(\Phi)}(g)$. We now find all conjugacy class representatives of $C_C(g)$ using `LMGClasses` and search for all classes intersecting 40DE. There are 28 such class representatives, of which 12 generate unique cyclic subgroups. We are now able to utilise the code in Section 9.2.3.

Let b be any 3-group obtained from running Code 9.2.3 and let $x \in 40DE$ be the corresponding element of order 40 that was input into the program. Then for all groups considered here, and all such x, we have that $\dim(C_V(\langle b, x\rangle)) > 0$. Hence, we have that $0 < \dim(C_V(\langle b, x\rangle)) \leq \dim(C_V(\langle S, x\rangle)) = \dim(C_V(N))$ as required. □

Corollary 3.16. Let $H < G$ such that $F^*(H) = \langle N, t\rangle \cong L_2(81)$ and suppose $V|_H$ has composition series associated to feasible decomposition 1. Moreover, assume that $N < P(\Phi)$ where $L(\Phi)$ is isomorphic to one of $E_6(3)$, $2^2.\Omega_{12}^+(3)$, $2.\Omega_{14}^+(3)$, $E_6(3) \times SL_2(3)$, $2.E_7(3)$ Then H is not maximal in G.

Proof. By Lemma 3.15, we have that $\dim(C_V(N)) > 0$. The result now follows from Lemma 3.7, Lemma 2.5 and the fact that the multiplicity of the Steinberg module of $F^*(H)$ over K is 0 in the feasible decomposition. □

Code Outcomes and Non-Maximality

We now consider the outcome of running the code on the groups and representatives identified in Lemma 3.14. First we consider the 2 cases when $\Phi \cong D_5$. We know from Lemma 3.14 that we only have 1 C_{40}-class in each $L(\Phi)$, hence it suffices to choose any $x \in L(\Phi) \cap 40DE$. Table 14 shows the initial sizes of the sets `FinSub`, `BadSub` and `ActnGpDiff` after running Code 9.2.3.

Table 14: The outcome of running Code 9.2.3 for $\Phi \cong D_5$.

	Φ	
	{1,2,3,4,5}	{2,3,4,5,6}
FinSub	11	13
BadSub	4	4
ActnGpDiff	15	15

Let b be any 3-group contained in any set `FinSub`, `BadSub` or `ActnGpDiff` from Table 14 and let x denote the corresponding element of order 40. Then $\dim(C_V(\langle b, x\rangle)) > 0$. Hence by Lemma 3.7 and Lemma 2.5, any group $H = \langle S, x, t\rangle$ (and any subsequent automorphic extension) constructed from an elementary abelian group $S \leq b$ will not be maximal in G.

Now we consider the remaining four cases, when $\Phi \cong \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3$. As in the previous case, we have only 1 C_{40}-class in $L(\Phi)$, so we choose any $x \in L(\Phi) \cap 40\mathrm{DE}$. Table 15 shows the sizes of the sets `FinSub`, `BadSub` and `ActnGpDiff` after the first stage of the algorithm.

Table 15: The outcome of running the code in Section 9.2.3 for $\Phi \cong \mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3$.

	Φ			
	{1,2,4,5,7,8}	{1,2,4,6,7,8}	{1,2,3,5,6,7}	{1,2,3,6,7,8}
FinSub	7	7	9	5
BadSub	5	5	3	3
ActnGpDiff	8	9	14	7

As before, let b be any 3-group contained in any set `FinSub`, `BadSub` or `ActnGpDiff` from Table 15 and let x denote the corresponding element of order 40. Then $\dim(C_V(\langle b, x\rangle)) > 0$. Hence by the previous argument, any H with $F^*(H) \cong \mathrm{L}_2(81)$ constructed from these groups will not be maximal in G. This, along with all other work conducted in this section, allows us to conclude the following.

Corollary 3.17. Let $H < G$ with $F^*(H) = \langle N, t\rangle \cong \mathrm{L}_2(81)$. Then H is not maximal in G.

3.4 $\mathrm{L}_2(27)$

Using the methods described in Sections 3.2 and 3.3, we now attempt to establish whether subgroups $H \leq G$ with $F^*(H) \cong \mathrm{L}_2(27)$ can embed maximally in G. The Brauer character table and feasible decompositions for $\mathrm{L}_2(27)$ are given in Section 8.2. Using Lemma 2.7, we have that any group $H \cong \mathrm{L}_2(27)$ constructed inside G with composition series associated to either feasible decomposition 1 or 2 will be such that $C_V(H) \neq \{0\}$. Hence in these cases, by Lemma 2.5, H or any automorphic extension of H is not maximal in G. It remains to investigate the groups H associated with feasible decomposition 3. In this section we present the progress made on this so far, and detail the work yet to be done.

3.4.1 Methodology

Let $H \cong \mathrm{L}_2(27)$, $S \in \mathrm{Syl}_3(H)$ and $N = N_H(S)$. By Lemma 3.3, we have that $S \cong 3^3$ and $H = \langle S, x, t\rangle$ where $x \in N_H(S)$ is order 13 acting irreducibly on S and t is an involution inverting x. The Steinberg module of H over K has dimension 27 and is denoted by φ_5 in Section 8.2. We see that this module appears with multiplicity 1 inside feasible decomposition 3. Hence by Lemma 3.7 and Lemma 2.5, it suffices to show $\dim(C_V(N)) > 1$ in order to disprove the maximality of H in G. We follow the same method as with $\mathrm{L}_2(2187)$ and $\mathrm{L}_2(81)$, making some adaptations to the code in Section 9.2.3 to make it compatible with the different field size (as remarked in Section 3.3). Unlike with $\mathrm{L}_2(2187)$ and $\mathrm{L}_2(81)$, the sets `BadSub` and `BadSetNew` we now encounter could potentially be very large. As such, reducing the number of groups in these sets could save us countless hours of computation time. The following result allows us to do just that.

Lemma 3.18. Let $b_1, b_2 < G$ be 3-subgroups of G normalised by an element $x \in G$ and suppose that $b_1^g = b_2$ for some $g \in C_G(x)$. Then, if t_1 and t_2 are involutions inverting x and $S_i \leq b_i$ $(i = 1, 2)$ are elementary abelian 3-groups upon which x acts irreducibly, we have that $\langle S_1, x, t_1\rangle$ is G-conjugate to $\langle S_2, x, t_2\rangle$.

Proof. We have that g and g^{-1} both work as the conjugating element. Also, we remark that $C^*_G(x)^g = C^*_G(x)$. □

Given a set of groups `BadSub`, using Lemma 3.18 we are able to reduce the size of the set by considering the groups upto conjugacy in $C_G(x)$. In practice, we consider the groups upto conjugacy in the smaller centraliser $C_{L(\Phi)}(x)$ (where x is $L(\Phi)$-cuspidal), testing showed this was far more efficient. The Magma code used to check for this conjugacy is contained in Section 9.2.5 and it works as follows.

Suppose we have a set `BadSub` (or `BadSetNew`) of size n containing the 3-groups b_1, b_2, ..., b_n. Given a random element $h \in C_{L(\Phi)}(x)$, we check whether $b_1^h = b_i$ for all $i \in \{2, \dots, n\}$. If we have equality, then we store b_1 in a separate set and discard b_i; the group b_i will not be considered from here on. Then, we find a new random element $s \in C_{L(\Phi)}(x)$ and start checking whether $b_1^s = b_j$ for all $j \in \{2, \dots, i-1, i+1, \dots, n\}$, repeating the process if we find equality again. We remark that if $b_1^h = b_i$ for some i, then we don't need to check whether $b_1^h = b_m$ for all $m \geq i$ as all groups b_i are distinct, meaning $b_1^h \neq b_m$ for all $m \neq i$. If we find that $b_1^h \neq b_i$ for $i \in \{2, \dots, n\}$, then we find a different element $s \in C_{L(\Phi)}(x)$ and start looking for equality again, repeating the process above if we do. Overall, we choose 45 random elements $h \in C_{L(\Phi)}(x)$ and check whether $b_1^h = b_i$ for $i \in \{2, \dots, n\}$ for all of these elements. After all 45 elements have been considered, we add b_1 to our set of representatives and consider the next group in the set b_2, assuming that b_2 hasn't already been discarded. We then repeat this entire process but with b_2 in place of b_1. We emphasise that we may have discarded many groups by this stage, and consequently these groups are not considered for the rest of the algorithm (or indeed the rest of the construction of H entirely). Hence, if equality is found, the set `BadSub` decreases in size and as a result the code speeds up. Lemma 3.18 and Code 9.2.5 are used frequently throughout Sections 3.4 and 6. In many cases, no conjugacy relations between the groups in `BadSub` are found. We do not mention every time Lemma 3.18 was used and no conjugacy was found, but this does not mean it was not considered. When we are able to find conjugacy, the size of `BadSub` is often greatly reduced, often removing hundreds of groups and saving potentially weeks of computation time.

Before we begin to look at the results of running Code 9.2.3 and 9.2.6, we first establish which parabolic subgroups must be considered. From the fusion possibility for feasible decomposition 3, we have that $x \in$ 13F. By Table 3, we have $\dim(C_V(x)) = 20$ and the Brauer character value of x on V is 1. Table 16 lists all isomorphism types for $L(\Phi)$ which contain elements of order 13. As with $L_2(81)$, we now consider all of these in order to establish whether they intersect the class 13F non-trivially.

Table 16: Levi-complements of standard parabolic subgroups containing elements of order 13.

$\|\Phi\|$	Structure of $L(\Phi)$
2	$L_3(3)$
3	$SL_2(3) \times L_3(3)$, $SL_4(3)$
4	$L_3(3)^2$, $SL_2(3)^2 \times L_3(3)$, $L_5(3)$, $SL_2(3) \times SL_4(3)$, $2^{2\cdot}\Omega_8^+(3)$
5	$L_3(3) \times SL_2(3)^3$, $L_3(3)^2 \times SL_2(3)$, $2^{\cdot}\Omega_{10}^+(3)$, $SL_2(3) \times L_5(3)$, $L_3(3) \times SL_4(3)$, $SL_6(3)$, $SL_2(3) \times 2^{2\cdot}\Omega_8^+(3)$, $SL_2(3)^2 \times SL_4(3)$
6	$2^{\cdot}\Omega_{10}^+(3) \times SL_2(3)$, $L_3(3) \times SL_2(3) \times SL_4(3)$, $SL_4(3)^2$, $E_6(3)$, $2^2.\Omega_{12}^+(3)$, $L_3(3) \times 2^{2\cdot}\Omega_8^+(3)$, $SL_2(3) \times SL_6(3)$, $SL_2(3)^2 \times L_5(3)$, $L_7(3)$, $L_3(3) \times L_5(3)$
7	$2.\Omega_{14}^+(3)$, $SL_8(3)$, $SL_2(3) \times L_7(3)$, $L_3(3) \times SL_2(3) \times L_5(3)$, $L_5(3) \times SL_4(3)$, $2^{\cdot}\Omega_{10}^+(3) \times L_3(3)$, $E_6(3) \times SL_2(3)$, $2.E_7(3)$

Lemma 3.19. Set

$$\mathfrak{J} = \{\{1,3,5,6\}, \{1,3,6,7\}, \{1,3,7,8\}, \{2,4,6,7\}, \{2,4,7,8\}, \{3,4,6,7\}, \{3,4,7,8\}, \{4,5,7,8\}\}$$

and let $g \in L(\Phi) \cap 13F$ for any $\Phi \in \mathfrak{J}$. Then $L(\Phi) \cong L_3(3)^2$ for all $\Phi \in \mathfrak{J}$ and $\langle g \rangle$ is $L(\Phi)$-cuspidal. Furthermore, for $\Phi = \{1,2,3,4,5,6\} \cong E_6$, we have that $P \in \mathrm{Syl}_{13}(L(\Phi))$ is elementary abelian of order 13^3 and P contains 864 elements belonging to the class 13F.

Proof. We have that $\langle g \rangle \cong C_{13}$ and $g \in 13F$. We now consider each isomorphism type of Levi-subgroup given in Table 16.

We are able to rule out a lot of groups listed in Table 16 by looking at how the conjugacy classes of elements of order 13 fuse into G. Suppose $L(\Phi)$ has structure $L_3(3)$, $SL_2(3) \times L_3(3)$, $SL_4(3)$, $SL_2(3)^2 \times L_3(3)$, $L_5(3)$, $SL_2(3) \times SL_4(3)$, $2^{2\cdot}\Omega_8^+(3)$, $2^{\cdot}\Omega_{10}^+(3)$, $SL_2(3) \times L_5(3)$, $L_3(3) \times SL_2(3)^3$, $SL_2(3)^2 \times SL_4(3)$, $SL_2(3) \times 2^{2\cdot}\Omega_8^+(3)$, $2^{\cdot}\Omega_{10}^+(3) \times SL_2(3)$, $SL_2(3)^2 \times L_5(3)$ and $g \in L(\Phi)$ is order 13. Then $g \in 13AB$, hence none of these groups are of interest to us here.

We now consider the remaining groups. Suppose $L(\Phi) \cong L_3(3)^2$. We find that $L(\Phi)$ has 6 classes of C_{13}-subgroups and of these classes, 2 have a representative of the form $\langle g \rangle$ where $g \in 13F$. As we cannot conjugate these subgroups into any smaller parabolic subgroups, we have that $\langle g \rangle$ is $L(\Phi)$-cuspidal for all $\Phi \in \mathfrak{J}$. Using Sylow's theorem, we are able to use this fact to rule out many larger groups in Table 16. Indeed, in general, suppose $K \leq G$ and $P \in \mathrm{Syl}_p(K)$ such that P is also a Sylow p-subgroup of G. Then by Sylow's theorem, we may conjugate any cyclic group $\langle g \rangle \leq G$ of order p into K. For $P \in \mathrm{Syl}_{13}(L(\Phi))$, we have $P \cong 13^2$. Hence we can rule out any group K in Table 16 such that $L(\Phi) < K$ and for $P \in \mathrm{Syl}_{13}(K)$, we have $P \cong 13^2$. This allows us to rule out the following isomorphism types: $L_3(3)^2 \times SL_2(3)$, $L_3(3) \times SL_4(3)$, $SL_6(3)$, $L_3(3) \times SL_2(3) \times SL_4(3)$, $SL_4(3)^2$, $2^2.\Omega_{12}^+(3)$, $L_3(3) \times 2^{2\cdot}\Omega_8^+(3)$, $SL_2(3) \times SL_6(3)$, $L_3(3) \times L_5(3)$, $2.\Omega_{14}^+(3)$, $SL_8(3)$, $L_7(3)$, $SL_2(3) \times L_7(3)$, $L_3(3) \times SL_2(3) \times L_5(3)$, $L_5(3) \times SL_4(3)$, $2^{\cdot}\Omega_{10}^+(3) \times L_3(3)$.

We may apply the same idea for $E_6(3)$. Let $\Phi = \{1,2,3,4,5,6\} \cong E_6$, then $L(\Phi) \cong E_6(3)$ and $P \in \mathrm{Syl}_{13}(L(\Phi))$ has structure $P \cong 13^3$. By the previous argument, we can rule out any group in Table 16 that contains $L(\Phi)$ and has an elementary abelian Sylow 13-subgroup of order 13^3. Therefore, we need not consider isomorphism types $E_6(3) \times SL_2(3)$ and $2.E_7(3)$. This leaves us with only $E_6(3)$ left to consider.

Due to the size of $L(\Phi)$ and its structure, we are unable to gather a complete list of representatives for the $L(\Phi)$-classes of elements of order 13. However, by Sylow's theorem, we know that P must contain all the representatives we need to consider. Unfortunately, again due to the size and structure of $L(\Phi)$, we are unable to use the function `LMGSylow(L,13)` to directly obtain P inside the 248-dimensional representation. Although, through the use of `FindCent`, we can obtain P from a centraliser of any element of order 13 inside $L(\Phi)$ (as P is abelian). We remark that we need not construct the entire centraliser in $L(\Phi)$, we stop the code when the centraliser we are constructing has order divisible by 13^3. Having constructed a subgroup K such that $P \leq K < L(\Phi)$ (and crucially $|K|$ is much smaller than $|L(\Phi)|$), we are able to use `LMGSylow(K,13)` in order to obtain P. Inside P, we search for all elements that have a fixed space of dimension 20. Of the 13^3 elements, 864 satisfy this requirement. □

Lemma 3.19 gives us 9 cases we must consider, which may be divided into $\Phi \cong 2A_2$ or $\Phi \cong E_6$. We consider each of these Dynkin types separately.

3.4.2 $\Phi = \{1, 2, 3, 4, 5, 6\} \cong E_6$

Suppose $\Phi = \{1, 2, 3, 4, 5, 6\} \cong E_6$ and let $P \in \mathrm{Syl}_{13}(L(\Phi))$. By Lemma 3.19 we have that $P \cong 13^3$ and $|P \cap 13F| = 864$. As our interest lies in the $L(\Phi)$-classes of C_{13}-subgroups, it suffices to consider only the elements that generate unique cyclic subgroups. We find that this reduces the number of cases down from 864 to 72. On Magma, we gather our 72 elements and loop through them, running Code 9.2.3 on each one. In particular, before each iteration `x13` is set to be one of our 72 elements and we create new sets `FinSub`, `BadSub` and `ActnGpDiff` after each loop. After these 72 iterations, we have added 381 groups to `FinSub` and 288 groups to `BadSub`; the set `ActnGpDiff` remained empty throughout. If b denotes any of these groups, then $\dim(C_V(\langle b, x\rangle)) \geq 2$ where x denotes the corresponding element of order 13 used to obtain b. Hence, by Lemma 3.7, any H constructed from such groups would be such that $C_V(H) \neq \{0\}$ and consequently not maximal in G by Lemma 2.5.

Due to the size and structure of $L(\Phi) \cong E_6(3)$, we were unable to directly determine the number $L(\Phi)$-classes of $\langle x\rangle$-subgroups and also whether or not these classes contain $L(\Phi)$-cuspidal subgroups. However, by Sylow's theorem, we know that P must contain all the representatives we need, hence the above method is sufficient.

3.4.3 $\Phi \cong 2A_2$

We now consider the 8 cases listed in Lemma 3.19 where $\Phi \cong 2A_2$. From Lemma 3.19, we know that each $L(\Phi) \cong L_3(3)^2$ has two classes of $\langle x\rangle$-subgroups; these representatives must be considered separately. Throughout this section and the following subsections, we let x_1 and x_2 denote representative elements from the two classes of $\langle x\rangle$-subgroups. In each case, we run Code 9.2.3 and 9.2.6, starting with Code 9.2.3 on both x_1, x_2. Following this, we must deal with the outputted `BadSub`; this is where the vast majority of the work is undertaken. The results of running Code 9.2.3 on both representatives x_1 and x_2 for all $\Phi \in \mathfrak{I}$ are shown in Table 17. We remark that `ActnGpDiff` remained empty throughout this process, hence its exclusion from Table 17. As with $L_2(81)$ and $L_2(2187)$, we omit the mention of `ActnGpDiff` and `SetKeepZero` if no groups are added to these sets.

Table 17: The sizes of sets FinSub and BadSub for each $\Phi \cong 2\mathrm{A}_2$ and each representative x_1, x_2 after using Code 9.2.3.

	Representative			
	x_1		x_2	
Φ	FinSub	BadSub	FinSub	BadSub
{1,3,5,6}	19	18	20	19
{1,3,6,7}	22	20	22	18
{1,3,7,8}	13	16	13	18
{2,4,6,7}	21	22	22	20
{2,4,7,8}	19	19	22	19
{3,4,6,7}	25	22	21	26
{3,4,7,8}	20	28	22	23
{4,5,7,8}	18	17	19	25

Let F_i denote any group in any set `FinSub` from Table 17 which corresponds to representative x_i. Then $\dim(C_V(\langle F_i, x_i\rangle)) > 1$ for $i \in \{1,2\}$. Hence any H constructed from F_i would not be maximal in G by Lemma 3.7 and Lemma 2.5. It remains to consider the groups in each `BadSub`.

$\Phi = \{1,3,5,6\}$

Firstly, we consider $\Phi = \{1,3,5,6\}$. From Table 17, we see that there are 2 sets `BadSub` to consider, consisting of 18 and 19 groups respectively. Let b be any 3-group contained in either of these sets and let x_i denote the corresponding cuspidal element of order 13. Then $\dim(C_V(\langle b, x_i\rangle)) > 1$. Hence any group $H \cong \mathrm{L}_2(27)$ constructed from b would not be maximal by Lemma 3.7 and Lemma 2.5.

It now remains to consider the remaining 7 cases of $\Phi \cong 2\mathrm{A}_2$. Unfortunately, these require much more work. We give a detailed write up for the case $\Phi = \{1,3,6,7\}$ on how we implement the method shown in Section 3.2.1. As the work for all the other cases is largely very similar to this, we omit detailed reports in these sections and instead provide brief summaries of the work conducted so far.

Before proceeding to the other cases we have some remarks. As the following work was undertaken, it became apparent that data management would become a key logistical battle. To minimise the amount of data being stored, it was decided that any 3-group b encountered while running through the algorithm (either using Code 9.2.3 or 9.2.6) such that $(C_V(\langle b, x_i\rangle)) > 1$ for $i \in \{1,2\}$ would be immediately discarded and not added to any sets `FinSub`, `BadSub`, `ActnGpDiff` or `BadSetNew`. Even after the removal of such groups, it will become very common to encounter sets `BadSub` and `BadSetNew` containing thousands of groups. Although we do not deviate from the method explained in Section 3.2.1, simply setting the code off running on a set of this size in a single Magma session would be extremely inefficient. As such, the sets `BadSub` and `BadSetNew` are often partitioned into subsets and considered via parallel processing; we do not mention every time that we do this.

$\Phi = \{1, 3, 6, 7\}$

x_1

From Table 17, we have a `BadSub` of size 20 for representative x_1 and $\Phi = \{1, 3, 6, 7\}$. Of these 20 groups, 19 are such that $\dim(C_V(\langle b, x_1 \rangle)) > 1$. Hence, any H constructed from these cases would not be maximal in G by Lemma 3.7 and Lemma 2.5. Let b denote the remaining group. We have that $|b| = 3^{67}$ and $b/\Phi(b) \cong V_1 \oplus ... \oplus V_8$ where $V_i \cong V_j$ for all $i, j \in \{1, ..., 8\}$. As in Procedure 3.11, we consider overgroups of $\Phi(b)$ in order to control the size of `SetKeep`; we set `CN=3` to obtain a `SetKeep` of size 757. Let t denote the pre-image of any of these vectors. After considering all groups of the form $\langle F, t^{\langle x \rangle} \rangle$, 729 groups are added to `BadSetNew` and none elsewhere. We now wish to use Lemma 3.18 in order to reduce the size of this set. Firstly, we construct $C_{L(\Phi)}(x_1)$ and then we use Code 9.2.5. After running this several times, we are able to find a representative set for `BadSetNew` of size 57. In particular, all 729 groups from the original `BadSetNew` are $C_{L(\Phi)}(x_1)$-conjugate to one of these 57 groups. We discard the old `BadSetNew` and keep only these 57 groups. We remark that for each of these groups, we have $\dim(C_V(\langle b, x_1 \rangle)) = 1$.

Let b be any group in `BadSetNew`. Then $|b| = 3^{61}$ and $b/\Phi(b) \cong V_1 \oplus ... \oplus V_6$ where $V_i \cong V_j$ for all $i, j \in \{1, ..., 6\}$. We now wish to run the program on all these groups, just as we did for the single group we had in `BadSub` initially. As there are only 57 to deal with, we need not consider them in separate Magma sessions. Hence we set `BadSub := BadSetNew; BadSetNew := {@@}` in preparation for new groups to be added to `BadSetNew`. As before, for each of these groups we must consider b/F where F is some overgroup of $\Phi(b)$. This time, for each b in `BadSub`, we set `CN=2` and then use Code 9.2.6 which yields a `SetKeep` of size 28 for each group b. We find that no groups are added to either `FinSub` or `ActnGpDiff` but 1464 new groups are added to `BadSetNew`. These all have order 3^{58} and have a Frattini quotient which is isomorphic to a direct sum of 5 irreducible, isomorphic 3-dimensional $\langle x_1 \rangle$-submodules. Moreover, for each of these groups b, we have that $\dim(C_V(\langle b, x_1 \rangle)) = 1$. Unfortunately, we are unable to trim this down to a smaller set of representatives using Lemma 3.18, meaning we must consider them all. We partition this set into subsets $S_1, \ldots, S_{15}$ of size 100 (and one subset of size 64) and load each subset into a separate Magma session. On each screen, we set `CN=2` and use Code 9.2.6, this maintains a `SetKeep` of size 28 in all cases. Across all 15 sessions, sets `BadSetNew` of sizes 2700, 2700, 2700, 1764, 1140, 2440, 2336, 2700, 2076, 2700, 2076, 1556, 2596, 2700, 1728 are returned. Giving a detailed write up on each of these sets would not only result in a very long thesis, but this would also become extremely repetitive. For this reason, we give a detailed write up for only one of these sets and then a much more concise report in the other cases. We remark that saving the generators for these 33912 groups requires approximately 113gb of storage.

We consider the set `BadSetNew` containing 1140 groups, which was created from the subset S_5. Code 9.2.5 fails to reduce the size of this set using conjugacy, meaning all groups must be considered in some capacity. These 1140 groups are split into 100 groups of order 3^{51} and 1040 groups of order 3^{55}, all of which have a Frattini quotient isomorphic to the direct sum of 4 irreducible, isomorphic 3-dimensional $\langle x_1 \rangle$-submodules.

Firstly we consider the 100 groups of order 3^{51}. After gathering these groups into a single `BadSub` and using Code 9.2.6 with `CN=2`, we find that `FinSub` remains empty while 1274 groups are added to `BadSetNew`. Let b denote any of these 1274 groups. Then $|b| = 3^{45}$ and $b/\Phi(b)$ is isomorphic to the direct sum of 3 irreducible isomorphic 3-dimensional $\langle x_1 \rangle$-submodules. Unfortunately, we are unable to reduce this set using conjugacy relations. After partitioning these 1274 groups into subsets of size 50 and using Code 9.2.4, we find that a total of 92 groups are added to `BadSetNew` across all screens before an empty `BadSetNew` is returned on the next loop. No groups are added to `FinSub`.

Now we consider the 1040 groups of order 3^{55}. Setting `CN=2` and using Code 9.2.6 maintains a `SetKeep` of size 28 across all groups, but in total only 42 distinct groups are added to `BadSetNew`. If b denotes any of these 42 groups, then we have that $b/\Phi(b) \cong V_1 \oplus V_2$ where $V_1 \cong V_2$. After one loop of the algorithm without preimages, an empty `BadSetNew` and `FinSub` are returned, thus concluding our work on the original `BadSub` of size 1140.

It remains to consider the other 14 sets of groups, recall these have sizes 2700, 2700, 2700, 1764, 2440, 2336, 2700, 2076, 2700, 2076, 1556, 2596, 2700, 1728. These were all completed using a similar method to that described above. Table 18 indicates the sizes of the sets `BadSetNew` that were returned throughout this process. The column headed B_0 states the size of the starting `BadSub` and the column headed B_i indicates the size of `BadSetNew` at the i^{th} stage of the algorithm. No groups were added to `FinSub`, `ActnGpDiff` or `SetKeepZero` at any point. We remark that parallel processing was absolutely essential to completing this work.

Table 18: The size of BadSetNew at each stage of the algorithm for each of the 14 BadSubs described above.

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7
2700	2600	149	0	-	-	-	-
2700	3080	16876	1013	635	304	55	0
2700	3224	16889	1324	0	-	-	-
1764	2522	75	1710	0	-	-	-
2440	3198	17002	1819	0	-	-	-
2336	2574	151	714	0	-	-	-
2700	2624	702	0	-	-	-	-
2076	2002	1475	55	0	-	-	-
2700	2600	113	0	-	-	-	-
2076	2470	170	940	0	-	-	-
1556	1456	59	0	-	-	-	-
2596	2496	92	0	-	-	-	-
2700	2600	97	0	-	-	-	-
1728	1664	37	0	-	-	-	-

This completes the x_1 case; no suitable elementary abelian groups were found at any point of the process.

x_2

As in the x_1 case, there is only 1 group of order 3^{67} to consider here. No further work has been completed on this group.

$\Phi \cong \{1, 3, 7, 8\}$

x_1

From Table 17, we see there is a `BadSub` containing 16 groups to consider. Of these 16 groups, 14 are such that $\dim(C_V(\langle b, x_1 \rangle)) > 1$, which leaves 2 groups of orders 3^{67} and 3^{70} left to consider. The 3^{70} remains incomplete, however the group of order 3^{67} has been completed and this work is summarised below.

Let b denote this group of order 3^{67}. We have that $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_8$. After setting `CN=3` and using Code 9.2.6, we arrive at a `BadSetNew` containing 729 groups. Using Lemma 3.18, we are able to reduce this set to 57 conjugacy representatives. All these groups have order 3^{61} and a Frattini quotient corresponding to 6 irreducible x_1-modules. We now perform another loop of Code 9.2.6 with `CN=2`. This returns a `BadSetNew` containing 1551 groups which cannot be reduced further using conjugacy. This set is partitioned into subsets of size 50 and these subsets are loaded into separate MAGMA sessions. The size of the created `BadSetNew` on each screen is given in the column headed B_1 in Table 19. We deal with these sets in a very similar way to those shown in $\Phi = \{1, 3, 6, 7\}$, with parallel processing playing a key role. A combination of Code 9.2.4 and Code 9.2.6 (with `CN=2` or `CN=3`) was used until an empty `BadSetNew` was returned. Table 19 uses the same notation as Table 18, with B_i denoting the size of `BadSetNew` at the i^{th} stage of the algorithm.

Table 19: The size of BadSetNew at each stage of the algorithm for the 1551 groups described above when split into subsets of size 50.

B_1	B_2	B_3	B_4	B_1	B_2	B_3	B_4
1376	33739	2758	0	1400	39364	554	0
1377	34432	13498	0	1372	5395	161	0
1390	36111	19363	0	1400	25594	2775	0
1389	35652	19905	0	1400	22039	1441	0
1381	34942	15715	0	1400	22041	702	0
1400	37232	22162	0	1400	22605	1478	0
1374	34622	14208	0	1405	23680	2092	0
1395	37196	24469	0	1399	21687	652	0
1372	34431	13796	0	1361	6967	683	0
1390	36232	20410	0	1380	13843	1082	0
1381	34931	15123	0	1412	26524	0	-
1400	37259	22819	0	1380	10292	621	0
1400	37179	21776	0	1400	20684	655	0
1400	36449	21582	0	1371	10657	1471	0
1400	21846	869	0	1394	21026	2485	0
1404	1255	108	0				

No groups are added to `FinSub` at any point, thus concluding this work.

x_2

As with x_1, there are 2 groups of orders 3^{67} and 3^{70} to consider here. If b_1 denotes the group of order 3^{67} and b_2 denotes the other group, then we have that $\dim(C_V(\langle b_1, x_2\rangle)) = 1$ and $\dim(C_V(\langle b_2, x_2\rangle)) = 0$. No further work has been completed on these groups.

$\Phi \cong \{2, 4, 6, 7\}$

x_1

From Table 17, we see `BadSub` contains 22 groups, of which 2 satisfy $\dim(C_V(\langle b, x_1\rangle)) \le 1$. These groups have orders 3^{98} and 3^{67} with Frattini quotients corresponding to 4 and 8 irreducible x_1-modules respectively.

No further work has been conducted on the 3^{98} case, however the group of order 3^{67} can be dealt with quickly. Indeed, setting `CN=2` and using Code 9.2.6 returns an empty `FinSub` and `BadSetNew` after only one iteration. Hence no suitable elementary abelian groups are to be found here.

x_2

Only 2 of the 20 groups in `BadSub` are such that $\dim(C_V(\langle b, x_2\rangle)) \leq 1$. These have orders 3^{67} and 3^{104} and we only deal with the larger group here. Let b denote this group. Then $|b| = 3^{104}$ and $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_6$. Setting `CN=3` and performing one loop of Code 9.2.6 outputs a `BadSetNew` containing 702 groups, which can be reduced to 54 conjugacy representatives using Lemma 3.18. Now Code 9.2.6 with `CN=2` returns a `BadSetNew` containing 1458 groups after one loop, all of which have order 3^{95} and a Frattini quotient corresponding to 3 irreducible x_2-modules. After 4 loops of Code 9.2.4 on these 1458 groups, sets `BadSetNew` of sizes 2835, 6723, 3875, 854 are found before an empty set is returned on the next loop, thus terminating the process. No groups are added to `FinSub`.

$\Phi \cong \{2, 4, 7, 8\}$

x_1

There are 19 groups in `BadSub` and 3 are such that $\dim(C_V(\langle b, x_1\rangle)) \leq 1$. These groups have orders 3^{98}, 3^{70} and 3^{67}. Let b denote the group of order 3^{98}, then $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_4$. After setting `CN=2` and using Code 9.2.6, a `BadSetNew` containing 26 groups is returned. Using Lemma 3.18 and Code 9.2.5, this set can be reduced to 2 groups, which both have order 3^{98} and a Frattini quotient corresponding to 4 irreducible x_1-modules. Again setting `CN=2` and using Code 9.2.6 on these 2 groups, we find a `BadSetNew` containing 2 distinct groups, which are both of the form $|b| = 3^{77}$ with $b/\Phi(b) \cong V_1 \oplus V_2$. After one further loop of the algorithm, an empty `BadSetNew` and `FinSub` are returned. The groups of order 3^{70} and 3^{67} mentioned at the beginning remain incomplete cases.

x_2

Of the 19 groups in `BadSub`, 3 satisfy the condition $\dim(C_V(\langle b, x_2\rangle)) \leq 1$. These groups have orders 3^{104}, 3^{70} and 3^{67} and we only consider the group of largest order. This group exhibits the same properties as the group of the same order described in $\Phi = \{2, 4, 6, 7\}$. Indeed, we set `CN=3` and use Code 9.2.6 to obtain a `BadSetNew` containing 774 groups, which can be reduced to 54 groups using conjugacy. Now setting `CN=2` for this set of 54 groups outputs a `BadSetNew` of size 1406. By following the same method described in the case $\Phi = \{2, 4, 6, 7\}$, from this set we obtain sets `BadSetNew` of sizes 2588, 5895, 6390, 4755 and 27 before an empty `BadSetNew` and `FinSub` are returned.

$\Phi \cong \{3, 4, 6, 7\}$

x_1

There are 3 groups satisfying $\dim(C_V(\langle b, x_1\rangle)) \leq 1$ to consider here; they have orders 3^{88}, 3^{60} and 3^{67}. Firstly, let b denote the group of order 3^{88}, then $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_5$. Using Code 9.2.6 with `CN=2`, we find a `BadSetNew` containing 702 groups after one iteration. We are able to reduce this set to 54 groups using Lemma 3.18. Successive loops of Code 9.2.6 is now sufficient to complete this work, with `CN` being defined appropriately at each stage to ensure `SetKeep` contained 28 vectors (this was either `CN=2`, `CN=3` or

CN=4). Sets BadSetNew of sizes 54, 54, 2, 51, 1296, 1 are found before an empty set is returned. No groups are added to FinSub.

Now let b denote the group of order 3^{60}. We have that $b/\Phi(b) \cong V_1 \oplus ... \oplus V_7$. After setting CN=4 and using Code 9.2.6, an empty BadSetNew and FinSub are returned, thus finishing this case. No further work has been done on the 3^{67} case.

x_2

Of the 26 groups in BadSub, only 3 are such that $\dim(C_V(\langle b, x_2\rangle)) \leq 1$. There are 2 groups of order 3^{60} and 1 group of order 3^{67}. We consider only the groups of order 3^{60}, which both have a Frattini quotient corresponding to 7 irreducible x_2-modules. After 6 loops of the algorithm (with preimages being used when appropriate) we arrive at an empty BadSetNew after encountering sets of sizes 705, 1430, 516, 1296, 5562. No groups are added to FinSub.

$\Phi \cong \{3, 4, 7, 8\}$

x_1

There are 4 groups satisfying $\dim(C_V(\langle b, x_1\rangle)) \leq 1$ to consider here. There are 2 groups of order 3^{60} whilst the other 2 groups have orders 3^{67} and 3^{70}; we only deal with the groups of order 3^{60} and they are considered in a very similar way. For the first group, we set CN=3 and use Code 9.2.6 to obtain a BadSetNew of size 729; this can be reduced to 57 groups using conjugacy. This set consists of groups of order 3^{54}, 3^{43} and 3^{47} with all groups having a Frattini quotient corresponding to 5 x_1-modules. Using Code 9.2.6 with CN=2 on these 57 groups yields a BadSetNew containing 1434 groups. After partitioning into smaller subsets and running the algorithm in parallel, a total of 560 groups are added to BadSetNew after the next loop before an empty BadSetNew and FinSub are returned.

For the second group of order 3^{60}, we follow a nearly identical method. After using Code 9.2.6 and Lemma 3.18, we arrive at a BadSetNew containing 57 groups as above. We remark that there was no overlap between these 2 sets. At this point, setting CN=2 and using Code 9.2.6 gives a BadSetNew containing 1384 groups. As before, we partition this set into smaller subsets and then use a combination of Code 9.2.6 and 9.2.4. After summing the sizes of the BadSetNew's found on each screen, a total of 20357, 6446, 25029 groups are found at each successive stage of the algorithm before an empty BadSetNew and FinSub are returned.

x_2

As with x_1, there are 4 groups satisfying $\dim(C_V(\langle b, x_2\rangle)) \leq 1$ here. These have orders 3^{88}, 3^{60}, 3^{67} and 3^{70}. Firstly, we consider the group of order 3^{88}, which is of the form $b/\Phi(b) \cong V_1 \oplus ... \oplus V_5$. Setting CN=3 gives a BadSetNew containing 702 groups, which can be reduced to 54 conjugacy representatives by using Code 3.18. After a loop of Code 9.2.6 with CN=2 followed by a standard loop of the algorithm, we find sets BadSetNew containing 56 and 1510 groups. After partitioning this set into smaller subsets and performing multiple loops of the algorithm, at each stage a total of 20330, 5213, 72385 groups are added to BadSetNew before an empty set is returned. No groups are added to FinSub.

Now we consider the group of order 3^{60}, which has a Frattini quotient corresponding to 7 x_2-modules. As in many of the previous cases, we set CN=3 and use Code 9.2.6 and Lemma 3.18 to obtain a BadSetNew consisting of 57 conjugacy representatives. There are 3 distinct orders occurring here, which are 3^{54}, 3^{47} and 3^{43}. After using Code 9.2.6 with CN=2 on these 57 groups, we obtain a BadSetNew containing 1382 groups. As before, we partition this set and run the code in parallel. We find sets `BadSetNew` containing 1355 and 22961 groups before an empty `BadSetNew` and `FinSub` are returned. The groups of orders 3^{67} and 3^{70} remain incomplete.

$\Phi \cong \{4, 5, 7, 8\}$

x_1

Here `BadSub` contains 17 groups with only 3 satisfying $\dim(C_V(\langle b, x_1 \rangle)) \leq 1$. These groups have orders 3^{107}, 3^{67} and 3^{70} and we only discuss the partially completed work on the group of order 3^{107} here. This group has a Frattini quotient corresponding to 4 x_1-modules. By setting `CN=3` and using Code 9.2.6, we find a `SetKeep` of size 757, but only 28 distinct groups are added to `BadSetNew`. By using Lemma 3.18 and Code 9.2.5, we are able to reduce this set to 4 conjugacy representatives, these have orders 3^{67}, 3^{82}, 3^{82}, 3^{60}. Firstly, we consider the 2 groups of order 3^{82}, which are both of the form $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_4$. After gathering these 2 groups into a single `BadSub` and using Code 9.2.6 with `CN=3`, we get a `BadSetNew` containing 1458 groups, all of which have order 3^{73} and a Frattini quotient corresponding to 2 x_1-modules. After one more loop of the algorithm, without preimages, no more groups are added to `BadSetNew` or `FinSub`. Next we consider the group of order 3^{60}. Unfortunately, this group does not appear to behave like the previous groups we have encountered of the same order. By setting `CN=3` and using Code 9.2.6, we get a `BadSetNew` of size 757; these groups all have order 3^{54}. After partitioning this set into 8 subsets and running one loop of the algorithm, we find 8 `BadSetNew`'s which have sizes 2800, 2800, 2799, 2799, 2800, 2800, 2798, 1596. Initial testing suggests all these groups have order 3^{51} and a Frattini quotient corresponding to 4 x_1-modules, but the order of every group has not been calculated here. After multiple repeats of Code 9.2.4 (ran in parallel), the first set of size 2800 gave `BadSetNew`'s of sizes 69770, 291863, 131249 before an empty set was returned. Similarly, the set containing 1596 groups gave sets containing 37527, 50655, 101433 groups before the process terminated. No groups were added to `FinSub`. Due to the number of groups being found, this process was very slow and hence is not finished. Considering only these two sets took months and thousands of separate screens were used during this time. The original groups of order 3^{67} and 3^{70} remain unconsidered, as does the group of order 3^{67} found later in the process and the 6 sets described above.

x_2

Of the 25 groups in `BadSub`, 3 are such that $\dim(C_V(\langle b, x_2 \rangle)) \leq 1$. Two of the these groups have order 3^{67} whilst the other has order 3^{70}. No further work has been conducted on these cases.

3.4.4 Construction

Throughout this entire section, we did not encounter any suitable elementary abelian groups S from which we needed to construct $H = \langle S, x, t \rangle \cong \mathrm{L}_2(27)$. However, if we did (or do in the future), we would then need to find all inverting involutions t of x in G. As we saw in Section 3.2.1, all such t are to be found in the extended centraliser $C_G^*(x)$. Moreover, we have that $C_G^*(x) = \langle C_G(x), t \rangle$ where t is any inverting involution of x. In Section 9.1.2, we construct the centraliser $C_G(x)$ ($x \in 13\mathrm{F}$) using root subgroups. By changing the roots used, we are able to construct the centraliser $C_G(x)$ for all Levi-cuspidal x considered in this section. All that remains is to find an involution t inverting x.

Set $\Phi_1 = \{r_1, r_3, r_4, r_5, r_6, r_7, r_8, r_{69}\}$, $\Phi_2 = \{r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_{120}\}$. Then for all Φ given in Table 17, we have either $\Phi \subset \Phi_1$ or $\Phi \subset \Phi_2$. Hence, all Levi-cuspidal elements x considered in this section are contained in either $G(\Phi_1)$ or $G(\Phi_2)$. Using `LMGNormaliser`, we can easily find an inverting involution

t inside $N_{G(\Phi_i)}(x)$ for $i = 1, 2$ and then construct $C^*_G(x) = \langle C_G(x), t\rangle \sim C_G(x).2$. We find that $C^*_G(x)$ contains 4 conjugacy classes of involutions, of which only 1 inverts x. In total, there are 9253764 involutions inverting x inside $C^*_G(x)$. Should any suitable elementary abelian groups S be found in the future, we would then follow a similar procedure to that described in Section 3.3.2.

3.4.5 Future work

To conclude this section, we outline work yet to be completed. Table 20 lists the groups that remain in each of the cases. The third column indicates the orders of the outstanding groups and the number of irreducible modules in the decomposition of the Frattini quotient.

Table 20: The remaining work for $L_2(27) < E_8(3)$.

Φ	Representative	Remaining Work
$\{1,3,6,7\}$	x_2	3^{67} (8 modules)
$\{1,3,7,8\}$	x_1	3^{70} (9 modules)
	x_2	3^{67} (8 modules), 3^{70} (9 modules)
$\{2,4,6,7\}$	x_1	3^{98} (4 modules)
	x_2	3^{67} (8 modules)
$\{2,4,7,8\}$	x_1	3^{67} (8 modules), 3^{70} (9 modules)
	x_2	3^{67} (8 modules), 3^{70} (9 modules)
$\{3,4,6,7\}$	x_1	3^{67} (8 modules)
	x_2	3^{67} (8 modules)
$\{3,4,7,8\}$	x_1	3^{67} (8 modules), 3^{70} (9 modules)
	x_2	3^{67} (8 modules), 3^{70} (9 modules)
$\{4,5,7,8\}$	x_1	3^{67} (8 modules), 3^{67} (8 modules), 3^{70} (9 modules), 3^{60} (7 modules)
	x_2	3^{67} (8 modules), 3^{67} (8 modules), 3^{70} (9 modules)

4 $L_3(5)$

Our attention now turns to subgroups $H < G$ such that $F^*(H) \cong L_3(5)$. In this section, we shall prove the following result.

Lemma 4.1. Let $H < G$ such that $F^*(H) \cong L_3(5)$. Then $H \cong L_3(5)$, H acts irreducibly on V, H is maximal in G and H is unique upto G-conjugacy.

4.1 Methodology and Construction

The feasible decompositions and Brauer character table for $L_3(5)$ are given in Section 8.4. We see there is only 1 feasible decomposition to consider and this consists of a single composition factor, meaning that the corresponding embedding into G is irreducible. With no trivial composition factors, we cannot use Lemma 2.7. Instead, as we did for $L_2(81)$ in Section 3.3.2, we must construct all copies of $L_3(5)$ upto G-conjugacy. The following result is our cornerstone in the construction of such groups.

Lemma 4.2. Let $H \cong L_3(5)$ and $P \in \mathrm{Syl}_5(H)$. Then $P \cong 5^{1+2}_+$ and P contains 6 unique elementary abelian subgroups of order 5^2. Moreover, there exist $X_1, X_2 < P$ such that $X_1 \cong X_2 \cong 5^2$, $N_H(X_1) \cong N_H(X_2) \cong 5^2 :$

$\mathrm{GL}_2(5)$ and $H = \langle N_H(X_1), N_H(X_2)\rangle$. Furthermore, let $x \in X_1 \cap X_2$ and $X_1 = \langle x, x_1\rangle$, $X_2 = \langle x, x_2\rangle$. Then $N_H(X_i) = \langle N_{C_H(x_i)}(X_i),\ N_{C_H(x)}(X_i)\rangle$ for $i = 1, 2$.

Using Lemma 4.2, we now detail a method to construct representatives from all $\mathrm{L}_3(5)$-conjugacy classes inside G. This method hinges on working with the Sylow 5-subgroup of H. Indeed, the first thing we do is construct all groups of the form 5_+^{1+2} in G upto conjugacy. By Sylow's theorem, it suffices to construct all such groups in some Sylow 5-subgroup $S \in \mathrm{Syl}_5(G)$. Details of how S is constructed are given in Section 9.1.2. Let P be as in Lemma 4.2. As S is sufficiently small, we can use Magma to directly obtain representatives from all conjugacy classes of subgroups in G isomorphic to P. We find that S contains 5 conjugacy classes of subgroups isomorphic to P. By the fusion pattern for $\mathrm{L}_3(5)$ shown in Section 8.4, any P we construct must consist entirely of elements from the class 5B.

We now wish to utilise Lemma 4.2 to find all copies of H for each P we have found. For each P, we must consider all possible pairs $\{A, B\}$ where $A, B < P$ and $A \cong B \cong 5^2$. As there are 6 possibilities for A, we have 15 pairs to consider for each P. Unfortunately, when working in G, we have no way of immediately identifying which pairs of subgroups satisfy the conditions shown in Lemma 4.2, hence all pairs must be considered in some capacity. However, it suffices to consider these pairs upto G-conjugacy. Searching through all of G for elements that conjugate our pairs to each other is an impossible task, hence we concentrate our search to a much smaller subgroup. We consider the normaliser $N_S(P)$ which has order 5^4. We find that our 15 pairs of elementary abelian subgroups split into 3 $N_S(P)$-orbits, meaning we need only consider a representative from each of these 3 classes; this is true for all of our 5 groups P. Furthermore, for each P, the 3 pairs we consider are of the form $\{X, A\}$, $\{X, B\}$, $\{X, C\}$. For each group X, A, B, C, we wish to find the normaliser defined at the end of Lemma 4.2.

We shall consider the group $X \cong 5^2$ from above, but the following method works and was utilised for all A, B and C. Suppose that $X = \langle x_1, x_2\rangle$. Then, by Lemma 4.2, we have that

$$N_H(X) = \langle N_{C_H(x_1)}(X), N_{C_H(x_2)}(X)\rangle \leq \langle N_{C_G(x_1)}(X), N_{C_G(x_2)}(X)\rangle.$$

We wish to obtain this overgroup. Firstly, we find both $C_G(x_1)$ and $C_G(x_2)$. Luckily, by the construction of S, we can choose x_1 to be such that we have $C_G(x_1)$ readily available. Indeed, S is the Sylow 5-subgroup of a centraliser of an order 5 element. Moreover, given $P \in \mathrm{Syl}_5(H)$, we have $Z(P) = Z(S) = \langle z\rangle$. By the structure of P, we have that z is contained in all groups X,A,B,C, hence we can choose $x_1 = z$. It remains to find $C_G(x_2)$. To begin, we find some involution $t_1 \in C_G(z) \cap 2\mathrm{B}$ such that t_1 centralises x_2 and $t_1x_2 \in 10\mathrm{E}$. We then use `CentraliserOfInvolution` to calculate $C_G(t_1)$. Using `LMGCentraliser`, we then find $C_{C_G(t_1)}(x_2) = C_G(t_1x_2) \leq C_G(x_2)$. Now choose $t_2 \in C_G(t_1x_2)$ such that $C_G(t_1) \neq C_G(t_2)$. Again using `LMGCentraliser`, we find $C_{C_G(t_2)}(x_2) = C_G(t_2x_2) \leq C_G(x_2)$. Then $C_G(x_2) = \langle C_G(t_1x_2), C_G(t_2x_2)\rangle$.

Having now acquired both $C_G(z)$ and $C_G(x_2)$, we proceed to calculate the normaliser of X inside both these groups using `LMGNormaliser`. Let $N_1 = N_{C_G(z)}(X)$ and $N_2 = N_{C_G(x_2)}(X)$. We find that $|N_1| = |N_2| = 2^6 \cdot 5^5$ and $|\langle N_1, N_2\rangle| = 2^9 \cdot 3 \cdot 5^5$. We remark that this is true for all groups X, A, B and C that we encounter in this section. From Lemma 4.2, we wish to look inside $\langle N_1, N_2\rangle$ for all subgroups isomorphic to $N_H(X) \cong 5^2 : \mathrm{GL}_2(5)$. As $\langle N_1, N_2\rangle$ is relatively small, we can use the `Subgroups` command with the added parameter `OrderEqual := 2^5*3*5^3` to obtain all conjugacy classes of subgroups of this order. We find 2 such classes and both of these contain 400 groups isomorphic to $N_H(X)$. We now repeat this process again, but replacing X with any of A, B or C. Finding all possibilities for $N_H(X)$ for a given P takes approximately 2 days of computation time. Luckily, using G-conjugacy, we can shorten this process slightly.

For a given elementary abelian subgroup, using X again as an example, we wish to look for conjugacy between elements of X and the central element $z \in X$. Should we find this conjugacy, say $z^e = x_2$ for some element $e \in G$, then we need not go through the long procedure of constructing $C_G(x_2)$ as we have that $C_G(x_2) = C_G(z)^e$. Our Sylow 5-subgroup S contains a unique normal elementary abelian subgroup $E \cong 5^4$. From [48], we deduce that $N_G(E)$ contains a subgroup of shape $[10^4].((4 \circ 2^{1+4}).\mathrm{Alt}(6).2)$. We wish to search for these conjugating elements inside this subgroup; we construct it as follows. Following on from the notation given in the previous example, we have that $E \leq C_G(z)$. Moreover, E has 4 generators and the first of these generators can be chosen to be z. In particular, it is contained in $Z(C_G(z))$. Our aim is to find an involution inside $C_G(z)$ that centralisers all 4 generators. By our choice, any involution from $C_G(z)$ will centralise the first generator. Suppose y denotes another generator ($y \notin Z(C_G(z))$), we find $C_{C_G(z)}(y)$ using `LMGCentraliser`. Inside here, we easily find an involution, say t, centralising all 4 generators. Using the Bray method, we find the centraliser $C_G(t)$ and then using `LMGNormaliser`, we construct $N_E = N_{C_G(t)}(E) \sim [10^4].((4 \circ 2^{1+4}).\mathrm{Alt}(6).2)$. For any P, there exists exactly 1 elementary abelian subgroup A such that $x_2 = z^e$ for some $x_2 \in A$ ($x_2 \notin \langle z \rangle$) and some $e \in N_E$. Hence, we need only follow the preceding construction for the remaining elementary abelian subgroups, so X, B and C.

Using the previous arguments, for each P we will have obtained 800 possibilities for each normaliser $N_H(X)$, $N_H(A)$, $N_H(B)$ and $N_H(C)$ inside G. It remains to loop through the pairs of these possible normalisers and check whether the group generated by them both is isomorphic to $\mathrm{L}_3(5)$. For example, considering the pair $\{X, C\}$, we have 800 possibilities for both $N_H(X)$ and $N_H(C)$. We choose some possibility for $N_H(X)$ and then loop over all possibilities for $N_H(C)$ and check whether the pair of normalisers could generate a group isomorphic to $\mathrm{L}_3(5)$. This is repeated for all 800 choices of $N_H(X)$.

In practice, checking whether the groups we generate are isomorphic to $\mathrm{L}_3(5)$ is a little tricky. An easy and quick check that was employed in Section 3.3.2 was to check the number of composition factors of $V|_H$ to see whether this lined up with the corresponding feasible decomposition of H. Typically, the groups we generate either have the required number of composition factors or exactly 1 composition factor (and in the vast majority of cases the number of composition factors is 1). Previously, we could safely discard any group H such that $V|_H$ consists of 1 composition factor. Unfortunately, with $H \cong \mathrm{L}_3(5)$, we require $V|_H$ to have only 1 composition factor. Thus, checking the number of factors for each group we generate here is meaningless. Instead, as in Section 3.3.2, we consider element orders to try and discard groups that are not suitable.

Let H be one of the possible $\mathrm{L}_3(5)$'s and let $x \in H$. Then should H be isomorphic to $\mathrm{L}_3(5)$, it must be that the order of x is contained in $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24, 31\}$. As we did in Section 3.3.2, any group we generate that contains an element of an invalid order is discarded. For each pair of elementary abelian subgroups, we have 640000 possibilities for H. Hence for each of these, we initially check the order of only 1 element. Should the element we choose have a valid order, we save H for future consideration. After all 640000 groups have been considered, we then repeat this check but on 100 elements instead of just 1. Table 21 shows how these checks are able to discard large numbers of groups.

Table 21: The outcome, for each P, of checking the order of random elements for each possibility of H generated from the given pair of elementary abelian subgroups.

	First Check			Second Check		
P	$\{X, A\}$	$\{X, B\}$	$\{X, C\}$	$\{X, A\}$	$\{X, B\}$	$\{X, C\}$
1	115	89	91	2	2	2
2	93	123	104	2	2	2
3	117	86	81	2	2	2
4	97	127	107	2	2	2
5	94	81	86	2	2	2

Table 21 is structured as follows. The first column indicates which of the 5 possibilities for P is being considered. Columns 2-4 give the number of possibilities, for each pair, of H that remain after performing the 1 element check described earlier. Columns 5-7 give the number of possibilities for H that remain after performing the 100 element check.

After performing both checks on all pairs for each P, we are left with a total of 30 possibilities for H. Before checking isomorphism, we do one last check with element orders, this time checking 10000 elements. All 30 groups pass this check, meaning we can be happy to check isomorphism without much chance of running into any problems. Indeed, we find that all 30 groups are isomorphic to $L_3(5)$. Despite 30 being a manageable number, ideally we would like to work with as few groups as possible. Again, we wish to trim these down using G-conjugacy. We quickly find that our original 30 groups are reduced to only 4 due to conjugacy inside $C_{N_E}(z)$. Let H_1, H_2, H_3, H_4 denote these representatives with $X, A, B, C \leq H_1$ being the elementary abelian subgroups previously described. Using MAGMA, we find that there exist elements $g, h \in \langle N_{C_G(x_1)}(X), N_{C_G(x_2)}(X)\rangle$ (assuming $X = \langle x_1, x_2\rangle$) such that $H_1^g = H_3$ and $H_1^h = H_4$. Furthermore, if we assume $B = \langle y_1, y_2\rangle$, then there exists an element $k \in \langle N_{C_G(y_1)}(B), N_{C_G(y_2)}(B)\rangle$ such that $H_4^k = H_2$. As such, we conclude that there is 1 conjugacy class of subgroups in G isomorphic to $L_3(5)$. It now remains to determine the maximality in G of the subgroup we have constructed.

4.2 Maximality

To show the $H \cong L_3(5)$ we have constructed is maximal in G, we shall prove there exists no proper subgroup containing it. To do this, we consider all possible maximal subgroups of G (whether it be known they are maximal or not) and check whether H could live inside any of them. Such a list of subgroups can be formed by considering Theorem 1.1. As H acts irreducibly on V, any overgroup must also act irreducibly. This greatly restricts the possibilities that we have to consider. For example, we need not consider any maximal parabolic subgroup of G, as they do not act irreducibly. It is sufficient to consider only the groups in Tables 1 - 2 and the maximal rank subgroups given in Table 8 and [48]. Upon searching through all these groups, we find only 1 group which has order divisible by $|H|$, namely the Thompson group Th. It is well known that Th is maximal in G and acts irreducibly on V (see [20]). It now suffices to check whether H lies inside a maximal subgroup of Th; such subgroups are listed in [20]. We find that no maximal subgroup of Th has order divisible by $|H|$, hence we must have that $H \nless$ Th. From this, we can conclude that either H or some almost simple group containing H is maximal in G. As $\mathrm{Aut}(H) \sim L_3(5).2$, the only almost simple group containing H is $\mathrm{Aut}(H)$ itself. We shall show G contains no subgroups isomorphic to $\mathrm{Aut}(H)$, thus proving that H itself is maximal in G.

We attempt to determine the feasible decompositions for Aut(H). By our previous remarks, should Aut(H) embed into G, it must act irreducibly on V. Using MAGMA, we find that Aut(H) has 2 irreducible 248-dimensional modules over K, one of which we can rule out immediately as it exhibits invalid values for the Brauer characters. The remaining irreducible module has the same fusion pattern as the feasible decomposition for $L_3(5)$ shown in Section 8.4, however an extra class of involutions must be considered here that was not present when considering $L_3(5)$. Using Brauer characters, we find that this extra class of involutions does not fuse into either classes of involutions in G. Hence, there are no feasible decompositions for Aut(H) and so it cannot embed as a subgroup of G. Consequently, we have that H is maximal in G. This completes the proof of Lemma 4.1.

We conclude this section with a brief remark. Like $L_3(5)$, the Thompson group Th contains a maximal subgroup of shape $5^2 : GL_2(5)$. As such, we realised that the algorithm shown in Section 4.1 can also be used to construct Th inside G. Indeed, all that must be changed is the final element order check. Instead of checking that the selected random element has an order occurring inside $L_3(5)$, we check if the element order occurs inside Th. The rest of the algorithm and code remains identical. Using this method, one can quickly construct a copy of Th inside G.

5 $L_2(11)$ and M_{11}

We now turn our attention to subgroups $H < G$ such that $F^*(H)$ is isomorphic to either $L_2(11)$ or M_{11}. It is well known that $L_2(11)$ is a maximal subgroup of M_{11}. Hence, we look to construct M_{11} from $L_2(11)$. The Brauer character tables and feasible decompositions for both these groups are given in Sections 8.12 and 8.13. In this section, we shall prove the following result.

Lemma 5.1. Let $H < G$ such that $F^*(H)$ is isomorphic to either $L_2(11)$ or M_{11}. Then H is not maximal in G.

Firstly, we use the information given by the feasible decompositions to restrict the possibilities for how $L_2(11)$ embeds into G; we shall see that the only possible maximal embedding of $L_2(11)$ aligns with the only possible maximal embedding of M_{11}. As such, determining the non-maximality of both these groups hinges on constructing subgroups isomorphic to $L_2(11)$. The following Lemma will be our cornerstone in this regard.

Lemma 5.2. Let $H \cong L_2(11)$. Then H has a maximal subgroup $M \cong \langle g, f\rangle \cong 11 \rtimes 5$ where g is order 11 and f is order 5. Moreover, there exists an involution $t \in H$ such that t inverts f and $H = \langle g, f, t\rangle$.

Unless stated otherwise, for the remainder of this section we follow the notation used in Lemma 5.2. From Section 8.12, we see that any groups H associated to feasible decompositions 1, 2 or 3 are not maximal in G by Lemma 2.7 and Lemma 2.5. Hence, it suffices to assume H is associated with feasible decomposition 4. In particular, we have $f \in 5B$ and $t \in 2B$.

From Section 8.13, we see there are 10 feasible decompositions for M_{11} (labelled 3,5,6,8,12,13,15,17,18,20) which cannot be eliminated using Lemma 2.7 or Table 3. Thanks to the work of Joe in his thesis [64], only 2 of these 10 feasible decompositions remain, namely decompositions 13 and 15. A key method used by Joe was to work inside the Sylow 2-subgroup of G and attempt to construct, upto G-conjugacy, subgroups isomorphic to the Sylow 2-subgroup of M_{11} which obey the fusion pattern of a given feasible decomposition. If no subgroups were found obeying a particular fusion pattern, then this completely invalidates the associated feasible decomposition. In decompositions 13 and 15, all involutions and elements of order 5 fuse into classes

2B and 5B in G; this aligns with feasible decomposition 4 for H. Moreover, given that in decompositions 1, 2, 3 of H we have that either $t \in 2A$ or $f \in 5A$, we can conclude that this is the only feasible decomposition of H which could extend to the only remaining feasible decompositions for M_{11}. As such, we can simplify the problem of studying the maximality of M_{11} in G to determining the maximality of H. If we find that there are no subgroups isomorphic to H in G which follow decomposition 4, then it must be that there are no subgroups isomorphic to M_{11} whose fusion follows $2A \to 2B$, $5A \to 5B$. Details of how we attempted to construct H inside G are now given.

To begin, we fix an element $g \in G$ of order 11. As G contains a single conjugacy class of order 11 elements, it suffices to let g be any such element. However, in order to easily find all suitable elements f, we can make a better choice for g. Indeed, from Section 9.1.2, we see that $C = C_G(g) \cong L_5(3) \times 121$. Moreover, we see that $C_G(g)$ contains a full Sylow 11-subgroup $121 \times 121 \cong S_{11} \in \mathrm{Syl}_{11}(G)$ and itself is contained in a maximal rank subgroup $N < G$ of type $2A_4$ (the $2A_4$ corresponds to the roots $\{1,3,4,5\} \dot\cup \{69,7,8,120\}$). All suitable elements of order 5 that act non-trivially on $\langle g \rangle$ are found inside $\langle C, f \rangle$ where f is any such element. The centraliser C can be easily constructed using Magma (see Section 9.1.2 for explicit details). Also, we can easily construct the subgroup $N_N(S_{11})$ which has order $|N_N(S_{11})| = 5^2 \cdot 11^4$. Hence, we choose $g \in N_N(S_{11})$ and a suitable element f can also be found here. We find that $\langle C, f \rangle$ contains 9 conjugacy classes of elements order 5, in total giving 1441560875524 elements. However, as we are only interested in H upto G-conjugacy, it suffices to consider f upto $\langle C, f \rangle$-conjugacy. Moreover, we require $f \in 5B$ and of the 9 classes, only 4 consist of 5B elements. In fact, we need only consider representatives from each of the $\langle C, f \rangle$-conjugacy classes of cyclic subgroups order 5 that consist entirely of 5B elements; there is only 1 such class. Hence it suffices to fix f to be a generator for the representative cyclic subgroup of this class.

Having now found all the possibilities for f that we must consider, it now remains to find, inside G, all 2B involutions that invert f. These will be contained in the extended centraliser $C_G^*(f) = \langle C_G(f), t \rangle$ where t is any inverting involution of f. By construction, $f \in N$ and so we can calculate $N_N(\langle f \rangle)$; this has order $2^{10} \cdot 5^2$. Inside this group we find an inverting involution of our chosen f. To construct $C_G(f)$ it suffices to work inside the centralisers of commuting 2B involutions. There exist involutions $t_1, t_2 \in C_N(f) \cap 2B$ such that $C_G(t_1) \neq C_G(t_2)$ and $C_G(f) = \langle C_{C_G(t_1)}(f), C_{C_G(t_2)}(f) \rangle$; this equality was verified using the orders in Table 3. From Table 4, we see that $C_G(f) \cong SU_5(9)$ and $|C_G(f)| = 2^{11} \cdot 3^{20} \cdot 5^5 \cdot 41 \cdot 73 \cdot 1181$. The extended centraliser $C_G^*(f) = \langle C_G(f), t \rangle$ can now be calculated; we find that $|C_G^*(f)| = 2^{12} \cdot 3^{20} \cdot 5^5 \cdot 41 \cdot 73 \cdot 1181$.

It now remains to find all involutions that invert f inside $C_G^*(f)$. Due to the group structure and its large order, we are unable to directly calculate its conjugacy classes. However, using `LMGSylow`, we can obtain a Sylow 2-subgroup of this group. Inside here, we find 13 conjugacy classes of involutions, of which 7 invert f. Additionally, we find that all 7 of these classes consist of 2B elements. We conclude that any subgroup of G isomorphic to either $L_2(11)$ or M_{11} containing 5B elements must only contain involutions from the 2B class. From this, we deduce that the only feasible decompositions of H that could extend to any M_{11} are 1, 2 and 4. However, as mentioned previously, decompositions 1 and 2 can only extend to M_{11}'s that are not maximal.

Taking a representative involution t from each of the 7 classes, we find that $|C_{C_G^*(f)}(t)| = 2^{10} \cdot 3^8 \cdot 5^2 \cdot 41$ in all cases. Consequently, each class has 22908561466500 involutions that could potentially lead to a group $H = \langle g, f, t \rangle \cong L_2(11)$. Using the following procedure, we can show each of these 7 classes are conjugate inside $C_G^*(f)$, thus proving there is only one class of inverting involutions inside $C_G^*(f)$.

Procedure 5.3. Let $t_1, t_2 \in G$ be involutions in any group G. Then $D = \langle t_1, t_2 \rangle \leq G$ is a dihedral group. If $|D| = 2n$ where n is odd, then D contains 1 conjugacy class of involutions. In particular, t_1 and t_2 are D-conjugate. Providing n is sufficiently small, we can use this to quickly show involutions are conjugate and explicitly find a conjugating element. As n is bounded by the largest element order in G, this is often very quick, even in large groups like $E_8(3)$. We remark that if G is infinite, then D need not be finite. The code implementing this procedure into Magma is given in Section 9.2.13.

Using Procedure 5.3, we can quickly show that each of the 7 conjugacy classes of inverting involutions become conjugate in the larger group $C_G^*(f)$, meaning there is only one class of 22908561466500 involutions to consider. Given one of these involutions t, checking if $\langle g, f, t\rangle$ could give a $L_2(11)$ takes approximately $1/128 \approx 0.008$ seconds. Consequently, it would take around 5700 years to consider all such t in this manner. Hence, we need to orbit this conjugacy class using a suitably large subgroup of $C_G^*(f)$ such that it is sufficient to consider only the orbit representatives to determine whether $\langle g, f, t\rangle \cong L_2(11)$. To do this, we study the permutation module of $L_2(11)$.

The group $H \cong L_2(11)$ has a natural 2-transitive action on the set $\Omega = \{1, ..., 12\}$, giving a representation as a subgroup of Sym(12). For all $v \in \Omega$, the stabiliser $\mathrm{Stab}_H(v)$ is a Frobenius group of order 55. Using this, we want to use the following Lemma to study the potential action of $\langle g, f\rangle$ on the permutation module.

Lemma 5.4. Let H be a finite group and let V be a GF(p)-module for H (p a prime). For $v \in V$, set $K = \mathrm{Stab}_H(v)$ and $U = \langle v^H\rangle$. Then U is isomorphic to some quotient of the GF(p) permutation module with H acting on the right cosets of K in H.

Without loss of generality, let $v \in \Omega$ such that $\mathrm{Stab}_H(v) = \langle g, f\rangle$. Consider the pair (v, v^t) of v and v when acted on by some inverting involution t of f. As t inverts f, the vector v^t is still fixed by f. However, it is no longer fixed by the element of order 11, g. Now assume that $H = \langle g, f, t\rangle$ and consider the vector space $U = \langle v^H\rangle = \{v, v^t, (v^t)^g, (v^t)^{g^2}, ..., (v^t)^{g^{10}}\} \subseteq \Omega$. By Lemma 5.4, U is a quotient of the KH permutation module. Moreover, this permutation module is 12-dimensional and consists of three composition factors of dimensions 1,10,1 with structure 1/10/1. Hence, U has dimension either 1,11 or 12. If $\dim(U) < 12$, then it must be that H fixes a vector and so H is not maximal in G by Lemma 2.5. However, it could still be the case that H extends to a potentially maximal M_{11}, and so these cases must still be considered. In all situations, H acts invariantly on U. If we encounter an inverting involution t such that $\langle g, f, t\rangle$ does not act invariantly on the space $\langle v^{\langle g,f,t\rangle}\rangle$, then it cannot be that $\langle g, f, t\rangle \cong L_2(11)$.

Now let $S = \mathrm{Stab}_{C_G^*(f)}(v) < C_G^*(f)$. By construction, if $H = \langle g, f, t\rangle$ does not have an invariant action on the associated space $\langle v^H\rangle$, then all groups of the form $H' = \langle g, f, t^h\rangle$, for all $h \in S$, will also not act invariantly on $\langle v^{H'}\rangle$. Therefore, it is sufficient to consider only orbit representatives of the 22908561466500 inverting involutions under the action of S (via conjugation). Before we can start gathering these orbit representatives, the group S must be constructed in the 248-dimensional setting. We remark that it is not necessary to construct all of S, a suitably large subgroup of S will also be sufficient.

Define M to be the intersection of fixed spaces $C_V(g) \cap C_V(f)$. From Table 3, we see that $\dim(C_V(g)) = 28$ and $\dim(C_V(f)) = 48$. Using Magma, we find that $\dim(M) = 4$. From our previous work, it suffices to let v be any non-zero vector in M, but we shall choose a particular 1-dimensional subspace to aid in the construction of S. Using `LMGSylow`, we can obtain a Sylow 3-subgroup of $C_G^*(f)$. Following this, the command `UnipotentStabiliser` can be used (unipotent as we are working in characteristic 3) to quickly construct various stabilisers. Looping through all 40 1-dimensional subspaces of M, we call `UnipotentStabiliser` to find the stabiliser inside the Sylow 3-subgroup of order 3^{20}. We find that one of the 1-dimensional subspaces has a stabiliser of order 3^6 inside the Sylow 3-subgroup used, and so we choose v to be the generator of this particular subspace. It now remains to build up this stabiliser to minimise the number of orbits we have to consider. Unfortunately, we cannot directly use the `Stabiliser` command on all of $C_G^*(f)$, so we resort to working in smaller subgroups. Various tactics were used to find these smaller subgroups. A fast and efficient method for obtaining smaller subgroups is to use `CentraliserOfInvolution` successively. Additionally, one can simply take subgroups generated by some proper subset of a generating set for $C_G^*(f)$. Typically, we are looking for subgroups of order $< 10^8$ for `Stabiliser` to work quickly enough. The process of generating random smaller subgroups (using the described methods) and then calling `Stabiliser` was used thousands of times to try and construct S. A sample of how this was implemented in Magma is shown in Section 9.2.13.

Eventually, we arrive at a stabiliser subgroup of order $13623552000 = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 73$; it is highly likely that this is all of the stabiliser inside $C_G^*(f)$. We can expect approximately $\frac{22908561466500}{13623552000} \approx 1682$ orbits of inverting involutions using this stabiliser. For the remainder of this section, unless stated otherwise, we shall use S to denote this stabiliser $\mathrm{Stab}_{C_G^*(f)}(v)$ of order 13623552000. The process of gathering these orbit representatives is where the vast majority of the time and effort was spent working on this particular problem. The following sections detail the methods used to do this.

5.1 The 10-Dimensional Model

As I'm sure it is very clear by this point of the thesis, calculations inside 248-dimensional matrix groups can be very time consuming. In almost all situations, reducing the problem to working in a smaller degree representation or a permutation representation is a good idea. In this section, the construction of such a representation for $C_G^*(f)$ is detailed. Following this, a method is established to find all orbit representatives in this representation. We shall see that we can use this information to inform our work inside the 248-dimensional representation.

5.1.1 Construction of the 10-dimensional representation

Since $C_G(f) \cong \mathrm{SU}_5(9)$, there is a natural 5-dimensional matrix representation readily available to use for calculations. However, it is not at all clear how the extended centraliser $C_G^*(f) \sim \mathrm{SU}_5(9).2$ could be modelled in the 5-dimensional setting. From [7], it must be that the inverting involution t acts on $\mathrm{SU}_5(9)$ in the same way as the graph automorphism, by inverse transpose. A similar situation is encountered in [66], where the authors are trying to construct a matrix which represents this action. It turns out the best course of action is to consider a matrix representation of slightly larger dimension. Indeed, let U denote the natural 5-dimensional GF(9)-module for $\mathrm{SU}_5(9)$ and set $M = U \oplus U^*$ to be the direct sum of U and its dual U^*. By letting $\mathrm{SU}_5(9)$ act upon M, we obtain a 10-dimensional matrix representation. Moreover, the inverse transpose map can be represented by the matrix $t = \left(\begin{array}{c|c} 0 & I_5 \\ \hline I_5 & 0 \end{array}\right)$ (see [66]). By taking the group generated by the 10-dimensional representation for $\mathrm{SU}_5(9)$ and our newly found matrix t, we acquire a 10-dimensional representation for the extended centraliser $C_G^*(f)$. The code implementing this into Magma is given in Section 9.2.13.

Having constructed a 10-dimensional model for the 248-dimensional setup, we need to map over the stabilizer S and the element of order 5 f. The latter can be done easily, given that f is the element of order 5 in the center of $\mathrm{SU}_5(9)$. The stabilizer is a bit more tricky. We find that $Z(S) = \langle z \rangle \cong 2$ and $|C_{C_G^*(f)}(z)| = 2^{11} \cdot 3^8 \cdot 5^4 \cdot 73$. Moreover, S is maximal in $C_{C_G^*(f)}(z)$. Hence, to find a 10-dimensional representation of S, it suffices to find a centraliser of an involution of the given order and find S as a maximal subgroup.

5.1.2 Finding orbit representatives

Having now identified all the relevant structures in the 10-dimensional representation, we can start finding the orbit representatives for the inverting involutions. Initially, a random search is effective. We find any inverting involution t and then conjugate this by some random element $h \in C_G^*(f)$. We then check if t and t^h are S-conjugate; if they are not, t^h is added to a set alongside t. The process is then repeated, with a new element h being generated and then S-conjugacy is tested for each element in our representative set. Should we find that t^h is S-conjugate to one of our orbit representatives, it is immediately discarded and the process restarts by generating a new h. Checking conjugacy in the 10-dimensional case is quick and so this procedure is an effective way of building a set of orbit representatives. However, in the 248-dimensional case, this check is significantly slower. Using the following Lemma, we can rule out conjugacy before calling `IsConjugate`.

Lemma 5.5. Suppose g, h are inverting involutions of f such that g and h lie in the same S-orbit, ie, there exists some $x \in S$ such that $g^x = h$. Then gz and hz have the same order.

Proof. Suppose $(hz)^n = (g^x z)^n = 1$. As $x \in S$, z and x commute and so $g^x z = x^{-1}gxz = x^{-1}gzx = (gz)^x$. In particular, $(gz)^x$ has order n and thus gz also has order n. □

Checking the orders of gz and hz is very quick (in both 10 and 248 dimensions) and often we find that they have differing orders. Thus, Lemma 5.5 offers an efficient way of disproving S-conjugacy without needing to use `IsConjugate`.

This random search equipped with Lemma 5.5 was employed to find approximately 90% of the orbit representatives in the 10-dimensional case. As more representatives are gathered, the time for each complete iteration is increased as conjugacy has to be checked more often. Due to the random nature of the process, representatives from small orbits are much more difficult to find. Moreover, once we have a large proportion of the orbits, it becomes more and more unlikely to find new ones.

As such, we wish to employ a more targeted approach to find the remaining inverters. Let $K < C_G^*(f)$ denote an overgroup of S and suppose K acts on the set of inverting involutions of f. As $S < K$, each S-orbit will fuse into some K-orbit. Indeed, each K-orbit is the disjoint union of some number of S-orbits. So given a K-orbit representative k and a S-orbit representative s, we check for K-conjugacy. If these elements turn out to be conjugate, the S-orbit containing s is entirely contained in the K-orbit containing k. We check this for all s obtained so far and see if we can completely partition the K-orbit. If we cannot, then there are undiscovered S-orbit representatives to be found inside the incomplete K-orbits.

To utilise this method, we first need to find a suitable K and all the subsequent K-orbit representatives (which in itself is a difficult task). An obvious and suitable candidate for K is $C_{C_G^*(f)}(z)$. To find the K-orbit representatives, we initially use the random process described before. As we are expecting to find less representatives (using the orbit-stabilizer theorem we can estimate there will be thousands of S-orbits but only a few hundred K-orbits), we can expect the random process to find the majority of the orbits. Indeed, using parallelisation, we are able to find the vast majority of the K-orbit representatives. By summing the sizes of all orbits obtained after this initial search, we find there are 42573600 inverting involutions not accounted for. As this number divides $|K|$, it is highly likely this is a single orbit with stabilizer size 14400. By direct search, we find 2193 subgroups of K of order 14400 (upto K-conjugacy). Given one of these subgroups, we find its centraliser in $C_G^*(f)$ and then look for the remaining inverting involution in here. To find this centraliser, we take successive centralisers of generating elements until all generators have been considered. Ideally, we find a generating set containing commuting involutions in order to utilise the very quick `CentraliserOfInvolution`. Looping through the 2193 subgroups of order 14400, we find a centraliser containing an inverting involution. This involution is not guaranteed to have a stabiliser of size 14400; the size of the stabiliser will be some multiple of 14400. Hence, it is required to explicitly check the size of the stabiliser. As we have no other orbits of this size, we know that this inverter is not K-conjugate to any of the other representatives and so we have found all the K-orbits. We remark that if we had other orbits of the same size, we would have to check for K-conjugacy with all these other representatives. This is something that is considered when utilising this method again in the 248-dimensional case.

In total, there are 180 K-orbits. We now search for all incomplete K-orbits and then consider only the $C_G^*(f)$-conjugates of the representatives from these orbits. It suffices to check S-conjugacy against only the other S-orbit representatives that lie inside the same K-orbit. Using this method, we very quickly find the remaining S-orbits, of which there are 4012. These can be sorted by the size of their respective stabilisers. A very useful observation is that all 4012 orbit representatives can be found in the centralisers of appropriate stabilisers and that we may consider these stabilisers upto S-conjugacy. For example, we find that of the 4012 involutions, 148 of them have a stabiliser of size 16. Upon gathering these 148 elements and constructing all their stabilisers, it turns out that they all dihedral and conjugate in S. Hence all 148 of these elements can be found in the centraliser, in $C_G^*(f)$, of any one of these stabilisers. Calculation shows this centraliser has order $2^7 \cdot 3^6 \cdot 5^3 \cdot 73$. This gives us a more targeted approach at finding the orbit representatives in the 248-dimensional case, providing we can map across the S-conjugacy class representatives of the stabilisers. If there is more than one class of stabilisers for a given order, then the inverting involutions will be split across the centralisers of the representative stabilisers.

Constructing the centraliser of a given stabiliser is often a non-trivial task. Computations in the 10-dimensional case are very quick and it is sufficient to simply take successive centralisers of generating elements of the stabiliser inside $C_G^*(f)$. However, this will not suffice for 248-dimensional setting. Indeed, we often need to find a generating set for each stabiliser with certain properties which allow for an easier computation of the centraliser. Central involutions of the stabiliser are fantastic elements to include in the generating set as centralisers of involutions can be found very quickly in any representation by using `CentraliserOfInvolution`. Often, taking successive centralisers of two involutions reduces the group size enough for the standard `Centraliser` function to work effectively with the remaining generating elements (which are often chosen at random after finding suitable involutions). We remark that `LMGCentraliser` does not work in the case $C_G(h)$ for $h \notin G$, which is the situation we have here. However, the standard function `Centraliser` does work. The following table gives all stabiliser sizes that occur, the number of orbit representatives that have the given stabiliser size, the number of S-conjugacy classes of the stabilisers, the size of the corresponding centralisers in $C_G^*(f)$ and how the orbit representatives split across the centralisers of the stabiliser representatives.

Table 22: Stabiliser information for the 4012 orbit representatives.

Stabiliser Order	Reps	# Stabiliser Classes	$\lvert C_{C_G^*(f)}(\text{Stabiliser})\rvert$	Distribution of reps
1	148	1	$\lvert C_G^*(f)\rvert$	148

2	2741	5	$2^{12} \cdot 3^{12} \cdot 5^4 \cdot 41 \cdot 73$	2676
			$2^{10} \cdot 3^8 \cdot 5^2 \cdot 41$	14
			$2^{11} \cdot 3^8 \cdot 5^4 \cdot 73$	32
			$2^{10} \cdot 3^8 \cdot 5^2 \cdot 41$	18
			$2^{10} \cdot 3^8 \cdot 5^2 \cdot 41$	1
4	528	10	$2^{11} \cdot 3^4 \cdot 5^4$	222
			$2^7 \cdot 3^4 \cdot 5 \cdot 41$	74
			$2^7 \cdot 3^4 \cdot 5 \cdot 41$	64
			$2^{10} \cdot 3^4 \cdot 5^2$	42
			$2^{10} \cdot 3^4 \cdot 5^2$	74
			$2^{10} \cdot 3^8 \cdot 5^2 \cdot 41$	22
			$2^9 \cdot 3^2 \cdot 5$	3
			$2^7 \cdot 3^4 \cdot 5 \cdot 41$	10
			$2^7 \cdot 3^2 \cdot 5^2$	4
			$2^{10} \cdot 3^4 \cdot 5^2$	13
8	60	6	$2^6 \cdot 5$	4
			$2^6 \cdot 3^2 \cdot 5^2$	32
			$2^5 \cdot 5^2$	6
			$2^7 \cdot 5$	12
			2^9	3
			$2^8 \cdot 3^2 \cdot 5$	3
16	148	1	$2^7 \cdot 3^6 \cdot 5^3 \cdot 73$	148
18	32	1	$2^6 \cdot 3^{12} \cdot 5^2$	32
20	148	1	$2^7 \cdot 3^6 \cdot 5^3 \cdot 73$	148
32	72	8	$2^6 \cdot 3^2 \cdot 5$	18
			$2^7 \cdot 3^2 \cdot 5^3$	32
			$2^6 \cdot 3^2 \cdot 5$	14
			$2^3 \cdot 5$	2
			$2^3 \cdot 5$	2
			2^5	2
			2^5	1
			$2^6 \cdot 3^2 \cdot 5$	1
36	34	4	$2^5 \cdot 3^8 \cdot 5$	10
			$2^3 \cdot 3^6 \cdot 5^2$	8
			$2^5 \cdot 3^4$	8
			$2^3 \cdot 3^4 \cdot 5$	8
40	65	4	$2^7 \cdot 3^2 \cdot 5^3$	32
			$2^6 \cdot 3^2 \cdot 5$	14
			$2^6 \cdot 3^2 \cdot 5$	18
			$2^6 \cdot 3^2 \cdot 5$	1
64	7	2	2^8	3
			$2^4 \cdot 5$	4

72	6	4	$2^3 \cdot 3^2$ $2^4 \cdot 3^2$ $2^3 \cdot 3^2$ $2^4 \cdot 3^2$	2 1 2 1
80	7	2	$2^4 \cdot 5$ 2^6	4 3
144	1	1	$2^4 \cdot 3^2$	1
320	1	1	2^3	1
2880	7	2	$2^5 \cdot 3^2 \cdot 5$ $2^2 \cdot 5$	3 4
5760	6	3	$2^2 \cdot 5$ 2^3 2^4	4 1 1
23040	1	1	2^2	1

Unfortunately, we cannot directly obtain an isomorphism between the 10 and 248-dimensional representations of $C_G^*(f)$. However, we can obtain an isomorphism between the 10 and 248-dimensional representations of S, meaning we can map conjugacy class representatives of stabilisers over to the 248-dimensional setting. After doing so, we can use Table 22 to find the corresponding inverting involutions. This method is particularly effective at finding orbit representatives from small orbits. By mapping across stabilisers, constructing their centralisers and searching for inverting involutions of the desired stabiliser order, we were able to find all orbit representatives in the 248-dimensional case apart from the 222 representatives with stabiliser order 4 and the 2676 representatives with stabiliser order 2. In these cases, we use properties of the centraliser to further partition the orbits.

First we consider the 222 representatives of stabiliser size 4 whose centraliser in $C_G^*(f)$ has order $2^{11} \cdot 3^4 \cdot 5^4$. We remark that of the 10 conjugacy classes of stabilisers of order 4, 9 of them have an elementary abelian representative whilst the remaining class is cyclic. For the 222 inverting involutions here, the stabiliser is elementary abelian. The centraliser of this subgroup is easily found (in both 10 and 248 dimensions) by using `CentraliserOfInvolution` successively on the involution generators. By working in 10-dimensions and finding the conjugacy classes of this centraliser, we can find how the 222 inverting involutions are distributed across the classes of the inverting involutions. This further partitions the set of orbit representatives and thus allows for parallelisation of this process and a more targeted search; constructing smaller disjoint sets is significantly quicker than attempting to find all 222 representatives in one large set.

In this case, we find there are 4 conjugacy classes of inverting involutions with corresponding centralisers of order 800, 640, 640 and 512. As these representatives were gathered as part of a random search, it is not the case that the 222 representatives all live inside the same centraliser. If they did, it would mean we can find the distribution in a much simpler way. However, we can still find this distribution without too much extra work. Indeed, let x be one of our 222 elements, S_x be the stabiliser of x inside S (we know that $S_x \cong 2^2$) and let $C = C_{C_G^*(f)}(S_x)$. From our earlier findings, the group $C_C(x)$ will have order either 800, 640 or 512; the value of $|C_C(x)|$ decides the distribution of all such x. After considering all 222 orbit representatives x and finding the centraliser $C_C(x)$, we find that 38 are such that $|C_C(x)| = 800$, 118 are such that $|C_C(x)| = 640$

and 66 have $|C_C(x)| = 512$. We remark that there are 2 conjugacy classes with a centraliser of order 640 here, but it is not necessary to determine exactly which of these classes x belongs to (this is more work than its worth). Moving back to 248-dimensions, we now identify representatives from these classes and generate random conjugates of these to start building up sets of orbit representatives. The centraliser is relatively small, meaning we can construct conjugacy class representatives directly using `LMGClasses`. As the 3 sets are small, obtaining all 222 inverting involutions is a fairly quick task (1-2 days of computation time) using this method.

Unfortunately, the same cannot be said for the 2676 involutions of stabiliser size 2. From Table 22, these can all be found in a centraliser of order $2^{12} \cdot 3^{12} \cdot 5^4 \cdot 41 \cdot 73$. We need to determine how these elements are distributed across the conjugacy classes of inverting involutions inside this centraliser. By using `LMGClasses` in the 10-dimensional case, we see that the centraliser has 2 conjugacy classes of inverting involutions and they have centralisers of orders 2125440 and 2073600 respectively. We must deal with the same subtlety as before, in that in the 10-dimensional model, all 2676 orbit representatives are not contained in the same centraliser. This can be dealt with in the same way as before. Using the same notation, we have that either $|C_C(x)| = 2125440$ or $|C_C(x)| = 2073600$ and this order determines the distribution of the 2676 elements amongst the 2 conjugacy classes in the larger centraliser. We find that 1322 are such that $|C_C(x)| = 2125440$ and the remaining 1354 lie inside the class with centraliser order 2073600. For notational purposes, we shall label the class with centraliser order 2125440 as C_1 and the class with centraliser order 2073600 as C_2. Despite successfully partitioning the 2676 involutions into two smaller sets, these sets themselves are still too large to construct (in a feasible amount of computation time). These sets can be further broken down by studying the product order with the central element $z \in Z(S)$. From Lemma 5.5, we know S-conjugate inverting involutions have the same product order with the central element z. Using the 10-dimensional model, we can calculate all these product orders for our 2 sets.

Table 23: Product Orders for C_1 and C_2.

Class	Product order with z	Number of elements with this product order
C_1	16	88
	20	160
	32	90
	40	64
	80	160
	82	400
	160	360
C_2	10	32
	16	42
	20	50
	24	40
	30	80
	40	160
	48	90
	60	90
	80	320
	164	450

Using Table 23, we can further partition the two sets of sizes 1322 and 1354 into much smaller subsets. For example, in the C_1 case, we see there are 160 elements that have product order 80 with z. It suffices to construct this set of 160 orbit representatives (providing they have all the right properties, not just the correct product order) independently from all the other elements, as by Lemma 5.5, we know any elements with different product orders cannot be S-conjugate. As before, this allows for the process to be split up and ran in parallel which greatly increases computation speed.

It now remains to use this information to find all the 2676 orbit representatives inside the 248-dimensional setting. First we construct the centraliser of order $2^{12} \cdot 3^{12} \cdot 5^4 \cdot 41 \cdot 73$; we know this exists as a centraliser $C_{C^*_G(f)}(S_x)$ where $S_x \cong 2$ is the stabiliser inside S of some inverting involution x. Using the isomorphism between the 10 and 248-dimensional representations of S, we can map across S_x and then find the corresponding 248-dimensional centraliser $C = C_{C^*_G(f)}(S_x)$. As S_x is generated by an involution, this can be done very quickly using `CentraliserOfInvolution`. We remark that `CentraliserOfInvolution` often failed to return the full centraliser after one calling and so had to be successively called multiple times to ensure the entire centraliser was obtained. Having now found C, representatives from the two conjugacy classes of inverting involutions must be identified. Again, thanks to `CentraliserOfInvolution`, this can be done quickly. Indeed, it suffices to generate random involutions until we find 2 that invert f and have centralisers of order 2125440 and 2073600. The process of building up the sets of sizes 1322 and 1354 can now begin.

In practice, we do not use Table 23 straight away. As before, generating random conjugates of the 2 class representatives works well as an initial approach. Due to the large size of these sets, we wish to run as much code as possible in parallel in simultaneous Magma sessions. As such, many screens were used to construct sets of orbit representatives (in both the 2125440 and 2073600 cases). On each screen, a set of pairwise non S-conjugate orbit representatives was constructed. After each set had gathered a sufficient number of elements, the random processes were stopped and each set was merged; this could also be done in parallel. For example, say two sets X_1 and X_2 have been gathered in the 2125440 case, each containing 100 elements. The set X_2 can be partitioned into 5 element disjoint subsets. Now on 20 different screens, we can load one of these 5 element subsets of X_2 and all of X_1. These two sets are now merged into one set of pairwise non S-conjugate orbit representatives. After all 20 screens have finished, we go through each one to find the elements of the subset of X_2 which are non-conjugate to all elements in X_1; we save these elements to a file as we go. After considering all 20 screens, we will have found all elements of X_2 which are non-conjugate to all elements in X_1. This process is used multiple times, for many different sets. Ideally, we start with 16 (or some other power of 2) separate screens constructing sets, then this is reduced to 8 sets, then 4, 2 and finally we will arrive at 1 large set of pairwise non-conjugate orbit representatives. The hope is that after this process has finished, we will be left with a set of orbit representatives which is nearly all of the total amount required (whether that be 1322 or 1354). At this point, Table 23 is used to find the remaining elements. We illustrate this with some empirical data from the construction of the set of size 1322. On 5 different screens, disjoint sets of sizes 540, 534, 548, 542, 533 were gathered. After combining all of these into one set, we arrive at a set of size 1273. To find the remaining 49 elements, we use Table 23. The 1273 representatives found so far are split into sets based on their product order with z. The focus is then changed to building up each of these subsets until they reach the required size as specified in Table 23. Random conjugates are taken of elements in each set to find orbit representatives with the correct product order and stabiliser size. As only 49 elements are missing, each subset is only missing a few elements. If the subset contains less than 100 elements, gathering the missing orbit representatives is quite quick and does not require multiple screens to complete. For the larger sets (we see for the 1322 case there are sets of size 360 and 400), multiple screens are used to find the remaining elements. Indeed, suppose for example we are missing 3 elements from the set of size 400. Then multiple screens are started, each trying to find any of the missing representatives. If a particular screen finds a missing element, this is saved and loaded onto all other screens, meaning each screen is now looking for 1 of 2 missing elements. This is repeated until all representatives of this product order are found.

Using the methods described above, all 4012 orbit representatives of inverting involutions were found over a period of approximately 3 months. It now remains to see whether any representative could lead to a subgroup isomorphic to $L_2(11)$. As mentioned previously, for each orbit representative t, we need to check whether $H = \langle g, f, t\rangle$ leaves the vector space $U = \langle v^H \rangle = \{v, v^t, (v^t)^g, (v^t)^{g^2}, ..., (v^t)^{g^{10}}\}$ invariant. If it does not, then H and any subgroup of the form $\langle g, f, t^h\rangle$ (for $h \in S$) is not isomorphic to $L_2(11)$. This is a quick check, and all 4012 involutions can be considered in a few minutes. However, as always when working in these large matrix groups, there is a small subtlety that must be considered. It is highly likely that the group $\langle g, f, t\rangle$ will actually be very large (all of G in most cases), so checking whether $U^H = U$ when this is the case will run for all of eternity. Although, it is clear that U is g-invariant by construction, and so it suffices to check $U^{\langle f,t\rangle} = U$ instead. As t always inverts f, this subgroup is always order 55 and so this is a very quick check. The code demonstrating this process is found in Section 9.2.13.

Only 3 of the 4012 orbit representatives give a subgroup $\langle g, f, t\rangle$ which leaves U invariant. In all 3 cases, the involution t stabilises v, meaning that U is 1-dimensional. By prior remarks, we know that these elements will not lead to maximal subgroups isomorphic to $L_2(11)$, however, they could eventually yield maximal

M_{11} subgroups. As such, we need to construct the 3 S-orbits of these elements. Given that all 3 elements live inside S, we can easily construct these orbits by using the `Class` function. Moreover, using `LMGClasses`, we see that the three orbits have sizes 591300, 2365200 and 2365200; these are small enough to loop through and consider each involution directly. For each element t in these classes, we check whether $\langle g, f, t\rangle$ has 22 composition factors on the 248-dimensional module (we are expecting 22 factors from the feasible decomposition). In all cases, we do not find a group with this property.

From the work conducted here, we can conclude that there are no subgroups of G which are isomorphic to $L_2(11)$ and follow feasible decomposition 4. As this was the only feasible decomposition that could lead to a potentially maximal $L_2(11)$, it must be that $L_2(11)$ is not a maximal subgroup of G. Furthermore, as this feasible decomposition was the only possibility for an extension of $L_2(11)$ to a maximal M_{11}, it must also be the case that M_{11} is not a maximal subgroup of G. This concludes the proof of Lemma 5.1.

6 $L_2(8) < E_8(2)$

Whilst working on $L_2(27) < E_8(3)$, I was asked to try and finish the analogous case in the $E_8(2)$ setting; this is the only work left to be completed in the classification of the maximal subgroups of $E_8(2)$. We remark that this work is completely analogous to the work conducted in Section 3.4; the methodology and algorithms used here are exactly the same as in Sections 3.2 - 3.4. Similarly, the code is essentially the same, with obvious modifications being made to consider the new field GF(2) and the new element x of order $2^3 - 1 = 7$. Furthermore, we use notation analogous to that seen in Section 2 for $E_8(2)$ and the computational setup is identical (but we are working in $GL_{248}(2)$ here instead). In particular, we shall use V to denote the minimal irreducible 248-dimensional adjoint GF(2)-module for $E_8(2)$. Also, for all of this work, we let $G \cong E_8(2)$.

The initial setup for this work is nearly identical to that seen in Lemma 3.19 (with the exception that we don't need to consider E_6 here). In particular, there were initially 8 cases, all corresponding to root systems of type $2A_2$. These are the same root systems described in Lemma 3.19, namely $\mathfrak{I} = \{\{1,3,5,6\},\{1,3,6,7\},\{1,3,7,8\},\{2,4,6,7\},\{2,4,7,8\},\{3,4,6,7\},\{3,4,7,8\},\{4,5,7,8\}\}$. Furthermore, each of these gives rise to two classes of $\langle x\rangle$-subgroups.

The cases not yet completed are: $\{1,3,5,6\},\{1,3,6,7\},\{3,4,6,7\},\{3,4,7,8\}$ (in some cases, partial work was done on these). Moreover, in each case, we only have one class of $\langle x\rangle$-subgroups to consider as the other class has already been completed. The rest of this section details the work completed by myself on these outstanding cases and also what is yet to be completed. Unless specified otherwise, we follow the notation used in Section 3. Much of the code used here is contained in Mahah's thesis [41] and can be obtained from the code in Sections 9.2.3 and 9.2.6 by making the appropriate modifications. As in Section 3, we do not mention `ActnGpDiff` nor `SetKeepZero` if they remain empty.

The Brauer character table and feasible decompositions for $L_2(8)$ (over GF(2)) can be found in [41]. There are 3 feasible decompositions which we must consider, these are:

$$28\varphi_1 + 14\varphi_2 + 5\varphi_3 + 8\varphi_4 \ \ (3A \to 3C, 7AC \to 7B, 9AC \to 9D),$$

$$30\varphi_1 + 15\varphi_2 + 4\varphi_3 + 8\varphi_4 \ \ (3A \to 3C, 7AC \to 7B, 9AC \to 9C),$$

$$34\varphi_1 + 17\varphi_2 + 2\varphi_3 + 8\varphi_4 \ \ (3A \to 3C, 7AC \to 7B, 9AC \to 9B).$$

The projective 8-dimensional Steinberg module is denoted by φ_3. Hence, by Lemma 3.7 and Lemma 2.5, any 2-group b that we encounter with the property that $\dim(C_V(\langle b, x\rangle)) > 5$ will only lead to non-maximal copies of $L_2(8)$. As in Section 3.4, any 2-group we find with this property is immediately discarded and not added to any set to speed up the process.

6.1 $\Phi = \{3, 4, 6, 7\}$

This case was partially completed, meaning we are left with various 2-groups that have not yet been considered. We divide these groups into 5 sets, which we shall label $\mathfrak{B}_1, \ldots, \mathfrak{B}_5$.

$\mathfrak{B}_1$

This set contains 8 groups, all of which have order 2^{68} and are such that $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. After running one loop of Code 9.2.4 on each group, we find 8 sets `BadSetNew`, all of size 8841. These groups have order $< 2^{40}$ and a Frattini quotient of the form $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_r$ where $r \leq 7$. In the following stage of the algorithm (again using 9.2.4), each group gave a `SetSub2` of size ≈ 600, with each iteration through these 600 groups taking around half a second. These 8 sets were partitioned and considered via parallel processing. No additional groups were added to `BadSetNew` after this stage, and we were left with 8 elementary abelian groups in `FinSub`, all of which have order 2^{27}. Upon running the code in Section 9.2.7, 19173961 minimal x-submodules are found for each group. Let f be any of the elementary abelian groups of order 2^3 corresponding to one of these minimal x-submodules, then $\dim(C_V(\langle f, x\rangle)) > 5$ and so by Lemma 3.7 and Lemma 2.5, any subgroup of the form $\langle f, x, t\rangle \cong L_2(8)$ will not be maximal in G.

$\mathfrak{B}_2$

Here we have 2 groups, both of which have order 2^{65} and have a Frattini quotient isomorphic to the direct sum of 7 irreducible x-modules. Using no pre-images, one iteration of the algorithm yields a `BadSetNew` containing 10402 groups. An initial inspection indicates that these groups all have order $< 2^{37}$ and a Frattini quotient of the form $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_r$ where $r \leq 7$. As such, it is sufficient to proceed in the algorithm without using pre-images. Many of these groups contributed very few groups to `SetSub2` (as low as 10), meaning we were able to partition them into larger subsets of size 500. Generally, iterating through the set `SetSub2` here was fast, with times of ≈ 0.2 seconds per iteration. After one more passing of the code, we found that the 10402 groups added nothing to `BadSetNew`, and 1 group of order 2^{24} to `FinSub`. Using Code 9.2.7, we find that this group has 2396745 minimal x-submodules and none of them correspond to a suitable elementary abelian group.

$\mathfrak{B}_3$

There is only 1 group to consider here, which we shall denote by b. We have that $|b| = 2^{65}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_8$. We use Procedure 3.11 with `CN=4` to get a `SetKeep` of size 585 and subsequently a `BadSetNew` of size 585 with 1 group also being added to `FinSub`. These 585 groups were partitioned into 11 subsets of size 50 and 1 subset of size 35 and then considered via parallel processing. On the 12 screens, the next passing of the algorithm (with no pre-images being used) gave sets `BadSetNew` of sizes 4050, 4050, 4050, 4050, 4050, 4050, 4050, 4050, 4050, 4050, 3862, 2702 and 1 group being added to `FinSub` on each screen. Running another pass of the algorithm on these sets terminates the process with no further groups being added to `FinSub`. Finally, there are 13 groups in `FinSub` to deal with, each having order 2^{24}. After running Code 9.2.7, we find that each group contains 2396745 minimal x-submodules and none of them yield a suitable elementary abelian group, thus finishing this case.

$\mathfrak{B}_4$

As in the previous section, there is only 1 group here, which we again denote by b. We have that $|b| = 2^{62}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_{12}$. By using Code 9.2.6 with `CN=4`, we obtain a `SetKeep` of size 585 and a `BadSetNew` of size 585 after one passing of the algorithm. These groups have order 2^{53} and have a Frattini quotient isomorphic to the direct sum of 9 modules. As such, to avoid `SetKeep` becoming too large, we must use Procedure 3.11 again. These 585 groups are partitioned into sets of size 5 and then considered in parallel with `CN=4` again. As before, each individual group yields a `BadSetNew` of size 585 (meaning there are now $585^2 = 342225$ groups to consider). Also, 1 group is added to `FinSub` on each screen. The groups at this stage of the algorithm have a Frattini quotient isomorphic to the direct sum of 6 modules or less, meaning that we no longer need to use pre-images in our calculations. The algorithm was restarted on all the screens with no further partitioning required. After this stage, no more groups were added to `BadSetNew` or `FinSub` on any of the screens. With each screen contributing one group to `FinSub`, we find there are 117 elementary abelian groups of order 2^{21} to consider. After running Code 9.2.7, which takes approximately 5 minutes to finish for each group, we find each group contains 299593 minimal x-submodules but none of them give any suitable elementary abelian groups.

$\mathfrak{B}_5$

This set contains 2340 groups of order 2^{56}, all of which have a Frattini quotient isomorphic to the direct sum of 9 irreducible x-modules. We split these 2340 groups into sets of 5 and then use Procedure 3.11 with `CN=4` on each set; this gives a `BadSetNew` of size 585 for each group (so 2925 groups on each screen). Upon brief inspection, it appears these groups have order 2^{47} and have a Frattini quotient isomorphic to the direct sum of 6 irreducible x-modules. As such, in the next pass of the algorithm no pre-images are required and no further partitioning is required. No additional groups are added to `BadSetNew` after the next stage. However, across all screens, there are a total of 256 groups in `FinSub` and also 106 groups in `ActnGpDiff`. As mentioned in Section 3.2.1, for any group $b \in$ `ActnGpDiff`, any elementary abelian groups of interest will lie inside $\Phi(b)$. We find that $|\Phi(b)| = 2^{29}$ for all $b \in$ `ActnGpDiff`, thus $\Phi(b)$ can be completely considered with only a few iterations of the algorithm and with minimal contributions to `BadSetNew`. After gathering all the Frattini subgroups of the 106 groups into a single `BadSub` and using Code 9.2.4, we find a further 1009 groups are added to `FinSub`, meaning there are now 1265 elementary abelian groups to deal with. These groups either have order 2^{24} / 2^{21} and 2396745 / 299593 minimal x-submodules respectively. After using Code 9.2.7, no suitable elementary abelian groups of order 2^3 are found here, thus concluding this case.

6.2 $\Phi = \{3, 4, 7, 8\}$

No prior work was completed on this case, hence we use Code 9.2.3 to create an initial `BadSub` and `FinSub`. We find that no groups are added to `FinSub` and 10 groups are added to `BadSub`. As before, we split these 10 groups into 5 sets $\mathfrak{B}_1, \ldots, \mathfrak{B}_5$.

$\mathfrak{B}_1$

There are 5 groups in this set and they have orders $2^{30}, 2^{53}, 2^{60}, 2^{47}, 2^{36}$. The groups of order 2^{30}, 2^{53} and 2^{60} have Frattini quotients isomorphic to the direct sum of 6, 6 and 7 irreducible x-modules respectively. After one pass of the algorithm on these 3 groups, 2288 groups are added to `BadSetNew` and 4631 groups are added to `FinSub`. The groups in `BadSetNew` have order either 2^{18} or 2^{21} with a Frattini quotient isomorphic to the direct sum of 5 modules. Running another pass of the algorithm returns an empty `BadSetNew` but an additional 823296 groups are added to FinSub. After using Code 9.2.7, we find a total of 4460544 minimal x-submodules, none of which correspond to a suitable elementary abelian subgroup.

Now we consider the groups of order 2^{47} and 2^{36}. Both of these have Frattini quotients isomorphic to the direct sum of 9 irreducible x-modules. As such, we must use pre-images and Code 9.2.6. In the 2^{36} case, we set CN=5 to obtain a SetKeep of size 4681 and a BadSetNew of the same size. For the group of order 2^{47}, we set CN=4 to obtain a smaller SetKeep of size 585 and a BadSetNew of size 585. In total, this gives a BadSetNew containing 5266 groups, all of which have a relatively small order. After another iteration of Code 9.2.4, no groups are added to BadSetNew or FinSub, thus concluding our work on $\mathfrak{B}_1$.

$\mathfrak{B}_2$

Here we have 2 groups, which have orders 2^{62} and 2^{76}. We first consider the group of order 2^{76}, which we label b. We have that $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_5$. After one iteration of Code 9.2.4, we get a set SetKeep of size 34 and subsequently a BadSetNew containing 39 groups. Of these groups, 5 have order 2^{39} and a Frattini quotient isomorphic to the direct sum of 9 irreducible x-modules. One iteration of Code 9.2.6 with CN=4 on these 5 groups yields a BadSetNew of size 2925. These groups have order 2^{30} and have a Frattini quotient consisting of 6 x-submodules. Each group adds approximately 300 vectors to SetKeep, with no groups being added to SetSub2. Hence after another pass of the algorithm, an empty FinSub and BadSetNew is returned. Now we turn out attention to the remaining 34 groups. The orders present here are $2^{40}, 2^{37}, 2^{41}, 2^{43}$ with all groups having a Frattini quotient consisting of 6 irreducible x-submodules. After one pass of the algorithm without using pre-images, we obtain a FinSub containing 41266 groups and a BadSetNew of cardinality 20160. The groups in BadSetNew have order 2^{18} and one more iteration of the algorithm on these groups gives us no further groups to consider. After using Code 9.2.7 on FinSub, a total of 61849658 minimal x-submodules are found but none correspond to an elementary abelian group of interest.

We now consider the remaining group $b \in \mathfrak{B}_2$ of order 2^{62}. We have that $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_9$. Setting CN=4 and using Code 9.2.6 gives a BadSetNew consisting of 585 groups. All groups contained here have a Frattini quotient consisting of ≤ 7 irreducible x-modules apart from one. This outlier has order 2^{39} and a Frattini quotient consisting of 9 irreducible x-modules. Using Code 9.2.6 with CN=3 returns a BadSetNew containing 73 groups (2^{33}, 7 modules), none of which output anything after another iteration of the algorithm. Returning to the other 584 groups, they are composed of 448 groups of order 2^{53}, 128 of order 2^{50} and 8 of order 2^{49}. After partitioning these groups into smaller sets and performing one iteration Code 9.2.4, a total of 184024 groups are added to BadSetNew. These groups appear to have order $\leq 2^{21}$ and a Frattini quotient corresponding to a direct sum of 5 irreducible x-modules. We remark that these properties were not calculated for all groups here, as this would take far too long. After another passing of Code 9.2.4 (and partitioning these groups onto thousands of screens), no additional groups are found.

$\mathfrak{B}_3$

There is only one group to consider here. We have that $|b| = 2^{73}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. Setting CN=3 and using Code 9.2.6 returns a BadSetNew containing 42 groups; we label this set as **B**. Eight of these groups have order 2^{21} and a Frattini quotient which decomposes into a direct sum of 6 irreducible x-modules. Setting CN=2 and using Code 9.2.6 gives a BadSetNew containing 72 groups. Running the algorithm as normal on these 72 groups gives sets BadSetNew of sizes 627, 4861 and 16323 before terminating. In total, 206 groups are added to FinSub, none of which contain any suitable elementary abelian groups.

There are 17 groups in **B** which have order 2^{27} and have a Frattini quotient isomorphic to the direct sum of

6 irreducible x-modules. Using Code 9.2.6 with `CN=4` gives a `BadSetNew` containing 7555 groups. After one iteration of the Code 9.2.4, an empty `BadSetNew` is returned (considering all 7555 groups took less than 12 hours). A total of 202 groups are added to `FinSub`, but after using Code 9.2.7, we find nothing of interest.

There is a single group of order 2^{63} in $\mathbf{B}$ with $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. Setting `CN=4` outputs a `BadSetNew` containing 104 groups. Only 6 of these groups pose a problem to us, with the other 98 returning an empty `FinSub` and `BadSetNew` after another iteration of the algorithm. For these 6 problem groups we set `CN=5` to get a `BadSetNew` of size 28086 which is reduced to 0 after one more pass of the algorithm. `FinSub` remains empty throughout this process.

Next we consider the 7 groups of order 2^{64} in $\mathbf{B}$, all of which are such that $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. After three iterations of Code 9.2.4, we find sets `BadSetNew` of sizes 238, 130761 and 0, thus terminating the process. Moreover, no groups are added to `FinSub`. We remark that parallel processing was used extensively here.

Finally, we deal with the last remaining 9 groups in $\mathbf{B}$, which all have order 2^{59} and a Frattini quotient isomorphic to the direct sum of 8 or 9 irreducible x-modules. Setting `CN=5` gives a `SetKeep` containing approximately 700 vectors for each group, which then outputs a `BadSetNew` of cardinality 5791, containing groups of orders $2^{24}, 2^{40}, 2^{36}, 2^{44}, 2^{33}, 2^{41}, 2^{43}$. Working through this set took months of computation time and involved using Procedure 9.2.6 and Code 9.2.4 extensively. We briefly summarise the findings here. Any group we encountered with a Frattini quotient isomorphic to a direct sum of more than 7 modules was considered by setting `CN=4`, which gave a `BadSetNew` of size 585 for each group. All other groups were considered using the standard algorithm and no preimages. After one pass of the algorithm, we obtain a `BadSetNew` containing 1479601 groups and a `FinSub` of size 786. Luckily, all groups found here have order $\leq 2^{30}$ meaning that one more pass of the algorithm was enough to return an empty `BadSetNew`. Similarly, the largest group in `FinSub` had order 2^{12}, meaning that all groups were quickly considered using Code 9.2.7. No suitable elementary abelian groups were found.

$\mathfrak{B}_4$

There is also one group to work with here. We have that $|b| = 2^{88}$, $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_8$, and one pass of Code 9.2.6 with `CN=3` yields a `BadSetNew` of size 122. The orders of these groups and the multiplicity in which they occur (which is given in brackets) are 2^{82} (56), 2^{65} (1), 2^{27} (8), 2^{74} (8), 2^{76} (8), 2^{41} (24), 2^{44} (17). Dealing with these groups took months of computation time and we do not give all the intricate details here. However, the process followed was the same as what has already been described, involving parallel processing with a combination of Code 9.2.4 and 9.2.6. As mentioned in the previous section, any group we encountered with a Frattini quotient isomorphic to a direct sum of more than 7 modules was considered by taking preimages by setting $\texttt{CN} \geq 2$. Thousands of separate screens were used to complete this process, and the accumulated sizes of the sets `BadSetNew` and `FinSub` are as follows. From the original 122 groups, we get sets `BadSetNew` of sizes 150703, 177951, 385250 and then 0. Overall, 318602 groups were added to `FinSub`. Upon running Code 9.2.7, a total of 1233905563 minimal x-submodules were found but none gave a suitable elementary abelian group. Furthermore, 11598 groups were added to `SetKeepZero`. As described in Section 3.2.1, in these cases, any elementary abelian groups of interest will lie inside the Frattini subgroups. Upon gathering the distinct Frattini subgroups, we find only 33 groups to consider, all of which have order 2^{32}. One iteration of Code 9.2.6 with `CN=2` returns an empty `BadSetNew`, thus finishing this case.

$\mathfrak{B}_5$

The final set we consider contains one group of order 2^{100}, the largest group encountered so far. Luckily, the Frattini quotient is isomorphic to a direct sum of only 4 irreducible x-modules, making this case more manageable. As with $\mathfrak{B}_4$, we give a summary of our findings here. Given that there are only 4 irreducible x-modules here, we need not take preimages, and one pass of Code 9.2.4 outputs a `BadSetNew` containing 52 groups. The orders of these groups and their multiplicity in the set are 2^{72} (2), 2^{75} (7), 2^{30} (8), 2^{69} (1), 2^{33} (3), 2^{42} (19), 2^{27} (1), 2^{62} (2), 2^{67} (9). After much parallel processing and iterations of Code 9.2.6 and 9.2.4, `BadSetNew`'s of sizes 82165, 645087 and 2429 are outputted before an empty set is returned. Moreover, 8536 groups are added to `SetKeepZero`. After using Code 9.2.4 but with `BadSub` defined to be the set of all Frattini subgroups of these 8536 groups, an empty `BadSetNew` is returned after only one pass of the algorithm. In total, 1547157 elementary abelian groups were added to `FinSub` throughout this process. After using Code 9.2.7, a total of 136174961 minimal x-submodules were found, but none gave a suitable elementary abelian group.

6.3 $\Phi = \{1, 3, 5, 6\}$

As with $\Phi = \{3, 4, 7, 8\}$, no prior work has been conducted on this case. After running the initial phase of the algorithm, 2 groups are added to `FinSub` and 11 to `BadSub`. Upon running Code 9.2.7 on the 2 groups in `FinSub`, we find 2696338 minimal x-submodules but none give a suitable elementary abelian group. We split the 11 groups in `BadSub` into 4 sets $\mathfrak{B}_1, \ldots, \mathfrak{B}_4$.

$\mathfrak{B}_1$

This set contains all cases which do not require the use of preimages. There are 2 groups to consider here, which have orders 2^{65}, and 2^{53}. The group of order 2^{65} has a Frattini quotient isomorphic to the direct sum of 9 irreducible x-modules. We remark this is the only 9 module case where using no preimages was a viable option. The other group in `BadSub` has a Frattini quotient of the form $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_6$. After one pass of Code 9.2.4 on these 2 groups, 6336 groups are added to `BadSetNew` and 211 to `FinSub`. Upon a brief initial test, the groups in `BadSetNew` appear to have order $< 2^{30}$ and a Frattini quotient isomorphic to 6 modules. As such, it is sufficient to run Code 9.2.4 in parallel on sets of size 500; this process was completed in approximately 24 hours. No additional groups were added to either `FinSub` or `BadSetNew`, meaning there are now only 211 elementary abelian groups in `FinSub` to deal with. After using Code 9.2.7, a total of 60517786 minimal x-submodules were found but none corresponded to a suitable elementary abelian group of order 8.

$\mathfrak{B}_2$

This set contains 6 groups; the orders of these groups and the number of x-modules in the direct sum corresponding to the Frattini quotient (which is given in brackets) are 2^{67} (9), 2^{53} (10), 2^{44} (7), 2^{35} (10), 2^{41} (9), 2^{36} (8). For the groups of orders 2^{53}, 2^{35} and 2^{41}, we set `CN=4` and use Code 9.2.6 to obtain a `BadSetNew` containing 585 groups in each case. 1170 of these groups have a Frattini quotient consisting of 6 modules, and one further pass of the algorithm (with no preimages being used) reveals that there is nothing to be found in these cases. The remaining 585 groups have a Frattini quotient consisting of 7 modules and we use Procedure 3.11 to deal with these as they contribute too many groups to `BadSetNew` when preimages are not used. As such, we set `CN=2` and use Code 9.2.6 on these groups to obtain a `BadSetNew` containing 5282 groups. A further pass of the algorithm at this point returns an empty `BadSetNew` with no contributions being made to `FinSub` at any point.

We now turn our attention to the groups of order 2^{44} and 2^{36}, which we gather into a single `BadSub`. We set `CN=3` and use Code 9.2.6 to obtain a `BadSetNew` containing 146 groups before performing another pass of the algorithm (without preimages) to find an empty `BadSetNew` is returned. In total, 12848 groups are added to `FinSub`. Running Code 9.2.7 reveals 60218193 minimal x-submodules, none of which lead to an appropriate elementary abelian group.

Finally, all that is left to consider is the group of order 2^{67}. With the Frattini quotient being isomorphic to a direct sum of only 6 irreducible x-modules, we need not use preimages straight away. After one pass of Code 9.2.4, we arrive at a set `BadSetNew` containing 136 groups. Of these groups, 48 have order 2^{29} and a Frattini quotient corresponding to 8 modules. In these cases, we set `CN=2` and use Code 9.2.6 whilst with the other 88 groups, we need not use preimages. After a further pass of the code, a `BadSetNew` containing 432 groups is returned. Unfortunately, these groups all have a Frattini quotient isomorphic to a direct sum of 7 irreducible x-modules and contribute thousands of groups each to `BadSetNew` when preimages are not used. As such, we must use preimages again. Setting `CN=2` as before yields a `BadSetNew` of size 3888. One more pass of the algorithm without preimages finally returns an empty `BadSetNew`. We remark that it will almost certainly have been more efficient to consider `CN=4` instead of `CN=2` in the first instance, as it may have lead to a `BadSetNew` where we need not use preimages again. However, given that the groups in question here had order $< 2^{30}$, this process was still very quick. In total, 16153 groups are added to `FinSub` and after running Code 9.2.7, 9434880 minimal x-submodules are found but none give any interesting elementary abelian groups.

$\mathfrak{B}_3$

There is only one group b to consider here. We have that $|b| = 2^{74}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_9$. After setting `CN=5` and using Code 9.2.6, we find that 841 vectors are added to `SetKeep` and subsequently 841 groups are added to `BadSetNew`. These groups all have order 2^{59} and a Frattini quotient corresponding to 8 modules. Setting `CN=4` yields a `BadSetNew` of size 585 for each group and this appears to be the most efficient method, given that Lemma 3.18 does not appear to reduce the amount of groups we need to consider at any stage. As such, we use parallel processing with each screen containing approximately 4 of these groups. After setting `CN=4` and using Code 9.2.6, each of the 4 groups contribute 2340 groups to `BadSetNew`, meaning 491985 groups are added to `BadSetNew` after one pass of the algorithm. These groups appear to have order 2^{24} and a Frattini quotient corresponding to 5 modules, meaning preimages are no longer required. After a further pass of Code 9.2.4, a total of 2321343 groups are added to `BadSetNew` across all the screens before an empty `BadSetNew` is returned on the next loop. Furthermore, 12671 groups are added to `SetKeepZero` but this only leads to 24 distinct Frattini subgroups. These subgroups have order 2^{32} and a Frattini quotient corresponding to 12 modules. Setting `CN=2` and using Code 9.2.6 leads to a `SetKeep` containing 9 vectors for each of the 24 groups, however no further groups are added to `BadSetNew`. Throughout this entire process, 33 groups are added to `FinSub` but after running Code 9.2.7, no suitable groups are found.

$\mathfrak{B}_4$

We conclude this section by discussing the work conducted so far on the 2 remaining groups from the original `BadSub`. Firstly, we consider the group b such that $|b| = 2^{97}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_5$. After one pass of Code 9.2.4, a `BadSetNew` containing 37 groups is outputted. The orders of these groups and the multiplicity in which they occur are 2^{44} (1), 2^{35} (1), 2^{41} (1), 2^{74} (10), 2^{80} (24). Some of the groups of smaller order can be dealt with quickly. Indeed, we first consider the groups of order 2^{44} and 2^{41}, which have Frattini quotients corresponding 7 and 9 irreducible x-modules respectively. After gathering these 2 groups into a single `BadSub` and running two loops of Code 9.2.6 with `CN=3`, we find sets `BadSetNew` of size 146 and 0. A total of 12848 groups are added to `FinSub`, the majority of which have order 2^{15}. After using Code 9.2.7, 60141488 minimal x-submodules are found but none correspond to a suitable elementary abelian group. Similarly, the group of order 2^{35} has a Frattini quotient which corresponds to 10 x-submodules and setting `CN=4` yields a `BadSetNew` containing 585 groups. Unfortunately, preimages must be used again here so we set `CN=2` and use Code 9.2.6 to obtain a `BadSetNew` containing 5264 groups, with each of the original 585 groups contributing approximately 9 groups. One more pass of the algorithm returns an empty `BadSetNew`, thus terminating this process. Only 1 group with 37449 minimal x-submodules is added to `FinSub`, but as before nothing interesting is to be found here. No further work has been conducted on any of the other groups mentioned in this case. Although, by using Code 9.2.5, the 24 groups of order 2^{80} can be reduced to 6 groups.

The final group b we consider in this section is such that $|b| = 2^{86}$ and $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. Setting `CN=3` and using Code 9.2.6 outputs a `BadSetNew` containing 107 groups. One of these groups has order 2^{35} and a Frattini quotient which corresponds to 10 irreducible x-modules. In this case, we set `CN=4` and use Code 9.2.6 to get a `BadSetNew` containing 628 groups. After partitioning these groups into subsets of size 50 and using `CN=2` to loop through the algorithm again, a total of 5652 groups are added to `BadSetNew`. After one more loop, with no preimages, an empty `BadSetNew` is returned. A further 17 groups of the 107 have order either 2^{27} or 2^{35} and a Frattini quotient of the form $b/\Phi(b) \cong V_1 \oplus \cdots \oplus V_7$. By using Lemma 3.18, these 17 groups can be reduced to a set of 11 conjugacy representatives. After gathering these 11 groups onto one screen, setting `CN=3` and using Code 9.2.6 outputs a `BadSetNew` containing 876 groups before an empty `BadSetNew` is returned on the next iteration of the algorithm. At no point were any groups added to `FinSub`. The remaining 73 groups of the 107 have order either 2^{77} or 2^{80} and a Frattini quotient which corresponds to 9 irreducible x-modules. Using Lemma 3.18, we can reduce these 73 groups to 19 conjugacy representatives. Only 4 of these 19 groups have been completed, and we consider them as follows. Initially, we set `CN=5` to get 4 sets `BadSetNew` of sizes 4681, 4681, 4682, 4682. Following this, the algorithm was ran as normal, without preimages. After 3 successive loops of Code 9.2.4, we find `BadSetNew`'s containing 158722, 60754, 0 groups (this is a cumulative total across the 4 sets just found). In total, 16 groups are added to `FinSub`, but none yield any suitable elementary abelian groups. We remark that completely considering only 10 groups from any of the 4 initial `BadSetNew`'s took upwards of 400 hours of computation time.

6.4 $\Phi = \{1, 3, 6, 7\}$

This case has been partially completed and as such, we are left with various groups obtained throughout the process. As before, we split these groups into 4 distinct sets $\mathfrak{B}_1, \ldots, \mathfrak{B}_4$.

$\mathfrak{B}_1$

There are 674 groups of varying order to consider here. All groups have order $\leq 2^{56}$ and a Frattini quotient corresponding to 7 modules or less. It is sufficient to use Code 9.2.4 with no preimages until an empty `BadSetNew` is returned. Only 2 iterations of the algorithm are required, with a `BadSetNew` of size 204384 being output before the algorithm terminates. A total of 29960 groups are added to `FinSub`. After using Code 9.2.7, 121175880 minimal x-submodules are found but none give a suitable elementary abelian group of order 8. We remark that these groups were partitioned into sets of size 10 and considered via parallel processing; it took approximately 50 days of computation time for the algorithm to run to completion.

$\mathfrak{B}_2$

Here we have 5 groups to consider, we shall label them as $b_1, \ldots, b_5$. Firstly, we have $|b_1| = 2^{41}$ and $b_1/\Phi(b_1) \cong V_1 \oplus \cdots \oplus V_7$. By setting `CN=6` and using Code 9.2.6, we arrive at an empty `BadSetNew` and a `FinSub` containing 1581 groups. After running Code 9.2.7, no suitable elementary abelian groups are found.

Next we have $|b_2| = 2^{39}$ with $b_2/\Phi(b_2) \cong V_1 \oplus \cdots \oplus V_9$. One loop of Code 9.2.6 with `CN=4` outputs a `BadSetNew` containing 585 groups and in the following iteration of the algorithm (with no preimages being used), an empty `BadSetNew` is returned. No groups are added to `FinSub`.

For b_3 we have $|b_3| = 2^{65}$ and $b_3/\Phi(b_3) \cong V_1 \oplus \cdots \oplus V_9$. Setting `CN=5` and using Code 9.2.6 yields a `BadSetNew` of size 4681. These groups have order 2^{53} and a Frattini quotient corresponding to 6 x-modules. Moreover, initial testing suggests that they each have a small `SetKeep` containing approximately 30 vectors. Hence, using Code 9.2.4 without preimages is sufficient. After two loops, sets `BadSetNew` of sizes 23645 and 0 are output. Only 1 group is added to `FinSub`, and Code 9.2.7 reveals nothing suitable is to be found here.

Similarly, we have that $|b_4| = 2^{62}$ with $b_4/\Phi(b_4) \cong V_1 \oplus \cdots \oplus V_9$. Setting `CN=3` gives a `SetKeep` containing 73 vectors and subsequently a `BadSetNew` of size 73. These groups have order 2^{56} and a Frattini quotient corresponding to 7 modules. As with b_3, these groups have a small `SetKeep`, meaning that using Code 9.2.4 without preimages is feasible. After two iterations the algorithm terminates, with sets `BadSetNew` of sizes 3800 and 0 being output. No groups are added to `FinSub`.

Finally, we consider b_5. We have that $|b_5| = 2^{67}$ and $b_5/\Phi(b_5) \cong V_1 \oplus \cdots \oplus V_6$. After 1 loop of the algorithm without preimages, a `BadSetNew` containing 136 groups is obtained. In the majority of these cases, it is sufficient to proceed without using preimages until termination. However, 48 of these 136 groups have a Frattini quotient corresponding to 8 x-modules, so preimages must be used here. After gathering these 48 groups together into a single `BadSub` and setting `CN=3`, we get a `BadSetNew` containing 3504 groups. After one further loop of the algorithm without preimages, an empty `BadSetNew` is returned. Throughout this process, 9625 groups are added to `FinSub`. Running Code 9.2.7 reveals 65534625 minimal x-submodules inside these groups, however none of them correspond to a suitable elementary abelian group.

$\mathfrak{B}_3$

Here we have 201 groups to deal with. There is 1 group of order 2^{53}, 88 groups of order 2^{56}, and 112 groups of order 2^{59}. These 201 groups are partitioned into 39 subsets of size 5 and 1 subset of size 6. For each of these subsets, we set `CN=4`, which results in each individual group contributing 585 groups to `BadSetNew`. Consequently, we are left with 40 sets `BadSetNew`, 39 of which contain 2925 groups and 1 contains 3510 groups. Each of these sets must be partitioned and considered in parallel; typically, we were able to partition these sets into subsets of size 100. However, in some cases, subsets of size 20 had to be used due to the speed of the process. Across all screens used to consider these 40 sets, a monumental total of 16227192 groups are added to `BadSetNew`. Only two loops of the algorithm were performed before an empty `BadSetNew` was returned. In the vast majority of cases, no preimages were used. Anytime a 2-group with a Frattini quotient corresponding to more than 7 irreducible x-modules was encountered, Code 9.2.6 was used with `CN=3`. Furthermore, a total of 323 elementary abelian groups were added to `FinSub`, but none gave anything suitable. The work described here was extremely computationally challenging and took months of computation time to complete.

$\mathfrak{B}_4$

There is only 1 group to deal with here, but as we shall see this does not mean this can be done quickly. The group in question has order 2^{94} and a Frattini quotient corresponding to 7 irreducible x-modules. After setting `CN=4` and performing one loop of Code 9.2.6, a `BadSetNew` containing 598 distinct groups is returned. Using Lemma 3.18 and Code 9.2.5, we trim this set down to 100 conjugacy representatives. We briefly summarise the work conducted so far on these groups.

There are 7 groups of orders 2^{46}, 2^{46}, 2^{50}, 2^{30}, 2^{53}, 2^{44}, 2^{33} which are all dealt with in a similar way. For the groups of order 2^{46}, we set `CN=4` and perform one loop of Code 9.2.6 followed by one standard iteration of the algorithm to obtain `BadSetNew`'s of size 9362 and 0. For the groups of orders 2^{50}, 2^{30}, 2^{53} and 2^{33}, we set `CN=4` to create a `BadSetNew` of size 585 for each group. In all cases, one loop of the algorithm returns an empty `BadSetNew` after this stage. Lastly, for the group of order 2^{44}, we do not use preimages. We find that 12848 groups are added to `FinSub`, but none of the 60141488 corresponding minimal x-submodules give any suitable elementary abelian groups.

We have 2 groups of order 2^{82} which both have a Frattini quotient corresponding to 6 irreducible x-modules. Without using preimages, one loop of the algorithm returns a `BadSetNew` containing 664 unique groups. Using Code 9.2.5, we are able to reduce this set down to 533 groups. By considering these groups in parallel using Code 9.2.4 and 9.2.6, we find sets `BadSetNew` of sizes 212102 and 657 before the algorithm terminates. Overall, 4739 groups are added to `FinSub` which corresponds to 46398811 minimal x-submodules, but none give any suitable elementary abelian groups.

There are 3 groups of order 2^{77}, 2 of which have a Frattini quotient corresponding to 7 x-modules whilst for the other group it is 9 x-modules. The 9 module case remains outstanding, however the other groups have been completed. Indeed, we gather these 2 groups together into a single `BadSub` and use Code 9.2.4 without preimages. We find that a `BadSetNew` of size 722 is outputted, which can be reduced to 119 conjugacy representatives by using Lemma 3.18. The orders of these groups and the multiplicity in which they occur are 2^{36} (3), 2^{46} (13), 2^{47} (69), 2^{50} (34). Using a combination of Code 9.2.4 and 9.2.6, we find `BadSetNew`'s of sizes 52413, 47282, 0. Anytime a group with a Frattini quotient corresponding to more than 7 irreducible x-modules was encountered, preimages were used with either `CN=3` or `CN=4`. 11 distinct groups are added to `FinSub`, which gives a total of 411939 minimal x-submodules to consider, however none give a suitable elementary abelian group.

There are 16 groups of order 2^{80} which are split into 8 groups with a Frattini quotient corresponding to 5 x-modules and 8 groups with a Frattini quotient corresponding to 8 x-modules. Considering first the 5 module groups, using Code 9.2.4 with no preimages outputs a `BadSetNew` containing 82 groups. Using Lemma 3.18 and Code 9.2.5, we find this can be reduced to 16 groups, which have orders 2^{35}, 2^{56}, 2^{59} or 2^{62}. For 14 of these groups (all groups which have order not equal to 2^{56}), we use Code 9.2.6 with `CN=5` and find that a total of 56437 are added to `BadSetNew` before an empty `BadSetNew` is returned on the next loop of the algorithm. Completing these two loops of the algorithm took as long as 450 hours for some of these groups, with all of them being considered in parallel on their own screens. The remaining 2 groups can be considered without the use of preimages. Indeed, 1186 groups are added to `BadSetNew` before the algorithm terminates. Overall, 19 groups are added to `FinSub`, none of which contain any suitable elementary abelian groups. The original groups of order 2^{80} which correspond to 8 x-modules remain outstanding.

Finally, we consider remaining 72 groups of the original 100 which all have order 2^{85}. We remark that of the original 598 groups that we found (before being reduced to 100 groups using conjugacy), 504 had order 2^{85}. These 504 groups were reduced to 72 by using Code 9.2.5 in parallel, as searching for conjugacy relations on a set of this size is not feasible. As such, the 504 groups were partitioned into subsets containing approximately 50 groups and then Code 9.2.5 was used on each individual set on different screens. After this process was completed on all subsets, the conjugacy representatives were taken from 2 separate screens and combined into a single set. Then, Code 9.2.5 was used again; this was repeated until we were left with only one set of conjugacy representatives.

After one iteration Code 9.2.4, the 72 groups contribute 1436 groups to `BadSetNew`. There is 1 group of order 2^{44} and another of order 2^{33}; these groups have a Frattini quotient corresponding to 7 and 8 modules respectively. Setting `CN=4` and running one loop of Code 9.2.6 followed by a standard iteration of the algorithm without preimages yields sets `BadSetNew`'s of sizes 1170 and 0. A total of 24107 groups are added to `FinSub`, which give 60153923 minimal x-submodules. As before, none of these give any suitable groups.

Of the 1436 groups, 217 have order 2^{59} and a Frattini quotient corresponding to 11 modules. Using Code 9.2.5, we can trim this set down to 91 conjugacy representatives. Each of these groups were loaded onto their own individual screen and Code 9.2.6 with `CN=5` was used to create a `BadSetNew` containing 4681 groups in all cases (so a total of 425971 groups). After one loop of the algorithm without preimages (these groups have order 2^{50} and a Frattini quotient corresponding to 7 modules), no further groups are added to `BadSetNew` on 90 of the 91 screens. In one of the sessions, we find 9 groups which have order 2^{38} and a Frattini quotient corresponding to 8 irreducible x-modules. In these cases, we set `CN=2` and use Code 9.2.6 to obtain a `BadSetNew` containing 81 groups. One more loop of the algorithm without preimages returns an empty `BadSetNew` here. Only 8 distinct groups are added to `FinSub`, but none contain any suitable elementary abelian groups. From start to finish, completely considering any of these 91 groups took anywhere from 400 to 950 hours of computation time.

Finally, we consider the remaining 1217 groups, which all have a Frattini quotient corresponding to 6 irreducible x-modules. At first, we attempted to deal with these groups directly by considering them in pairs via parallel processing. Unfortunately, this process was very slow. However, 170 of these groups were completed this way in 2 iterations of the algorithm, with a total of 64249 groups being added to `BadSetNew` before an empty `BadSetNew` was returned. We then attempted to reduce the size of this set using Lemma 3.18, but this was also extremely slow, even when parallelised as described before. We were able to reduce the set of 1047 groups down to 975 groups using conjugacy, but this process was stopped as it was taking too long. No further work has been conducted here.

6.5 Outstanding work

In this section, we outline the uncompleted work from Sections 6.3 and 6.4. For $\Phi = \{1, 3, 5, 6\}$, we have:

- 14 groups of order 2^{80} and 8 groups of order 2^{74}, all of which have a Frattini quotient corresponding to 9 irreducible x-modules.
- 9 groups of order 2^{74}, all of which have a Frattini quotient corresponding to 15 irreducible x-modules.

For $\Phi = \{1, 3, 6, 7\}$, we have:

- 919 groups of order 2^{65} and 56 groups of order 2^{62}, all of which have a Frattini quotient corresponding to 6 irreducible x-modules.
- 1 group of order 2^{77} which has a Frattini quotient corresponding to 9 irreducible x-modules.

- 8 groups of order 2^{80}, all of which have a Frattini quotient corresponding to 8 irreducible x-modules.

7 The Maximal Tori of Finite Exceptional Groups of Lie Type

In this section we take an excursion from investigating $E_8(3)$ to look at the work conducted by myself, Parkin, Javed and Rowley involving the maximal tori of the finite exceptional groups of Lie type. The work in this section is the same as that shown in [42], however extra examples are given here (in Section 7.6) highlighting the work I conducted in more detail.

Let $G(q)$ be a twisted or untwisted finite exceptional group of Lie type over GF(q), either adjoint or simply connected, and let W be the Weyl group of $G(q)$. In this section, we compile an extensive data set on the maximal tori of $G(q)$. A systematic labelling is given, via the corresponding conjugacy classes of W, together with a complete list of the structures of the maximal tori in terms of their torsion coefficient decomposition. Additionally the associated admissible graphs and the structure or shape of centralizers of elements in W are given as well as compatible computer files [67]. Also various apparent inconsistencies in the current literature on simply connected groups are discussed and resolved.

7.1 Introduction

Maximal tori of finite groups of Lie type frequently play a role in many facets of these groups. For example, they feature in the representation theory of groups of Lie type, and, indeed, are very much centre stage in Kazhdan-Lusztig theory (see [16]). Also, they arise in investigations into group structure as every semisimple element of a group of Lie type is conjugate to an element of some maximal torus. An important fact, first observed in [5, E.II.1.2], is the one-to-one correspondence between conjugacy (twisted conjugacy) classes of the Weyl group and conjugacy classes of maximal tori. In [15], Carter presents a unified approach to the conjugacy classes of Weyl groups making it feasible to systematically describe maximal tori. Further, Carter introduced the admissible graph associated with a given (untwisted) conjugacy class and showed that this graph determines the characteristic polynomial on the standard vector space of the class representative. In [17], [18], he described the maximal tori for finite classical groups (see also Buturlakin and Grechkoseeva [12]).

The aim of the paper [42] is to present an extensive, and comprehensive, collection of data on the maximal tori for the untwisted and twisted finite exceptional groups of Lie type. Most of this information is gathered in tables appearing in Sections 7.4 and 7.5. These tables rest on a systematic labelling of the conjugacy classes of the Weyl groups which we use to christen the corresponding conjugacy class of maximal tori. We take a leaf from the Atlas [20] in the naming and ordering of conjugacy classes. So for conjugacy class representatives w_1 and w_2 of the Weyl group W, w_1^W precedes w_2^W in the labelling if w_1 and w_2 have the same order and $|C_W(w_1)| > |C_W(w_2)|$. We resolve the conundrum of what to do when $|C_W(w_1)| = |C_W(w_2)|$ with recourse to specific Coxeter group properties of the two conjugacy classes. Section 7.3 covers these issues in detail.

We now give an overview of what appears in the tables. The compact tabular data in Sections 7.4 and 7.5 give, for a particular Weyl group W, a W-(twisted) conjugacy class representative w decomposed into a product of involutions, its name, the associated admissible graph Γ_w (only in the untwisted case), the structure or shape of both $C_W(w)$ ($C_{W,\sigma}(w)$ when twisted) and the corresponding maximal torus in its torsion coefficient decomposition. Describing a maximal torus in this manner not only exposes its torsion coefficients, it also gives it in an invariant form. Moreover, the data in Tables 29, 30 and 37 has been verified to hold for both the adjoint and simply connected groups of type E_6, ${}^2E_6$ and E_7. That is, the torus structures are the same for the the adjoint and simply connected groups of these types. This cannot be guaranteed for a general group of Lie type - the files of [67] contain counterexamples in types B and C.

Incomplete lists of maximal tori for untwisted simply connected exceptional groups of Lie type are available in Deriziotis and Fakiolas [26], while Galt and Staroletov [33] [34] [35] [36] give lists which are, in some cases, partial for $G_2(q)$, ${}^2G_2(q)$, ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$. Various lists are also to be found in Deriziotis and Michler [27], Fleischmann and Janiszczak [29], [30], Gager [32], Haller [38], Lawther [47], Shinoda [69], and Shoji [70]. Partial lists of admissible graphs may also be found in [26] (for E_6, E_7, E_8) and [32]. We remark that in [32], complete lists of both maximal tori (in the twisted and untwisted case) and admissible graphs are given for both $G_2(q)$ and $F_4(q)$. Moreover, lists of admissible graphs and corresponding tori structures are given for E_6, E_7 and E_8, but only for the semi-Coxeter and Coxeter classes of W. Carter [15] gives complete lists of admissible graphs, however, these are not directly linked to maximal tori like they are here. Also, related work on the spectra of exceptional groups of Lie type is given in Buturlakin [9], [10] and [11]. To enable backward compatibility, where appropriate, our tables have a column giving the corresponding label for the maximal tori in [26] (see Section 7.3 for how we label those not appearing in [26]).

All decompositions exhibited in Tables 27-38 below have been computed by hand, and, sometimes, agree with the output from the Magma [6] function `TwistedTorusOrder` or the lists of [32], and [69] and [26]. The ambiguity here lies in the fact that there are many ways to describe a finite abelian group as a direct product of cyclic groups. To address this confusion in general, we tabulate the torsion coefficient decompositions of the maximal tori in groups of exceptional type. By this, we mean that we express each torus $T_w(q)$ as a direct product $\lambda_1(q)\times\lambda_2(q)\times\cdots\times\lambda_n(q)$, with $\lambda_1(q)|\lambda_2(q)|\dots|\lambda_n(q)$ where $\lambda_i(q)$ denotes a cyclic group of order $\lambda_i(q)$. In the case of [32], [69] and [26], torus structures are obtained by diagonalizing relation matrices. These papers present many torus structures corresponding to diagonal matrices which are not in the canonical form, as detailed in 24. The function `TwistedTorusOrder` in Magma obtains torus structures at all prime powers $q \leq 20$ and produces a collection of polynomials in q which yield the cyclic factors of the maximal tori for groups over the field of q elements, on substitution of any $q \leq 20$. The polynomials produced by this algorithm are not necessarily presented in the torsion coefficient decomposition, and there is no reason they should yield the cyclic factors for the maximal tori of groups over the field of $q > 20$ elements. Although, using Tables 27 to 38, we present authors have verified the validity of the output of `TwistedTorusOrder` in the untwisted exceptional cases. A more detailed discussion of how `TwistedTorusOrder` works is included in Section 7.6. In [67] we include a function which takes a fixed q as an argument and is hence guaranteed to produce accurate torus structures, in torsion coefficient decomposition, for all Lie types. Additionally, unlike the built-in Magma function, our function accepts twisted root data, and can therefore be used to compute maximal torus structures in a wider range of groups than was previously possible.

In the literature reviewed above there are many maximal torus structures which seem at first glance to be in disagreement. These apparent infelicities are mostly to be observed in the cases covered by Tables 32 and 38. We now give an overview of these apparent disparities. Using the labelling system described in Section 7.3, the structures of tori with labels 3C, 6G (E_6), 2D, 2E, 4D, 4E, 4G, 4H, 6M, 6O (E_7), and 2D, 3C, 4F, 4G, 4H, 4I, 6E, 6S, 6W, 6X, 8D, 8E, 8G, 9B, 18B (E_8) agree with both those outputted by Magma and those given in [26]. However, in all these cases neither Magma nor [26] gives the torsion coefficient decomposition like we do here. Additionally, there are a number of structures in [26], again not given in torsion decomposition form, for which it is not clear that they yield isomorphic groups to those presented here. These are listed in Table 24.

Table 24: Discrepancies.

$G(q)$		$T_w(q)$	Structure given here	Structure in [26]
$E_6(q)$	6B	6	$(q-1)^2 \times (q^4+q^3-q-1)$	$(q-1)\times(q^2-1)\times(q^3-1)$
$E_7(q)$	6C	18′	$(q-1)^3 \times (q^4+q^3-q-1)$	$(q-1)\times(q^2+q+1)\times(q^4-q^3+q-1)$
	6F	−18′	$(q+1)^3 \times (q^4-q^3+q-1)$	$(q+1)\times(q^2-q+1)\times(q^4+q^3-q-1)$
	8C	30	$(q^2+1)\times(q^5-q^4+q-1)$	$(q^3-q^2+q-1)\times(q^4+1)$
	8D	-30	$(q^2+1)\times(q^5+q^4+q+1)$	$(q^3+q^2+q+1)\times(q^4+1)$
	12A	26	$(q-1)\times(q^6+q^5+q^4-q^2-q-1)$	$(q^3-1)\times(q^4-1)$
	12B	-26	$(q+1)\times(q^6-q^5+q^4-q^2+q-1)$	$(q^3+1)\times(q^4-1)$
$E_8(q)$	6C	6″	$(q-1)^4 \times (q^4+q^3-q-1)$	$(q-1)^3\times(q^2-1)\times(q^3-1)$
	6D	−6″	$(q+1)^4 \times (q^4-q^3+q-1)$	$(q+1)^3\times(q^2-1)\times(q^3+1)$
	6K	63	$(q^4+q^2+1)^2$	$(q^2-q+1)\times(q^2+q+1)\times(q^4+q^2+1)$
	6N	25′	(q^3-2q^2+2q-1) $\times(q^5-q^4+q^3-q^2+q-1)$	$(q-1)\times(q^2-q+1)$ $\times(q^5-q^4+q^3-q^2+q-1)$
	6O	−25′	(q^3+2q^2+2q+1) $\times(q^5+q^4+q^3+q^2+q+1)$	$(q+1)\times(q^2+q+1)$ $\times(q^5+q^4+q^3+q^2+q+1)$
	12H	26	$(q-1)^2\times(q^6+q^5+q^4-q^2-q-1)$	$(q-1)\times(q^3-1)\times(q^4-1)$
	12I	-26	$(q+1)^2\times(q^6-q^5+q^4-q^2+q-1)$	$(q+1)\times(q^3+1)\times(q^4-1)$
	12L	35	$(q^2-1)\times(q^6+q^5+q^4-q^2-q-1)$	$(q^4-1)\times(q^4+q^3-q-1)$
	12M	-35	$(q^2-1)\times(q^6-q^5+q^4-q^2+q-1)$	$(q^4-1)\times(q^4-q^3+q-1)$

The motivation for including the structure or shape of $C_W(w)$ is as follows. Let $G(q)$ be one of our finite groups of Lie type, W its Weyl group and $w \in W$. Then for the corresponding maximal torus, $T_w(q)$, the group $N_{G(q)}(T_w(q))/T_w(q)$ contains a subgroup isomorphic to $C_W(w)$. For more on this consult Proposition 3.3.6 of [16]. A final remark on the tables is that the list of conjugacy class representatives is linked to compatible computer files in [67].

Section 7.2 gives sufficient background on the finite exceptional groups of Lie type, followed by details concerning the root systems of the relevant Weyl group. At the end of this section we explain the notation used in describing group structures. The theme of notational conventions continues in Section 7.3 where we discuss at length the labelling of Weyl group conjugacy classes. Then, as already mentioned, Sections 7.4 and 7.5 display the various tables of maximal tori data. Our final section gives some sample calculations and also contains a discussion of the algorithm for maximal tori used in Magma.

7.2 Background

Suppose q is a power of a prime p and let $G(q)$ be a finite group of exceptional Lie type, either twisted or untwisted. Then there exists some algebraic group G of the corresponding untwisted type over $\overline{\mathrm{GF}(q^a)}$ for some $a \in \{1, 2, 3\}$, and a Steinberg endomorphism σ on G under which $G(q) = G_\sigma = \{g \in G \mid \sigma(g) = g\}$. Let $W = W(G)$ be the Weyl group of G. Choose some σ-stable maximal torus T of G and let $X(T)$ be the character group thereof. As in [59, Proposition 22.2], σ acts on $X(T) \otimes \mathbb{R}$ as $\sigma_q \phi$, where ϕ induces a permutation on the positive roots of the root system Φ of W and also σ induces an automorphism of W. Either ϕ is trivial and $\sigma_q = F_q$ is the standard Frobenius morphism, or G has type G_2 and $p = 3$, B_2 or F_4 and $p = 2$, D_4, or E_6. In these cases an additional 'twisted' possibility for σ with ϕ nontrivial yields the finite exceptional groups of type ${}^2B_2$, ${}^2G_2$, ${}^3D_4$, ${}^2F_4$ and ${}^2E_6$ as G_σ.

Following [59, 25.1], there is a well-known one-to-one correspondence between the $G(q)$-conjugacy classes of σ-stable maximal tori of G and the ϕ-classes, often referred to as twisted conjugacy classes, of the Weyl group W. We recall that $x, y \in W$ belong to the same ϕ-class if and only if there is some $g \in W$ with $y = \phi(g)xg^{-1}$. Note that when ϕ is trivial the ϕ-classes are nothing but the conjugacy classes of W. The twisted centralizer of $w \in W$ is defined to be

$$C_{W,\sigma}(w) = \{x \in W \mid \sigma(x)wx^{-1} = w\}.$$

Let $T_w(q) = T_\sigma = \{g \in T \mid \sigma(g) = g\}$ for some σ-stable maximal torus T of G whose G-conjugate maximal tori correspond to the ϕ-class of $w \in W$. Then by [59, Proposition 25.2] we have that

$$|T_w(q)| = |\det_{X(T)\otimes\mathbb{R}}(\sigma w - I)|.$$

In particular, whenever $\sigma = F_q$, the order $|T_w(q)|$ can be obtained by evaluating the characteristic polynomial of w at q. Moreover, by [32, Proposition 1.4], the action of $\sigma w - I$ on $X(T) \otimes \mathbb{R}$ gives the relation matrix for the torus $T_w(q)$, so that we may use w to compute the structure of $T_w(q)$. We remark that the results above hold regardless of the isogeny type of $G(q)$, and can therefore be used to obtain torus structures for groups of both adjoint and simply connected type.

For convenience, we often refer to *the* maximal torus of $G(q)$ corresponding to *the* Weyl group element w, rather than to the conjugacy (twisted conjugacy) class of each. We shall denote by w_0 the longest Weyl group element throughout.

In [32] and [26], a method of determining the cyclic structure of maximal tori is presented. In [32] q is considered as an indeterminant and then the polynomial matrix $q.w - I$ is a relation matrix for the corresponding maximal torus $T_w(q)$. Diagonalizing this relation matrix over $\mathbb{Z}[q]$ then reveals the cyclic structure of the maximal torus, as the cyclic parts of $T_w(q)$ may be read off the diagonal. However, $\mathbb{Z}[q]$ is not a principal ideal domain, and so not every relation matrix we encounter is diagonalizable. More detail and examples of how we deal with this are given in Section 7.6. Using this approach we find the torsion coefficient decomposition for all maximal tori of the finite groups of exceptional Lie type. That is, the unique decomposition into a direct product of cyclic groups $\lambda_1(q) \times \lambda_2(q) \times \ldots \times \lambda_n(q)$ where $\lambda_i(q)$ divides $\lambda_{i+1}(q)$ for $1 \le i < n$. We remark yet again that the structures given in [26] are not necessarily in this unique form, and were verified only for groups of simply connected type.

Each $w \in W$ can be expressed as a product of involutions $w = w_1 w_2$ where $w_1 = w_{i_1} \cdots w_{i_m}$, $w_2 = w_{j_1} \cdots w_{j_n}$ are products of reflections corresponding to mutually orthogonal roots. This decomposition is minimal, in that $L(w) = m + n$, where $L(w)$ denotes the reflection length of w. For each such decomposition of w, we define a graph Γ_w consisting of $L(w)$ nodes corresponding to the roots $r_1, \ldots, r_{m+n}$ where two distinct roots r_i and r_j are joined by an edge of weight $n_{ij}n_{ji}$ where

$$n_{ij} = \frac{2(r_i, r_j)}{(r_i, r_i)} \in \mathbb{Z}.$$

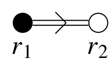
In Sections 7.2.2, 7.2.4, and 7.2.5, we define roots that allow us to express every class representative w in the above decomposition. See Section 7.3 for a detailed explanation of the notation we use to denote this decomposition in the later tables.

If w has graph Γ_w, then any conjugate of w also has graph Γ_w. We say such a graph is admissible if the nodes correspond to a set of linearly independent roots and each subgraph, which is a cycle, contains an even number of nodes. We define a subgraph to be a subset of the nodes and all edges joining such nodes. A cycle is a graph in which each node is joined to exactly two other nodes by edges of weight greater than zero. As remarked in [15], each conjugacy class of W has a corresponding admissible graph. However, this correspondence is not one-to-one; distinct classes may be associated with the same graph and an individual class may have more than one admissible graph. The graph Γ_w determines the characteristic polynomial of the action of w on $X(T) \otimes \mathbb{R}$ (see [15]). Consequently, by the results of [59], the admissible graph Γ_w determines the order $|T_w(q)|$.

Next we review the root systems we will encounter. In the Dynkin diagrams and graphs, in Sections 7.2.2 to 7.2.6, a black node denotes that the root is long, whilst a white node denotes that the root is short. The arrows all point from a long root to a short root. In a root system of type E_n, all roots are of equal length; hence we use a white node for all roots.

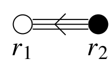
7.2.1 Roots for B_2

Let $\{r_1, r_2\}$ be the simple roots of a root system of type B_2.



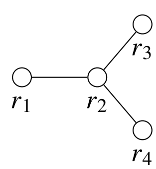
7.2.2 Roots for G_2

Let $\{r_1, r_2\}$ be the simple roots of a root system of type G_2, and define $r_6 = 3r_1 + 2r_2$.



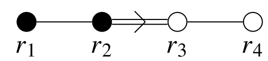
7.2.3 Roots for D_4

Let $\{r_1, r_2, r_3, r_4\}$ be the simple roots of a root system of type D_4 with Dynkin diagram



7.2.4 Roots for F_4

Let $\{r_1, r_2, r_3, r_4\}$ be the simple roots of a root system of type F_4 with Dynkin diagram

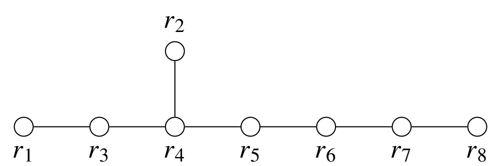


Now define

$$r_6 = r_2 + r_3 \qquad r_{13} = r_2 + 2r_3 + r_4 \qquad r_{14} = r_1 + 2r_2 + 2r_3$$
$$r_{20} = r_{14} + 2r_4 \qquad r_{21} = r_{20} + r_3 \qquad r_{22} = r_{20} + 2r_3$$

7.2.5 Roots for E_6, E_7 and E_8

Let G be a simple algebraic group of type E_6, E_7 or E_8 over the algebraic closure $\overline{GF(q)}$ with Weyl group $W = W(G)$. Let Φ be the root system of G. If G is of type E_8, we consider Φ to be embedded in $\mathbb{R}^8$ with orthonormal basis $\{\epsilon_i \mid 1 \le i \le 8\}$. Then Φ is generated by the simple roots $\{r_i \mid 1 \le i \le 8\}$. If G is of type E_6 (respectively E_7) then we consider Φ to be generated by the simple roots $\{r_i \mid 1 \le i \le 6\}$ (respectively $\{r_i \mid 1 \le i \le 7\}$). Root labels are chosen with respect to the following Dynkin diagram.



Now define

$$r_{17} = r_2 + r_3 + r_4 \qquad r_{18} = r_2 + r_4 + r_5 \qquad r_{29} = r_5 + r_6 + r_7 + r_8$$
$$r_{48} = r_{17} + r_4 + 2r_5 + r_6 \qquad r_{56} = r_4 + r_{17} + r_{29} \qquad r_{61} = r_{48} + r_6 + r_7$$
$$r_{69} = r_1 + r_{17} + r_{48} \qquad r_{74} = r_{48} + r_6 + 2r_7 + r_8 \qquad r_{82} = r_1 + r_3 + r_4 + r_5 + r_{61}$$
$$r_{97} = r_{17} + r_{82} \qquad r_{118} = r_8 + r_{61} + r_{97} \qquad r_{120} = r_7 + r_8 + r_{118}$$

We note that the roots r_{69}, r_{97} and r_{120} are the longest roots in E_6 , E_7 and E_8 respectively.

7.2.6 Root Graphs

Figures 1 and 2 illustrate the graphs described earlier in this section when considering the roots described in Sections 7.2.5 and 7.2.4 as vertices respectively. In particular, these graphs both contain all the admissible graphs listed in Tables 28 - 31 as subgraphs.

Figure 1: The graph of the roots for E_n (n=6,7,8) defined in Section 7.2.5.

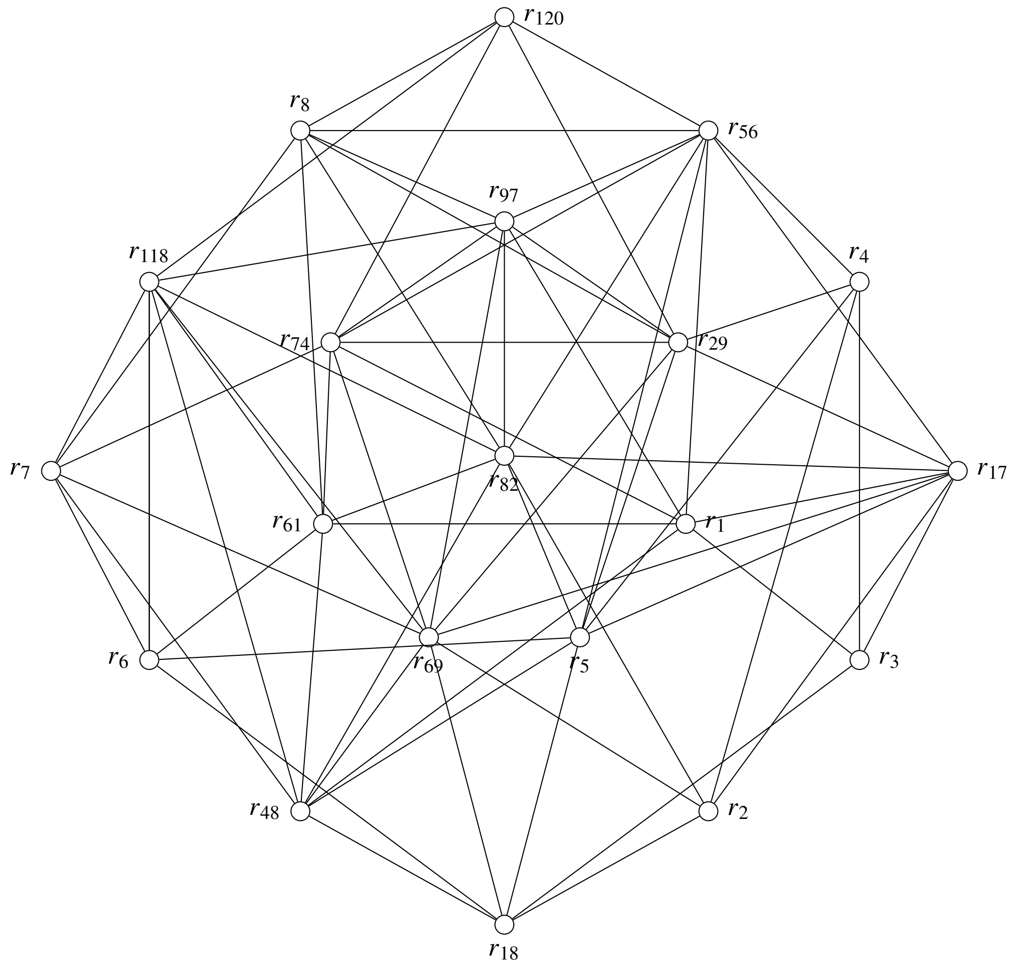
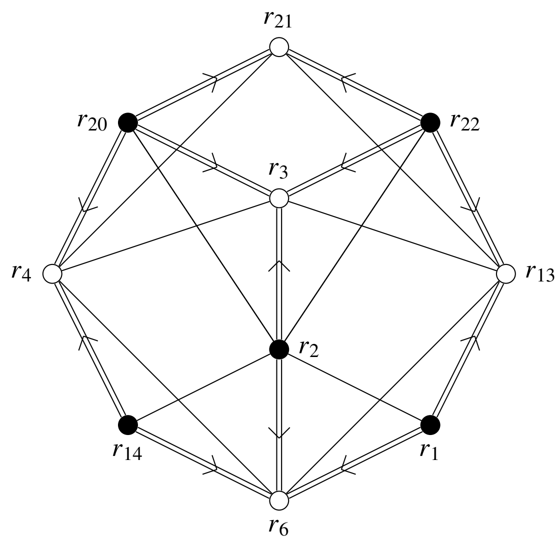


Figure 2: The graph of the roots for F_4 defined in Section 7.2.4.



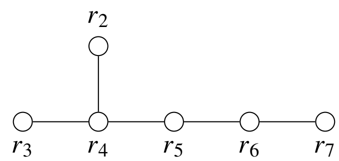
Finally we discuss the notation used to describe group structures. Broadly speaking we follow the Atlas [20] conventions and notation. The exceptions or additions are Sym(n) and Alt(n) which denote, respectively, the symmetric and alternating groups of degree n. While Dih(n) denotes the dihedral group of order n and $\mathrm{ASL}_n(q)$ the n-dimensional affine special linear group over GF(q). Further, we use the prefix C to denote the conformal version of a classical group.

7.3 Notational Conventions

First we consider the situation when $G(q)$ is untwisted. Let W denote the Weyl group of $G(q)$. The tables for Weyl groups of the untwisted groups are in Section 7.4 and are arranged as follows.

The first column gives a representative w for each conjugacy class of W, using $[i_1, i_2, \ldots i_m][j_1, j_2, \ldots j_n]$ to denote the decomposition $w = uv$ into a product of involutions where $u = w_{i_1} w_{i_1} \ldots w_{i_m}$ and $v = w_{j_1} w_{j_2} \ldots w_{j_n}$ in W. We use w_i to denote the reflection in the hyperplane orthogonal to the positive root r_i. In all cases, the set $\{i_1, i_2, \ldots i_m, j_1, j_2, \ldots j_n\}$ corresponds to a linearly independent set of roots that constitute the nodes of the associated admissible graph defined in the column headed Γ_w. The roots we use in such decompositions are listed in 7.2.2, 7.2.4 and 7.2.5. So for the Weyl groups of type, respectively, E_6, E_7 and E_8 we have $w_0 = r_{69}, w_0 = r_{97}$ and $w_0 = r_{120}$. If the root system Φ has roots of different lengths, then admissible graphs may also contain roots of different lengths. As in [32], we use a tilde to denote graphs consisting of only short roots. For example, the graph $\tilde{A}_3$ denotes the graph A_3 consisting of only short roots, whilst we simply write A_3 for the graph consisting of only long roots. See [15] and [32] for a detailed explanation of the notation used to describe the graphs Γ_w.

We give a short example illustrating the above notation. Consider the class 10E in $W(E_8)$. This has admissible graph D_6, which can be realised as a subgraph of the E_8 (and indeed the E_7) Dynkin diagram, by considering the following roots:



From this we can form two sets of mutually orthogonal roots, namely $[r_2, r_3, r_5, r_7]$ and $[r_4, r_6]$. Being a subset of the fundamental roots, this set of roots is linearly independent. Then, the class representative $w \in W(\mathrm{E}_8)$ of class 10E can be written as $w = uv$ where $u = w_2w_3w_5w_7$ and $v = w_4w_6$. This element can be explicitly constructed in MAGMA, in a permutation representation, using the command `w:=makew("E8",R,[2,3,5,7,4,6],"perm")` where `R := RootDatum("E8")` and the function `makew` may be found in the accompanying computer file [67]. A matrix representation can also be achieved here, by replacing `"perm"` with `"mat"`. A structure of the corresponding torus $T_w(q)$ may then be determined using the in-built MAGMA function `TwistedTorusOrder(R,w)` or by using the function `TorusStructure(R,0,w,q)` from [67] (note that `TwistedTorusOrder` takes `w` as a permutation and `TorusStructure` takes `w` as a matrix). We remark that, as this graph may also be realised in E_7, there is a conjugacy class of $W(\mathrm{E}_7)$ (namely 10C), that has admissible graph D_6 and representative given by the same product of reflections defined above.

In general, one must specify an isogeny type when calling `RootDatum()`. We note that, of the types considered in [42], isogeny need only be considered for E_6, ${}^2\mathrm{E}_6$ and E_7, see [15, 1.19].

Now moving back to the arrangement of the tables. The second column is the name of the conjugacy class. As mentioned in Section 7.1, in our tables we follow in the footsteps of the amazing ATLAS. Thus, those W-conjugacy classes having order n elements are labelled nA, nB, ... as in [20]. On one occasion we run out of alphabet letters for $\mathrm{E}_8(q)$ (see Table 31). There are 28 classes of elements of order 6 in W. So we use 6AA and 6BB. If for two such classes X_1 and X_2 we have $|X_1| = |X_2|$, we fall back on Coxeter group related properties to decide which class has precedence. The properties we use are related to the minimal Coxeter length elements of the classes. For X a W-conjugacy class, set $\min(X) = \min\{\ell(x) \mid x \in X\}$ and $X_{\min} = \{x \in X \mid \ell(x) = \min(X)\}$. Here $\ell(x)$ is the Coxeter length of x. The third column, headed Min, gives an ordered pair $(\min(X), |X_{\min}|)$ where X is a conjugacy class of W. Returning to X_1 and X_2 with $|X_1| = |X_2|$, if $\min(X_1) < \min(X_2)$ or if $\min(X_1) = \min(X_2)$ and $|X_{1\min}| < |X_{2\min}|$, then X_1 precedes X_2. These rules serve to order almost all the classes. How we deal with those for which $|X_1| = |X_2|$, $\min(X_1) = \min(X_2)$ and $|X_{1\min}| = |X_{2\min}|$ will be explained shortly after an example. In the $\mathrm{E}_7(q)$ table, the names 6L and 6M are given as 6L precedes 6M because the Min column tells us that $\min(6\mathrm{L}) = 5 = \min(6\mathrm{M})$ and $|6\mathrm{L}_{\min}| = 12 < 16 = |6\mathrm{M}_{\min}|$. All classes in $W(\mathrm{E}_6)$ and $W(\mathrm{E}_7)$ can be sorted in this way. We defer discussing $W(\mathrm{G}_2)$ and $W(\mathrm{F}_4)$ for the moment.

We now explain how we order the remainder of the classes in $W = W(\mathrm{E}_8)$. For $w \in W$, the excess of w (see [39]), $e(w)$, is defined as

$$e(w) = \min\{\ell(w) - \ell(x) - \ell(y) \mid w = xy,\ x^2 = y^2 = 1\}.$$

For X a W-conjugacy class, set

$$X^0_{\min} = \{x \in X \mid x \in X_{\min},\ e(x) = 0\}.$$

The remaining unordered classes of W are dealt with by considering $|X^0_{\min}|$. Let X_1 and X_2 be two W-conjugacy classes such that $|X_1| = |X_2|$, $\min(X_1) = \min(X_2)$ and $|X_{1\min}| = |X_{2\min}|$. Then, X_1 precedes X_2 in the tables if $|X^0_{1\min}| < |X^0_{2\min}|$. For such classes, the cardinality $|X^0_{\min}|$ is given in Table 25.

Table 25: Values of $|X^0_{\min}|$ for various classes X of $W(\mathrm{E}_8)$.

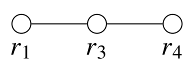
w	X	Min	$\lvert X^0_{\min}\rvert$
[3,2,5][4,7]	6P	(5,16)	4
[97,3,2,6][7]	6Q	(5,16)	16
[2,3,5,7][4,6]	10E	(6,32)	2
[1,4,6,8][7,120]	10F	(6,32)	8

The idea of choosing minimal sized sets (such as $|X_{\min}|$ and $|X^0_{\min}|$) to indicate pre-eminence follows the spirit of the Atlas which, in effect, uses the minimal size of conjugacy classes to decide the order of conjugacy classes whose elements have the same order.

The tables for $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$ and $\mathrm{E}_8(q)$ have an extra column labelled $T_w(q)$. In this column, we list numbers i corresponding to the torus T_i as given in [26]. When $w_0 \in Z(W)$, multiplication by w_0 sends a W-conjugacy class X to w_0X, also a W-conjugacy class. Hence in [26], when $X \neq w_0X$, only one of the corresponding maximal tori is listed. Assume X corresponds to T_i in [26] and $X \neq w_0X$. Then, as we wish to list all maximal tori, here we use $-i$ to denote the maximal torus associated to w_0X. In other words, if T_i is $T_w(q)$ for some $w \in W$ here, then T_{-i} corresponds to $T_{w_0w}(q)$. The use of the negative sign is prompted by the fact that when $w_0 \in Z(W)$, it is minus the identity matrix in the reflection representation.

Next we explain the occurrence of primes which decorate the entries of the fourth column in the $\mathrm{E}_7(q)$ and $\mathrm{E}_8(q)$ tables. Let $w \in W(\mathrm{E}_6)$ (respectively $W(\mathrm{E}_7)$) be such that the admissible graph of w is determined by roots orthogonal to r_7 (respectively r_8). Then, the torus with label i of $\mathrm{E}_6(q)$ (respectively $\mathrm{E}_7(q)$) gives rise to a torus which we label i' of $\mathrm{E}_7(q)$ (respectively $\mathrm{E}_8(q)$). The structure of the torus with label i' is given by $(q-1) \times T_w(q)$ where $T_w(q)$ is the torus with label i. A double prime in Table 31 indicates that the structure of the torus with label i'' may be determined from the structure of the torus with label i in Table 29 by multiplying by $(q-1)^2$.

We demonstrate this with an example. Consider the class 4C in $W(\mathrm{E}_6)$, with admissible graph A_3



that can be realised as a subgraph of the Dynkin diagram of E_6. As before, a representative of the corresponding class is given by $w = uv$ where $u = w_1w_4$ and $v = w_5$. The structure of $T_w(q)$ can be obtained using the methods shown in the previous example. This admissible graph can be realized as a subgraph of both the E_7 and E_8 Dynkin diagrams. Moreover, as roots r_1, r_3 and r_4 are all orthogonal to both r_7 and r_8, we have that the maximal torus $T_w(q)$ extends to a maximal torus $T_w(q)', T_w(q)''$ of both E_7 and E_8, by multiplying $T_w(q)$ by cyclic factors $(q-1)$ and $(q-1)^2$ respectively. Indeed, we see that classes 4C and 4D in E_7 and E_8 have torus $T_w(q)'$, $T_w(q)''$ respectively. The admissible graph for these classes is identical to the original graph A_3 in E_6 and thus all three of these classes have representative given by the same product of reflections. Note that when multiplying by a cyclic factor of $(q-1)$, the torsion coefficient decomposition of $T_w(q)$ may not be preserved. An example of this, and how we deal with it, is given in Section 7.6.

We must also consider the Weyl groups with unequal root lengths as these cannot be completely sorted in the same way. We note that in both $W(\mathrm{G}_2)$ and $W(\mathrm{F}_4)$, if two $W(G)$-conjugacy classes X_1 and X_2 both contain elements of the same order and $|X_1| = |X_2|$, then more often than not, we have $\min(X_1) = \min(X_2)$, $|X_{1\min}| = |X_{2\min}|$ and $|X^0_{1\min}| = |X^0_{2\min}|$. Hence we use a different method to order such classes. Let Λ denote the set of short roots and let $N^\Lambda(w) = N(w) \cap \Lambda$, where $N(w)$ is the set of positive roots sent negative by w. For a class X, we define $N^\Lambda_{\min}(X)$ to be

$$N^\Lambda_{\min}(X) = \min\{|N^\Lambda(w)| \mid w \in X\}.$$

Then for two W-conjugacy classes X_1 and X_2 with $\min(X_1) = \min(X_2)$, $|X_{1\min}| = |X_{2\min}|$ and $|X^0_{1\min}| = |X^0_{2\min}|$, we have X_1 preceding X_2 if $|N^\Lambda_{\min}(X_1)| < |N^\Lambda_{\min}(X_2)|$. Such classes X and the corresponding values $|N^\Lambda_{\min}(X)|$ can be found in Table 26. This is sufficient to resolve all of the remaining classes.

We remark that, as shown in [32], if τ is an automorphism of $F_4(q)$ which acts on the root system by interchanging long and short roots, and we have two conjugacy classes of W represented by w and w' such that $\Gamma_w = (\Gamma_{w'})^\tau$, then $T_w(q) \cong T_{w'}(q)$. In particular, if Γ_1, Γ_2 are admissible graphs associated with two distinct conjugacy classes and Γ_2 can be obtained from Γ_1 by interchanging long and short roots, then the maximal tori corresponding to both classes are isomorphic.

Table 26: Values of $|N^\Lambda_{\min}(X)|$ for the remaining unsorted classes in $W(G_2)$ and $W(F_4)$.

W	w	X	Min	$\lvert N^\Lambda_{\min}(X)\rvert$
$W(G_2)$	[2]	2B	(1,1)	0
	[1]	2C	(1,1)	1
$W(F_4)$	[3]	2B	(1,2)	0
	[1]	2C	(1,2)	1
	[3,14,21]	2D	(9,1)	3
	[1,3,14]	2E	(9,1)	6
	[3][4]	3B	(2,2)	0
	[1][2]	3C	(2,2)	2
	[6,21][13]	4D	(5,4)	2
	[14,20][2]	4E	(5,4)	3
	[1,14,20][2]	6B	(10,8)	4
	[3,6,21][13]	6C	(10,8)	6
	[2,21][4]	6D	(3,2)	1
	[2][14,21]	6E	(3,2)	2
	[4][3,14]	6F	(3,4)	1
	[2,4][14]	6G	(3,4)	2

We now turn to the twisted cases.

The tables for the twisted groups are in Section 7.5 and follow a similar structure to the tables in the untwisted case. In the second column, we describe each ϕ-class of a Weyl group W by its makeup in terms of the standard conjugacy classes of W. For example, in Table 34 the ϕ-class containing the Weyl group identity is labelled $1A_1 2A_1 3A_2 6A_2$ to indicate that it contains one element of the $W(G_2)$-conjugacy classes 1A and 2A, and two elements of $W(G_2)$-conjugacy classes 3A and 6A. Our labelling for the standard conjugacy classes of W is as given in Section 7.4. The ordering of the maximal tori in the twisted case is done in a similar way to the untwisted case. As before, the column headed Min gives the ordered pair (min(X), $|X_{\min}|$) where X is the twisted conjugacy class. The twisted classes are then sorted by the same conditions as in the untwisted case. We remark that considering the ordered pair (min(X), $|X_{\min}|$) is sufficient to order all classes in the twisted case.

7.4 Maximal Tori in Untwisted $G(q)$

Table 27: The maximal tori of $G_2(q)$.

w	Class	Min	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
Id	1A	(0,1)	$\emptyset$	Dih(12)	$(q-1)^2$
$[1,6] = w_0$	2A	(6,1)	$A_1 + \tilde{A}_1$		$(q+1)^2$
[2]	2B	(1,1)	A_1	2^2	$q^2 - 1$
[1]	2C	(1,1)	$\tilde{A}_1$		$q^2 - 1$
[2][6]	3A	(4,2)	A_2	6	$q^2 + q + 1$
[1][2]	6A	(2,2)	G_2		$q^2 - q + 1$

Table 28: The maximal tori of $\mathrm{F}_4(q)$.

w	Class	Min	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
Id	1A	(0,1)	$\emptyset$	$W(\mathrm{F}_4) \sim 2_+^{1+4} : \mathrm{Sym}(3)^2$	$(q-1)^4$
[1,14,20,22] $\sim w_0$	2A	(24,1)	$4\mathrm{A}_1, 4\tilde{\mathrm{A}}_1, 2\mathrm{A}_1+2\tilde{\mathrm{A}}_1$		$(q+1)^4$
[3]	2B	(1,2)	$\tilde{\mathrm{A}}_1$	$2^2 \times \mathrm{Sym}(4)$	$(q-1)^2 \times (q^2-1)$
[1]	2C	(1,2)	A_1		$(q-1)^2 \times (q^2-1)$
[3,14,21]	2D	(9,1)	$3\mathrm{A}_1, 2\tilde{\mathrm{A}}_1+\mathrm{A}_1$		$(q+1)^2 \times (q^2-1)$
[1,3,14]	2E	(9,1)	$3\tilde{\mathrm{A}}_1, 2\mathrm{A}_1+\tilde{\mathrm{A}}_1$		$(q+1)^2 \times (q^2-1)$
[1,20]	2F	(4,1)	$2\mathrm{A}_1, 2\tilde{\mathrm{A}}_1$	$\mathrm{Dih}(8)^2$	Table 32
[1,3]	2G	(2,3)	$\mathrm{A}_1+\tilde{\mathrm{A}}_1$	2^4	$(q^2-1)^2$
[1,21,2][4]	3A	(16,16)	$\mathrm{A}_2+\tilde{\mathrm{A}}_2$	$3 \times \mathrm{SL}_2(3)$	$(q^2+q+1)^2$
[3][4]	3B	(2,2)	$\tilde{\mathrm{A}}_2$	$6 \times \mathrm{Sym}(3)$	$(q-1) \times (q^3-1)$
[1][2]	3C	(2,2)	A_2		$(q-1) \times (q^3-1)$
[1,4][13,14]	4A	(12,12)	$2\mathrm{B}_2, \mathrm{D}_4(a_1)$	$\mathrm{GU}_2(3) \sim 4{\cdot}\mathrm{Sym}(4)$	$(q^2+1)^2$
[2][3]	4B	(2,2)	B_2	$4 \times \mathrm{Dih}(8)$	$(q-1) \times (q^3-q^2+q-1)$
[1,14][2,21]	4C	(14,16)	$\mathrm{A}_3+\tilde{\mathrm{A}}_1, \mathrm{B}_2+2\mathrm{A}_1$		$(q+1) \times (q^3+q^2+q+1)$
[6,21][13]	4D	(5,4)	$\tilde{\mathrm{A}}_3, \mathrm{A}_1+\mathrm{B}_2$	$2^2 \times 4$	Table 32
[14,20][2]	4E	(5,4)	$\mathrm{A}_3, \tilde{\mathrm{A}}_1+\mathrm{B}_2$		Table 32
[2,4][3,14]	6A	(8,16)	$\mathrm{F}_4(a_1)$	$3 \times \mathrm{SL}_2(3)$	$(q^2-q+1)^2$
[1,14,20][2]	6B	(10,8)	$\mathrm{D}_4, \mathrm{B}_3+\tilde{\mathrm{A}}_1$	$6 \times \mathrm{Sym}(3)$	$(q+1) \times (q^3+1)$
[3,6,21][13]	6C	(10,8)	$\tilde{\mathrm{D}}_4, \mathrm{C}_3+\mathrm{A}_1$		$(q+1) \times (q^3+1)$
[2,21][4]	6D	(3,2)	$\mathrm{A}_1+\tilde{\mathrm{A}}_2$	$2^2 \times 3$	q^4+q^3-q-1
[2][14,21]	6E	(3,2)	$\tilde{\mathrm{A}}_1+\mathrm{A}_2$		q^4+q^3-q-1
[4][3,14]	6F	(3,4)	C_3		q^4-q^3+q-1
[2,4][14]	6G	(3,4)	B_3		q^4-q^3+q-1
[1,14][2,4]	8A	(6,14)	$\mathrm{B}_4, \mathrm{C}_4$	8	q^4+1
[1,3][2,4]	12A	(4,8)	F_4	12	q^4-q^2+1

Table 29: The maximal tori of $E_6(q)$.

w	Class	Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
Id	1A	(0,1)	1	$\emptyset$	$W(E_6) \sim Sp_4(3):2$	$(q-1)^6$
[1]	2A	(1,6)	2	A_1	$2 \times Sym(6)$	$(q-1)^4 \times (q^2-1)$
[1,4,6,69] $\sim w_0$	2B	(12,1)	8	$4A_1$	$W(F_4) \sim 2_+^{1+4}:Sym(3)^2$	$(q+1)^2 \times (q^2-1)^2$
[1,2]	2C	(2,10)	3	$2A_1$	$Dih(8) \times Sym(4)$	$(q-1)^2 \times (q^2-1)^2$
[1,4,6]	2D	(3,5)	5	$3A_1$	$2^2 \times Sym(4)$	$(q^2-1)^3$
[1,2,5][3,6,69]	3A	(24,80)	21	$3A_2$	$3_+^{1+2}:SL_2(3)$	$(q^2+q+1)^3$
[1][3]	3B	(2,10)	4	A_2	$3 \times (Sym(3) \wr 2)$	$(q-1)^3 \times (q^3-1)$
[1,5][3,6]	3C	(4,4)	10	$2A_2$	$3 \times Sym(3)^2$	Table 32
[2,3][4,18]	4A	(6,12)	14	$D_4(a_1)$	$GU_2(3) \sim 4^{\cdot}Sym(4)$	$(q^3-q^2+q-1)^2$
[1,69,4,6][5]	4B	(13,16)	16	$2A_1+A_3$	$4 \times Sym(4)$	$(q+1)^2 \times (q^4-1)$
[1,4][3]	4C	(3,20)	7	A_3	$4 \times Dih(8)$	$(q-1)^2 \times (q^4-1)$
[1,4,6][5]	4D	(4,16)	11	A_1+A_3	$2^2 \times 4$	$(q^2-1) \times (q^4-1)$
[1,4][3,5]	5A	(4,32)	12	A_4	10	$(q-1) \times (q^5-1)$
[1,18,4][2,3,48]	6A	(12,144)	25	$E_6(a_2)$	$3 \times SL_2(3)$	$(q^2-q+1) \times (q^4+q^2+1)$
[1,4][5]	6B	(3,20)	6	A_1+A_2	$6 \times Sym(3)$	$(q-1)^2 \times (q^4+q^3-q-1)$
[3,2,5][4]	6C	(4,8)	13	D_4		$(q^2-1) \times (q^4-q^3+q-1)$
[1,2,5][3,6]	6D	(5,4)	15	A_1+2A_2		$(q^2+q+1) \times (q^4+q^3-q-1)$
[1,4,6,69][3,2]	6E	(14,48)	22	A_1+A_5, A_6		$(q+1) \times (q^5+q^4+q^3+q^2+q+1)$
[1,2,5][6]	6F	(4,10)	9	$2A_1+A_2$	$3 \times Dih(8)$	$(q^2-1) \times (q^4+q^3-q-1)$
[1,4,6][3,5]	6G	(5,16)	18	A_5	$2^2 \times 3$	Table 32
[1,4][3,2,5]	8A	(5,32)	19	D_5	8	$q^6-q^4+q^2-1$
[1,4,18][2,3,6]	9A	(8,80)	24	$E_6(a_1)$	9	q^6+q^3+1
[1,2,5][4,6]	10A	(5,16)	17	A_1+A_4	10	q^6+q^5-q-1
[1,4,6][3,2,5]	12A	(6,32)	23	E_6	12	$q^6+q^5-q^3+q+1$
[2,3][4,18,1]	12B	(7,64)	20	$D_5(a_1)$	12	$q^6-q^5+q^4-q^2+q-1$

Table 30: The maximal tori of $\mathrm{E}_7(q)$.

w		Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
Id	1A	(0,1)	$1'$	$\emptyset$	$W(\mathrm{E}_7) \sim 2.\mathrm{Sp}_6(2)$	$(q-1)^7$
w_0	2A	(63,1)	$-1'$	$7\mathrm{A}_1$		$(q+1)^7$
[1]	2B	(1,7)	$2'$	A_1	$2^6 : \mathrm{Sym}(6)$	$(q-1)^5 \times (q^2-1)$
[3,2,5,7,97,61]	2C	(30,1)	$-2'$	$6\mathrm{A}_1$		$(q+1)^5 \times (q^2-1)$
[4,17,97]	2D	(3,1)	-8	$3\mathrm{A}_1$	$(2_+^{1+4} : (2 \times 2^2)).\mathrm{Sym}(3)^2$	Table 32
[1,4,6,69]	2E	(12,1)	8	$4\mathrm{A}_1$		Table 32
[1,2]	2F	(2,15)	$3'$	$2\mathrm{A}_1$	$2^5 : (2^2 \times \mathrm{Sym}(4))$	$(q-1)^3 \times (q^2-1)^2$
[97,3,2,5,7]	2G	(13,1)	$-3'$	$5\mathrm{A}_1$		$(q+1)^3 \times (q^2-1)^2$
[1,4,6]	2H	(3,10)	$5'$	$3\mathrm{A}_1$	$2^7 : \mathrm{Sym}(3)$	$(q-1) \times (q^2-1)^3$
[2,3,5,7]	2I	(4,2)	$-5'$	$4\mathrm{A}_1$		$(q+1) \times (q^2-1)^3$
[1][3]	3A	(2,12)	$4'$	A_2	$6 \times \mathrm{Sym}(6)$	$(q-1)^4 \times (q^3-1)$
[1,2,5][3,6,69]	3B	(24,80)	21	$3\mathrm{A}_2$	$2 \times (3_+^{1+2} : \mathrm{SL}_2(3))$	$(q^2+q+1)^2 \times (q^3-1)$
[1,5][3,6]	3C	(4,16)	10	$2\mathrm{A}_2$	$6 \times \mathrm{Sym}(3)^2$	$(q-1) \times (q^3-1)^2$
[2,3][4,18]	4A	(6,12)	$14'$	$\mathrm{D}_4(a_1)$	$2^3 \times (4 \circ \mathrm{Dih}(8)).\mathrm{Sym}(3)$	$(q-1) \times (q^3-q^2+q-1)^2$
[97,3,5,7][1,6,2]	4B	(33,360)	$-14'$	$\mathrm{A}_1 + 2\mathrm{A}_3$		$(q+1) \times (q^3+q^2+q+1)^2$
[1,4][3]	4C	(3,24)	$7'$	A_3	$\mathrm{Sym}(4) \times 2^2 \times 4$	$(q-1)^3 \times (q^4-1)$
[2,5,7][4]	4D	(4,8)	-16	$\mathrm{A}_1 + \mathrm{A}_3$		Table 32
[1,69,4,6][5]	4E	(13,16)	16	$2\mathrm{A}_1 + \mathrm{A}_3$		Table 32
[97,3,2,5,7][6]	4F	(14,8)	$-7'$	$3\mathrm{A}_1 + \mathrm{A}_3$		$(q+1)^3 \times (q^4-1)$
[2,3][4,18,7]	4G	(7,12)	28	$\mathrm{A}_1 + \mathrm{D}_4(a_1)$	$(2^3 \times \mathrm{Dih}(8)) : 4$	Table 32
[97,3,5,7][1,6]	4H	(16,60)	-28	$2\mathrm{A}_3, \mathrm{A}_2 + \mathrm{D}_4(a_1)$		Table 32
[1,4,6][5]	4I	(4,36)	11	$\mathrm{A}_1 + \mathrm{A}_3$	$2^4 \times 4$	$(q-1) \times (q^2-1) \times (q^4-1)$
[1,7,2,5][4]	4J	(5,12)	-11	$2\mathrm{A}_1 + \mathrm{A}_3$		$(q+1) \times (q^2-1) \times (q^4-1)$
[1,4][3,5]	5A	(4,40)	$12'$	A_4	$10 \times \mathrm{Sym}(3)$	$(q-1)^2 \times (q^5-1)$
[97,3,2,5,7,61][4]	6A	(31,32)	$-4'$	$3\mathrm{A}_1 + \mathrm{D}_4$	$6 \times \mathrm{Sym}(6)$	$(q+1)^4 \times (q^3+1)$
[1,4,18,82][2,3,48]	6B	(21,800)	-21	$\mathrm{E}_7(a_4)$	$2 \times (3_+^{1+2} : \mathrm{SL}_2(3))$	$(q^2-q+1)^2 \times (q^3+1)$
[1,4][5]	6C	(3,36)	$18'$	$\mathrm{A}_1 + \mathrm{A}_2$	$2 \times 6 \times \mathrm{Sym}(4)$	$(q-1)^3 \times (q^4+q^3-q-1)$
[3,2,5][4]	6D	(4,8)	$13'$	D_4		$(q-1) \times (q^2-1) \times (q^4-q^3+q-1)$
[97,3,2,6][7]	6E	(5,2)	$-13'$	$3\mathrm{A}_1 + \mathrm{A}_2$		$(q+1) \times (q^2-1) \times (q^4+q^3-q-1)$

Table 30: The maximal tori of $E_7(q)$.

w		Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
[97,3,2,5,7][4]	6F	(14,16)	$-18'$	$2A_1+D_4$	$2\times 6\times \mathrm{Sym}(4)$	$(q+1)^3\times(q^4-q^3+q-1)$
[1,4,18,82][2,3,7]	6G	(23,708)	-10	$A_1+D_6(a_2)$	$6\times \mathrm{Sym}(3)^2$	$(q+1)\times(q^3+1)^2$
[1,18,4][2,3,48]	6H	(12,144)	25	$E_6(a_2)$	$6\times \mathrm{SL}_2(3)$	$(q^2-q+1)\times(q^5-q^4+q^3-q^2+q-1)$
[97,2,5,7][1,4,6]	6I	(25,420)	-25	A_2+A_5		$(q^2+q+1)\times(q^5+q^4+q^3+q^2+q+1)$
[1,2,5][6]	6J	(4,24)	$9'$	$2A_1+A_2$	$2\times 6\times \mathrm{Dih}(8)$	$(q-1)\times(q^2-1)\times(q^4+q^3-q-1)$
[3,2,5][4,7]	6K	(5,8)	$-9'$	A_1+D_4		$(q+1)\times(q^2-1)\times(q^4-q^3+q-1)$
[1,2,5][3,6]	6L	(5,12)	15	A_1+2A_2	$2\times 6\times \mathrm{Sym}(3)$	$(q^3-1)\times(q^4+q^3-q-1)$
[2,5,7][4,6]	6M	(5,16)	-22	A_5		Table 32
[1,4,18,82][2,3]	6N	(10,112)	-15	$D_6(a_2)$		$(q^3+1)\times(q^4-q^3+q-1)$
[1,4,6,69][3,2]	6O	(14,48)	22	A_1+A_5		Table 32
[1,4,6][3,5]	6P	(5,32)	18	A_5	$2^2\times 6$	$(q-1)\times(q^6-1)$
[1,2,5,7][4,6]	6Q	(6,16)	-18	A_1+A_5		$(q+1)\times(q^6-1)$
[1,4,6][3,5,7]	7A	(6,32)	29	A_6	14	q^7-1
[1,4][3,2,5]	8A	(5,32)	19	D_5	$2^2\times 8$	$(q-1)\times(q^6-q^4+q^2-1)$
[1,4][3,2,5,7]	8B	(6,16)	-19	A_1+D_5		$(q+1)\times(q^6-q^4+q^2-1)$
[2,3,97][1,4,18]	8C	(8,64)	30	$D_6(a_1)$		Table 32
[97,3,5,7][1,4,6]	8D	(17,316)	-30	A_7		Table 32
[1,4,18][2,3,6]	9A	(8,80)	24	$E_6(a_1)$	18	$q^7-q^6+q^4-q^3+q-1$
[2,3,5,7][4,6,97]	10A	(15,96)	$-12'$	A_1+D_6	$10\times \mathrm{Sym}(3)$	$(q+1)^2\times(q^5+1)$
[1,2,5][4,6]	10B	(5,40)	17	A_1+A_4	2×10	$(q-1)\times(q^6+q^5-q-1)$
[2,3,5,7][4,6]	10C	(6,32)	-17	D_6		$(q+1)\times(q^6-q^5+q-1)$
[1,5,7][3,6]	12A	(5,24)	26	A_2+A_3	$2^2\times 12$	$(q-1)\times(q^6+q^5+q^4-q^2-q-1)$
[1,2,5][97,3,6]	12B	(6,8)	$-20'$	$A_1+A_2+A_3$		$(q+1)\times(q^6+q^5+q^4-q^2-q-1)$
[2,3][4,18,1]	12C	(7,64)	$20'$	$D_5(a_1)$		$(q-1)\times(q^6-q^5+q^4-q^2+q-1)$
[2,3][4,18,1,7]	12D	(8,32)	-26	$A_1+D_5(a_1)$		$(q+1)\times(q^6-q^5+q^4-q^2+q-1)$
[1,4,6][3,2,5]	12E	(6,32)	23	E_6	2×12	$q^7-q^5-q^4+q^3+q^2-1$
[1,4,18,82][2,3,6]	12F	(11,280)	-23	$E_7(a_2), E_7(b_2)$		$q^7-q^5+q^4+q^3-q^2+1$
[1,4,18,7][3,2,6]	14A	(9,160)	-29	$E_7(a_1)$	14	q^7+1
[97,4,6][1,5,7]	15A	(6,16)	27	A_2+A_4	30	$q^7+q^6+q^5-q^2-q-1$

Table 30: The maximal tori of $E_7(q)$.

w		Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
[1,4,6][3,2,5,7]	18A	(7,64)	−24	E_7	18	$q^7 + q^6 - q^4 - q^3 + q + 1$
[1,4,18][2,3,48,97]	30A	(13,366)	−27	$E_7(a_3)$	30	$q^7 - q^6 + q^5 + q^2 - q + 1$

Table 31: The maximal tori of $E_8(q)$.

w	Class	Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
Id	1A	(0,1)	$1''$	$\emptyset$	$W(E_8) \sim 2^{\cdot}\Omega_8^+(2).2$	$(q-1)^8$
w_0	2A	(120,1)	$-1''$	$8A_1$		$(q+1)^8$
[1]	2B	(1,8)	$2''$	A_1	$2^2 \times Sp_6(2)$	$(q-1)^6 \times (q^2-1)$
[97,3,2,5,7,120,61]	2C	(63,1)	$-2''$	$7A_1$		$(q+1)^6 \times (q^2-1)$
[1,4,6,69]	2D	(12,1)	$8'$	$4A_1$	$2^{2\cdot}(2^8 : Sym(3)^3)$	Table 32
[1,2]	2E	(2,21)	$3''$	$2A_1$	$2^{6\cdot}(2^2 \times Sym(6))$	$(q-1)^4 \times (q^2-1)^2$
[97,3,2,5,7,120]	2F	(30,1)	$-3''$	$6A_1$		$(q+1)^4 \times (q^2-1)^2$
[1,4,6]	2G	(3,21)	$5''$	$3A_1$	$2^{5\cdot}(2^4 : Sym(3)^2)$	$(q-1)^2 \times (q^2-1)^3$
[97,3,2,5,7]	2H	(13,2)	$-5''$	$5A_1$		$(q+1)^2 \times (q^2-1)^3$
[2,3,5,7]	2I	(4,7)	31	$4A_1$	$2^{6\cdot}(Sym(4) \times 2^2)$	$(q^2-1)^4$
[1][3]	3A	(2,14)	$4''$	A_2	$(6 \times Sp_4(3)) : 2$	$(q-1)^5 \times (q^3-1)$
[1,2,5,8][3,69,6,120]	3B	(80,4480)	56	$4A_2$	$3 \times 2^{\cdot}U_4(2)$	$(q^2+q+1)^4$
[1,2,5][3,6,69]	3C	(24,80)	$21'$	$3A_2$	$(Dih(12) \times 6)^{\cdot}ASL_2(3)$	Table 32
[1,5][3,6]	3D	(4,32)	$10'$	$2A_2$	$(2 \times 3 \times Sym(3)^3) : 2$	$(q-1)^2 \times (q^3-1)^2$
[2,3,7,120][4,18,8,74]	4A	(60,15120)	59	$2D_4(a_1)$	$(4 \circ 2_+^{1+4}).Alt(6).2$	$(q^2+1)^4$
[2,3][4,18]	4B	(6,12)	$14''$	$D_4(a_1)$	$(2^3 : Sym(4)) : GU_2(3)$	$(q-1)^2 \times (q^3-q^2+q-1)^2$
[97,3,5,7,2,120][1,6]	4C	(66,1260)	$-14''$	$2A_1+2A_3$		$(q+1)^2 \times (q^3+q^2+q+1)^2$
[1,4][3]	4D	(3,28)	$7''$	A_3	$2^5 : (4 \times Sym(5))$	$(q-1)^4 \times (q^4-1)$
[97,3,2,5,7,120][1]	4E	(31,12)	$-7''$	$4A_1+A_3$	$\sim 2^5 : CO_3(5)$	$(q+1)^4 \times (q^4-1)$
[1,69,4,6][5]	4F	(13,16)	$16'$	$2A_1+A_3$	$(2 \times 4 \times Alt(4) \times Sym(4)) : 2$	Table 32
[2,3][4,18,7]	4G	(7,24)	28	$A_1+D_4(a_1)$	$2^3 : (2 \times GU_2(3))$	Table 32
[97,3,5,7][1,6,2]	4H	(33,360)	-28	A_1+2A_3		Table 32
[97,3,5,7][1,6]	4I	(16,60)	37	$2A_3$	$(2^4 \times Dih(8))^{\cdot}(2 \times 4)$	Table 32
[1,4,6][5]	4J	(4,80)	$11'$	A_1+A_3	$Sym(4) \times 2^3 \times 4$	$(q-1)^2 \times (q^2-1) \times (q^4-1)$
[97,3,2,5,7][6]	4K	(14,24)	$-11'$	$3A_1+A_3$		$(q+1)^2 \times (q^2-1) \times (q^4-1)$
[2,3,7,120][4,18,8]	4L	(21,384)	49	$A_3+D_4(a_1)$	$(2 \times 4^3) : Sym(3)$	$(q^2+1)^2 \times (q^4-1)$
[1,7,2,5][4]	4M	(5,40)	33	$2A_1+A_3$	$(Dih(8) \times 2^3)^{\cdot}2^2$	$(q^2-1)^2 \times (q^4-1)$
[3,5,7,120][4,8]	4N	(6,32)	36	$2A_3$	$(2 \times 4) \wr 2$	$(q^4-1)^2$
[1,4][3,5]	5A	(4,48)	$12''$	A_4	$10 \times Sym(5)$	$(q-1)^3 \times (q^5-1)$

Table 31: The maximal tori of $E_8(q)$.

w	Class	Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
[1,4,69,8][3,5,7,120]	5B	(48,7952)	57	$2A_4$	$5 \times SL_2(5)$	$(q^4+q^3+q^2+q+1)^2$
[97,3,2,5,7,61][4,120]	6A	(64,56)	$-4''$	$4A_1+D_4$	$(6 \times Sp_4(3)):2$	$(q+1)^5 \times (q^3+1)$
[1,4,18,82][2,3,48,56]	6B	(40,4480)	-56	$E_8(a_8)$	$3 \times 2^{\cdot}U_4(2)$	$(q^2-q+1)^4$
[1,4][5]	6C	(3,56)	$6''$	A_1+A_2	$2 \times 6 \times Sym(6)$	$(q-1)^4 \times (q^4+q^3-q-1)$
[97,3,2,5,7,61][4]	6D	(31,32)	$-6''$	$3A_1+D_4$		$(q+1)^4 \times (q^4-q^3+q-1)$
[1,4,18,82][2,3,48,120]	6E	(42,11592)	$-21'$	$A_1+E_7(a_4)$	$(2 \times SU_3(2) \times Sym(3)):3$	Table 32
[3,2,5][4]	6F	(4,8)	$13''$	D_4	$(2^2 \times 3).2^4.Sym(3)^2$	$(q-1)^2 \times (q^2-1) \times (q^4-q^3+q-1)$
[1,2,5,7,120][3]	6G	(14,2)	$-13''$	$4A_1+A_2$		$(q+1)^2 \times (q^2-1) \times (q^4+q^3-q-1)$
[1,4,18,82][2,3,48]	6H	(21,800)	-43	$E_7(a_4)$	$(2^2 \times SU_3(2)):3$	$(q^2-q+1)^2 \times (q^4-q^3+q-1)$
[1,5,8][3,6,120,2]	6I	(25,80)	43	A_1+3A_2		$(q^2+q+1)^2 \times (q^4+q^3-q-1)$
[1,4,18,82][2,3,7,120]	6J	(44,4070)	$-10'$	$2A_1+D_6(a_2)$	$(2 \times 3 \times Sym(3)^3):2$	$(q+1)^2 \times (q^3+1)^2$
[1,18,4,120][2,3,48,8]	6K	(44,16374)	63	$A_2+E_6(a_2)$	$3 \times SL_2(3)^2$	$(q^4+q^2+1)^2$
[1,2,5][6]	6L	(4,56)	$9''$	$2A_1+A_2$	$2 \times 3 \times Dih(8) \times Sym(4)$	$(q-1)^2 \times (q^2-1) \times (q^4+q^3-q-1)$
[97,3,2,5,7][4]	6M	(14,16)	$-9''$	$2A_1+D_4$		$(q+1)^2 \times (q^2-1) \times (q^4-q^3+q-1)$
[1,18,4][2,3,48]	6N	(12,144)	$25'$	$E_6(a_2)$	$2 \times 3 \times Sym(3) \times SL_2(3)$	(q^3-2q^2+2q-1) $\times(q^5-q^4+q^3-q^2+q-1)$
[1,2,5,7,120][3,6,8]	6O	(46,3752)	$-25'$	$A_1+A_2+A_5$		(q^3+2q^2+2q+1) $\times(q^5+q^4+q^3+q^2+q+1)$
[3,2,5][4,7]	6P	(5,16)	-32	A_1+D_4	$2^2 \times 6 \times Sym(4)$	$(q^2-1)^2 \times (q^4-q^3+q-1)$
[97,3,2,6][7]	6Q	(5,16)	32	$3A_1+A_2$		$(q^2-1)^2 \times (q^4+q^3-q-1)$
[1,2,5][3,6]	6R	(5,32)	$15'$	A_1+2A_2	$2 \times 6 \times Sym(3)^2$	$(q-1) \times (q^3-1) \times (q^4+q^3-q-1)$
[1,4,6,69][3,2]	6S	(14,48)	$22'$	A_1+A_5		Table 32
[1,4,18,82][2,3,120]	6T	(23,708)	$-15'$	$A_1+D_6(a_2)$		$(q+1) \times (q^3+1) \times (q^4-q^3+q-1)$
[1,2,5,8][3,6]	6U	(6,8)	34	$2A_1+2A_2$	$(2^2 \times 6 \times Sym(3)):2$	$(q^4+q^3-q-1)^2$
[1,4,18,82][2,3]	6V	(10,112)	-34	$D_6(a_2)$		$(q^4-q^3+q-1)^2$
[1,18,4][2,3,48,8]	6W	(13,114)	54	$A_1+E_6(a_2)$	$2 \times 6 \times SL_2(3)$	Table 32
[97,2,5,7][1,4,6]	6X	(25,420)	-54	A_2+A_5		Table 32
[2,3,5,7][4,8]	6Y	(6,16)	40	A_2+D_4	$6^2 \times Sym(3)$	$(q^2-1) \times (q^6-1)$
[1,4,6][3,5]	6Z	(5,64)	$18'$	A_5	$2^2 \times 6 \times Sym(3)$	$(q-1)^2 \times (q^6-1)$

Table 31: The maximal tori of $E_8(q)$.

w	Class	Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
[1,4,82,120,6][2,8]	6AA	(15,48)	$-18'$	$2A_1 + A_5$	$2^2 \times 6 \times \mathrm{Sym}(3)$	$(q+1)^2 \times (q^6-1)$
[1,2,5,7][4,6]	6BB	(6,48)	39	$A_1 + A_5$	$2^3 \times 6$	$(q^2-1) \times (q^6-1)$
[1,4,6][3,5,7]	7A	(6,96)	29	A_6	2×14	$(q-1) \times (q^7-1)$
[1,4][3,2,5]	8A	(5,32)	$19'$	D_5	$2:(8 \times \mathrm{Sym}(4))$	$(q-1)^2 \times (q^6-q^4+q^2-1)$
[1,4,6,8][7,82,120]	8B	(15,32)	$-19'$	$2A_1 + D_5$		$(q+1)^2 \times (q^6-q^4+q^2-1)$
[1,4,18,82][2,3,8,74]	8C	(30,7748)	61	$D_8(a_3)$	$\mathrm{CU}_2(3) \sim 8^{\cdot}\mathrm{Sym}(4)$	$(q^4+1)^2$
[2,3,97][1,4,18]	8D	(8,64)	30	$D_6(a_1)$	$8 \times (2^2:4)$	Table 32
[1,4,6,8][3,7,82,120]	8E	(34,2080)	-30	$A_3 + D_5$		Table 32
[1,4][3,2,5,7]	8F	(6,48)	42	$A_1 + D_5$	$2^3 \times 8$	$(q^2-1) \times (q^6-q^4+q^2-1)$
[97,3,5,7][1,4,6]	8G	(17,316)	48	A_7		Table 32
[1,4,6,8][3,5,7]	8H	(7,64)	47	A_7	2×8	q^8-1
[1,4,18][2,3,6]	9A	(8,80)	$24'$	$E_6(a_1)$	$18 \times \mathrm{Sym}(3)$	$(q-1) \times (q^7-q^6+q^4-q^3+q-1)$
[1,4,6,8][3,5,7,120]	9B	(28,2816)	58	A_8	3×18	Table 32
[2,3,5,7][4,6,97,120]	10A	(32,256)	$-12''$	$2A_1 + D_6$	$10 \times \mathrm{Sym}(5)$	$(q+1)^3 \times (q^5+1)$
[1,4,18,118][2,3,48,97]	10B	(24,3370)	-57	$E_8(a_6)$	$5 \times \mathrm{SL}_2(5)$	$(q^4-q^3+q^2-q+1)^2$
[1,2,5][4,6]	10C	(5,96)	$17'$	$A_1 + A_4$	$2 \times 10 \times \mathrm{Sym}(3)$	$(q-1)^2 \times (q^6+q^5-q-1)$
[2,3,5,7][4,6,97]	10D	(15,96)	$-17'$	$A_1 + D_6$		$(q+1)^2 \times (q^6-q^5+q-1)$
[2,3,5,7][4,6]	10E	(6,32)	-38	D_6	$10 \times \mathrm{Dih}(8)$	$(q^2-1) \times (q^6-q^5+q-1)$
[1,4,6,8][7,120]	10F	(6,32)	38	$2A_1 + A_4$		$(q^2-1) \times (q^6+q^5-q-1)$
[2,3][4,18,1]	12A	(7,64)	$20''$	$D_5(a_1)$	$4 \times 6 \times \mathrm{Sym}(4)$	$(q-1)^2 \times (q^6-q^5+q^4-q^2+q-1)$
[2,3,7][4,18,8]	12B	(8,24)	41	$A_2 + D_4(a_1)$	$2 \times 3 \times \mathrm{GU}_2(3)$	$(q^3-q^2+q-1) \times (q^5+q^3-q^2-1)$
[97,3,5,2,120][1,6]	12C	(15,16)	$-20''$	$2A_1 + A_2 + A_3$	$4 \times 6 \times \mathrm{Sym}(4)$	$(q+1)^2 \times (q^6+q^5+q^4-q^2-q-1)$
[2,3,120,7,61][4,18,8]	12D	(46,15134)	-41	$D_4 + D_4(a_1)$	$2 \times 3 \times \mathrm{GU}_2(3)$	$(q^3+q^2+q+1) \times (q^5+q^3+q^2+1)$
[1,4,18,82][2,3,6,29]	12E	(20,2696)	67	$E_8(a_3), E_8(b_3)$	$3 \times \mathrm{GU}_2(3)$	$(q^4-q^2+1)^2$
[1,4,18,82][2,3,48,8]	12F	(22,2360)	-62	$E_8(a_7)$	$12 \times \mathrm{SL}_2(3)$	$(q^2-q+1) \times (q^6-q^5+q^3-q+1)$
[1,4,6,8][3,2,5,120]	12G	(26,840)	62	$A_2 + E_6$		$(q^2+q+1) \times (q^6+q^5-q^3+q+1)$
[1,5,7][3,6]	12H	(5,80)	26	$A_2 + A_3$	$2 \times 12 \times \mathrm{Dih}(8)$	$(q-1)^2 \times (q^6+q^5+q^4-q^2-q-1)$
[1,4,7,82,120][3,8]	12I	(17,168)	-26	$A_3 + D_4$		$(q+1)^2 \times (q^6-q^5+q^4-q^2+q-1)$
[1,4,6][3,2,5]	12J	(6,32)	$23'$	E_6	$2 \times 12 \times \mathrm{Sym}(3)$	$(q-1) \times (q^7-q^5-q^4+q^3+q^2-1)$

Table 31: The maximal tori of $E_8(q)$.

w	Class	Min	$T_w(q)$	Γ_w	Shape of $C_W(w)$	Structure of $T_w(q)$
[1,4,18,82][2,3,6,120]	12K	(24,1758)	$-23'$	$A_1+E_7(a_2)$	$2\times 12\times \mathrm{Sym}(3)$	$(q+1)\times(q^7-q^5+q^4+q^3-q^2+1)$
[1,2,6,8][4,7]	12L	(6,32)	35	$A_1+A_2+A_3$	$2^3\times 12$	$(q^2-1)\times(q^6+q^5+q^4-q^2-q-1)$
[2,3][4,18,1,7]	12M	(8,96)	-35	$A_1+D_5(a_1)$		$(q^2-1)\times(q^6-q^5+q^4-q^2+q-1)$
[2,3,7][4,18,1,8]	12N	(9,64)	51	$A_2+D_5(a_1)$	6×12	$q^8+q^6-q^2-1$
[2,3,97,7][4,18,1,8]	12O	(22,2040)	60	$D_8(a_1)$	$12\times \mathrm{Sym}(3)$	$(q^2+1)\times(q^6+1)$
[1,4,6][3,2,5,120]	12P	(7,32)	52	A_1+E_6	$2^2\times 12$	$q^8+q^7-q^6-2q^5+2q^3+q^2-q-1$
[1,4,18,82][2,3,6]	12Q	(11,280)	-52	$E_7(a_2)$		$q^8-q^7-q^6+2q^5-2q^3+q^2+q-1$
[2,3,5,7][4,6,8]	12R	(7,64)	55	D_7	2×12	$q^8-q^6+q^2-1$
[1,4,6,8][5,7,120]	14A	(7,32)	46	A_1+A_6	2×14	q^8+q^7-q-1
[1,4,18,7][3,2,6]	14B	(9,160)	-46	$E_7(a_1)$		q^8-q^7+q-1
[2,3,5,7,120][4,6,8]	14C	(18,852)	-29	D_8		$(q+1)\times(q^7+1)$
[97,4,6][1,5,7]	15A	(6,64)	27	A_2+A_4	2×30	$(q-1)\times(q^7+q^6+q^5-q^2-q-1)$
[1,4,18,7][2,3,48,29]	15B	(16,1516)	-64	$E_8(a_5), E_8(b_5)$	30	$q^8-q^7+q^5-q^4+q^3-q+1$
[1,4,6][3,2,5,7,120]	18A	(16,192)	$-24'$	A_1+E_7	$18\times \mathrm{Sym}(3)$	$(q+1)\times(q^7+q^6-q^4-q^3+q+1)$
[1,4,18,8][2,3,48,97]	18B	(14,732)	-58	$E_8(a_4)$	3×18	Table 32
[1,4,6][3,2,5,7]	18C	(7,64)	-53	E_7	2×18	$q^8-q^6-q^5+q^3+q^2-1$
[1,4,18][2,3,6,8]	18D	(9,80)	53	$A_1+E_6(a_1)$		$q^8-q^6+q^5-q^3+q^2-1$
[1,4,6,8][3,7,120]	20A	(7,32)	45	A_3+A_4	2×20	$q^8+q^7+q^6+q^5-q^3-q^2-q-1$
[2,3,97][4,18,1,8]	20B	(9,128)	-45	$D_7(a_1)$		$q^8-q^7+q^6-q^5+q^3-q^2+q-1$
[1,4,18,7][3,2,6,29]	20C	(12,624)	66	$E_8(a_2)$	20	$q^8-q^6+q^4-q^2+1$
[1,4,7][3,2,5,8]	24A	(7,32)	50	A_2+D_5	2×24	$q^8+q^7-q^5+q^3-q-1$
[1,4,18,82][2,3,8]	24B	(11,256)	-50	$D_7(a_2)$		$q^8-q^7+q^5-q^3+q-1$
[1,4,18,7][3,2,6,8]	24C	(10,320)	65	$E_8(a_1)$	24	q^8-q^4+1
[1,2,6,8][4,7,120]	30A	(7,16)	44	$A_1+A_2+A_4$	2×30	$q^8+2q^7+2q^6+q^5-q^3-2q^2-2q-1$
[1,4,18][2,3,48,97]	30B	(13,366)	-44	$E_7(a_3)$		$q^8-2q^7+2q^6-q^5+q^3-2q^2+2q-1$
[1,4,18][2,3,48,97,120]	30C	(26,4996)	-27	$A_1+E_7(a_3)$		$(q+1)\times(q^7-q^6+q^5+q^2-q+1)$
[1,4,6,8][3,2,5,7]	30D	(8,128)	64	E_8	30	$q^8+q^7-q^5-q^4-q^3+q+1$

Table 32: The torsion coefficient decomposition of various maximal tori of $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$.

$G(q)$	Class	Torsion coefficient decomposition	Uniform structure
$F_4(q)$	2F	$\begin{cases}(q^2-1)^2, & q \text{ even}\\ 2\times\frac{(q^2-1)}{2}\times(q^2-1), & q \text{ odd}\end{cases}$	$(q-1)\times(q+1)\times(q^2-1)$
	4D	$\begin{cases}q^4-1, & q \text{ even}\\ 2\times\frac{(q^4-1)}{2}, & q \text{ odd}\end{cases}$	$(q^2-1)\times(q^2+1)$
	4E	$\begin{cases}q^4-1, & q \text{ even}\\ 2\times\frac{(q^4-1)}{2}, & q \text{ odd}\end{cases}$	$(q^2-1)\times(q^2+1)$
$E_6(q)$	3C	$\begin{cases}3\times\frac{(q^3-1)}{3}\times(q^3-1), & q\equiv 1 \bmod 3\\ (q^3-1)^2, & q\equiv 0,2 \bmod 3\end{cases}$	$(q-1)\times(q^2+q+1)\times(q^3-1)$
	6G	$\begin{cases}3\times\frac{(q^6-1)}{3}, & q\equiv 1 \bmod 3\\ (q^6-1), & q\equiv 0,2 \bmod 3\end{cases}$	$(q^2+q+1)\times(q^4-q^3+q-1)$
$E_7(q)$	2D	$\begin{cases}(q-1)\times(q^2-1)^3, & q \text{ even}\\ 2\times(q-1)\times\frac{(q^2-1)}{2}\times(q^2-1)^2, & q \text{ odd}\end{cases}$	$(q+1)\times(q-1)^2\times(q^2-1)^2$
	2E	$\begin{cases}(q+1)\times(q^2-1)^3, & q \text{ even}\\ 2\times(q+1)\times\frac{(q^2-1)}{2}\times(q^2-1)^2, & q \text{ odd}\end{cases}$	$(q-1)\times(q+1)^2\times(q^2-1)^2$
	4D	$\begin{cases}(q-1)\times(q^2-1)\times(q^4-1), & q \text{ even}\\ 2\times(q-1)\times\frac{(q^2-1)}{2}\times(q^4-1), & q \text{ odd}\end{cases}$	$(q+1)\times(q-1)^2\times(q^4-1)$
	4E	$\begin{cases}(q+1)\times(q^2-1)\times(q^4-1), & q \text{ even}\\ 2\times(q+1)\times\frac{(q^2-1)}{2}\times(q^4-1), & q \text{ odd}\end{cases}$	$(q-1)\times(q+1)^2\times(q^4-1)$
	4G	$\begin{cases}(q^3-q^2+q-1)\times(q^4-1), & q \text{ even}\\ 2\times(q^3-q^2+q-1)\times\frac{(q^4-1)}{2}, & q \text{ odd}\end{cases}$	$(q^2-1)\times(q^2+1)\times(q^3-q^2+q-1)$
	4H	$\begin{cases}(q^3+q^2+q+1)\times(q^4-1), & q \text{ even}\\ 2\times(q^3+q^2+q+1)\times\frac{(q^4-1)}{2}, & q \text{ odd}\end{cases}$	$(q^2-1)\times(q^2+1)\times(q^3+q^2+q+1)$
	6M	$\begin{cases}(q-1)\times(q^6-1), & q \text{ even}\\ 2\times(q-1)\times\frac{(q^6-1)}{2}, & q \text{ odd}\end{cases}$	$(q-1)\times(q^3-1)\times(q^3+1)$
	6O	$\begin{cases}(q+1)\times(q^6-1), & q \text{ even}\\ 2\times(q+1)\times\frac{(q^6-1)}{2}, & q \text{ odd}\end{cases}$	$(q+1)\times(q^3-1)\times(q^3+1)$

Table 32: The torsion coefficient decomposition of various maximal tori of $F_4(q)$, $E_6(q)$, $E_7(q)$ and $E_8(q)$.

	8C	$\begin{cases} (q^7-q^6+q^5-q^4+q^3-q^2+q-1), & q \text{ even} \\ 2\times\frac{(q^7-q^6+q^5-q^4+q^3-q^2+q-1)}{2}, & q \text{ odd} \end{cases}$	$(q^2+1)\times(q^5-q^4+q-1)$
	8D	$\begin{cases} (q^7+q^6+q^5+q^4+q^3+q^2+q+1), & q \text{ even} \\ 2\times\frac{(q^7+q^6+q^5+q^4+q^3+q^2+q+1)}{2}, & q \text{ odd} \end{cases}$	$(q^2+1)\times(q^5+q^4+q+1)$
$E_8(q)$	2D	$\begin{cases} (q^2-1)^4, & q \text{ even} \\ 2^2\times(\frac{(q^2-1)}{2})^2\times(q^2-1)^2, & q \text{ odd} \end{cases}$	$(q-1)^2\times(q+1)^2\times(q^2-1)^2$
	3C	$\begin{cases} 3\times(q^2+q+1)\times\frac{(q^3-1)}{3}\times(q^3-1), & q\equiv 1 \bmod 3 \\ (q^2+q+1)\times(q^3-1)^2, & q\equiv 0,2 \bmod 3 \end{cases}$	$(q-1)\times(q^2+q+1)^2\times(q^3-1)$
	4F	$\begin{cases} (q^2-1)^2\times(q^4-1), & q \text{ even} \\ 2^2\times(\frac{(q^2-1)}{2})^2\times(q^4-1), & q \text{ odd} \end{cases}$	$(q-1)^2\times(q+1)^2\times(q^4-1)$
	4G	$\begin{cases} (q-1)\times(q^3-q^2+q-1)\times(q^4-1), & q \text{ even} \\ (2q-2)\times(q^3-q^2+q-1)\times\frac{(q^4-1)}{2}, & q \text{ odd} \end{cases}$	$(q^2-1)\times(q^3-q^2+q-1)^2$
	4H	$\begin{cases} (q+1)\times(q^3+q^2+q+1)\times(q^4-1), & q \text{ even} \\ (2q+2)\times(q^3+q^2+q+1)\times\frac{(q^4-1)}{2}, & q \text{ odd} \end{cases}$	$(q^2-1)\times(q^3+q^2+q+1)^2$
	4I	$\begin{cases} (q^4-1)^2, & q \text{ even} \\ 2^2\times(\frac{(q^4-1)}{2})^2, & q \text{ odd} \end{cases}$	$(q^2-1)^2\times(q^2+1)^2$
	6E	$\begin{cases} 3\times(q^2-q+1)\times\frac{(q^3+1)}{3}\times(q^3+1), & q\equiv 2 \bmod 3 \\ (q^2-q+1)\times(q^3+1)^2, & q\equiv 0,1 \bmod 3 \end{cases}$	$(q+1)\times(q^2-q+1)^2\times(q^3+1)$
	6S	$\begin{cases} (q^2-1)\times(q^6-1), & q \text{ even} \\ 2^2\times\frac{(q^2-1)}{2}\times\frac{(q^6-1)}{2}, & q \text{ odd} \end{cases}$	$(q-1)\times(q+1)\times(q^3-1)\times(q^3+1)$
	6W	$\begin{cases} 3\times(q^2-q+1)\times\frac{(q^6-1)}{3}, & q\equiv 2 \bmod 3 \\ (q^2-q+1)\times(q^6-1), & q\equiv 0,1 \bmod 3 \end{cases}$	$(q^2-q+1)^2\times(q^4+q^3-q-1)$
	6X	$\begin{cases} 3\times(q^2+q+1)\times\frac{(q^6-1)}{3}, & q\equiv 1 \bmod 3 \\ (q^2+q+1)\times(q^6-1), & q\equiv 0,2 \bmod 3 \end{cases}$	$(q^2+q+1)^2\times(q^4-q^3+q-1)$
	8D	$\begin{cases} (q-1)\times(q^7-q^6+q^5-q^4+q^3-q^2+q-1), & q \text{ even} \\ (2q-2)\times\frac{(q^7-q^6+q^5-q^4+q^3-q^2+q-1)}{2}, & q \text{ odd} \end{cases}$	$(q^3-q^2+q-1)\times(q^5-q^4+q-1)$

Table 32: The torsion coefficient decomposition of various maximal tori of $\mathrm{F}_4(q)$, $\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$ and $\mathrm{E}_8(q)$.

	8E	$\begin{cases} (q+1)\times(q^7+q^6+q^5+q^4+q^3+q^2+q+1), & q \text{ even} \\ (2q+2)\times\frac{(q^7+q^6+q^5+q^4+q^3+q^2+q+1)}{2}, & q \text{ odd} \end{cases}$	$(q^3+q^2+q+1)\times(q^5+q^4+q+1)$
	8G	$\begin{cases} (q^8-1), & q \text{ even} \\ 2^2\times\frac{(q^8-1)}{4}, & q \text{ odd} \end{cases}$	$(q^2-1)\times(q^2+1)\times(q^4+1)$
	9B	$\begin{cases} (q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1), & q \text{ even} \\ 3\times\frac{(q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1)}{3}, & q \text{ odd} \end{cases}$	$(q^2+q+1)\times(q^6+q^3+1)$
	18B	$\begin{cases} (q^8-q^7+q^6-q^5+q^4-q^3+q^2-q+1), & q \text{ even} \\ 3\times\frac{(q^8-q^7+q^6-q^5+q^4-q^3+q^2-q+1)}{3}, & q \text{ odd} \end{cases}$	$(q^2-q+1)\times(q^6-q^3+1)$

7.5 Maximal Tori of Twisted $G(q)$

In order to compress the description of the σ-classes in Tables 34, 36, 37, we have the convention that, for example, $2A_{12}D_{72}4C_{72}6D_{96}G_{288}$ ($w = [69]$ in Table 37) means $2A_{12}2D_{72}4C_{72}6D_{96}6G_{288}$.

Table 33: The maximal tori of ${}^2B_2(q)$.

w	σ-class of w	Min	Shape of $C_{W,\sigma}(w)$	Structure of $T_w(q)$
[1]	$2B_1C_1$	(1,2)	4	$q - \sqrt{2q} + 1$
[1,2,1]	$2B_1C_1$	(3,2)		$q + \sqrt{2q} + 1$
Id_W	$1A_12A_14A_2$	(0,1)	2	$q - 1$

Table 34: The maximal tori of ${}^2G_2(q)$.

w	σ-class of w	Min	Shape of $C_{W,\sigma}(w)$	Structure of $T_w(q)$
[1]	$2B_1C_1$	(1,2)	6	$q - \sqrt{3q} + 1$
[6,2,1]	$2B_1C_1$	(3,2)		$2 \times \frac{(q+1)}{2}$
[6]	$2B_1C_1$	(5,2)		$q + \sqrt{3q} + 1$
Id_W	$1A_12A_13A_26A_2$	(0,1)	2	$q - 1$

Table 35: The maximal tori of ${}^3D_4(q)$.

w	σ-class of w	Min	Shape of $C_{W,\sigma}(w)$	Structure of $T_w(q)$
$[1,4,3,2]$	$3A_66A_2$	(4,8)	$SL(2,3)$	$(q^2 - q + 1)^2$
$[1,4,3,2,1,4,3,2]$	$3A_26A_6$	(8,8)		$(q^2 + q + 1)^2$
Id_W	$1A_12B_3C_3D_34A_6$	(0,1)	$\mathrm{Dih}(12)$	$(q-1) \times (q^3 - 1)$
$[1,4,2,3,4,2]$	$2A_1B_3C_3D_34A_6$	(6,9)		$(q+1) \times (q^3 + 1)$
[2]	$2E_9F_34B_{12}C_{12}D_{12}$	(1,1)	2^2	$q^4 + q^3 - q - 1$
[1]	$2E_3F_94B_{12}C_{12}D_{12}$	(1,3)		$q^4 - q^3 + q - 1$
[1,2]	$3A_{24}6A_{24}$	(2,6)	4	$q^4 - q^2 + 1$

Since $D_4(q)$ does not appear in our work, we have no fixed labelling of the standard conjugacy classes of the related Weyl group. This only impacts upon the labelling of the σ-classes represented by [2] and [1] in Table 35. We remark that the σ-class of [1] contains the simple reflections corresponding to [3] and [4], which is enough to differentiate it from that of [2].

Table 36: The maximal tori of ${}^2F_4(q)$.

w	σ-class of w	Min	Shape of $C_{W,\sigma}(w)$	Structure of $T_w(q)$
[14,1,2,3] [21,2,3,2]	$2F_2G_48A_212A_4$ $2F_2G_48A_212A_4$	(6,12) (18,12)	$GU_2(3) \sim 4{\cdot}Sym(4)$	$(q-\sqrt{2q}+1)^2$ $(q+\sqrt{2q}+1)^2$
[14,20]	$2F_2G_44A_28A_812A_8$	(12,24)	$GL_2(3)$	$(q+1)^2$
[20,1]	$2F_8G_83A_44A_4B_4C_46A_48A_{20}12A_{16}$	(8,22)	4^2	q^2+1
Id_W	$1A_12A_1F_4G_43A_44A_6B_8C_8$ $6A_48A_{16}12A_{16}$	(0,1)	Dih(16)	$(q-1)^2$
[3,1] [21,1]	$2G_{16}3B_8C_84B_4C_46B_8C_88A_{24}12A_{16}$ $2G_{16}3B_8C_84B_4C_46B_8C_88A_{24}12A_{16}$	(2,6) (10,16)	12	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$ $q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
[2] [21]	$2B_4C_4D_4E_44D_{16}E_{16}\ 6D_{24}E_{24}F_{24}G_{24}$ $2B_4C_4D_4E_44D_{16}E_{16}\ 6D_{24}E_{24}F_{24}G_{24}$	(1,2) (3,2)	2×4	$q^2-q\sqrt{2q}+\sqrt{2q}-1$ $q^2+q\sqrt{2q}-\sqrt{2q}-1$
[2,20]	$2G_{16}3A_8B_{16}C_{16}4B_{16}C_{16}$ $6A_8B_{16}C_{16}8A_{48}12A_{16}$	(4,10)	6	q^2-q+1
[1]	$2B_4C_4D_4E_44D_{40}E_{40}6D_{48}E_{48}F_{48}G_{48}$	(1,2)	2^2	q^2-1

Table 37, giving the maximal tori for ${}^2E_6(q)$, has an additional column. This column gives the corresponding $T_w(q)$ number in the $E_6(q)$ Table 29, so illuminating the one-to-one correspondence between the maximal tori of $E_6(q)$ and ${}^2E_6(q)$.

Table 37: The maximal tori of ${}^2\mathrm{E}_6(q)$.

w	σ-class of w	Min	$T_w(q)$	Shape of $C_{W,\sigma}(w)$	Structure of $T_w(q)$
w_0	$2\mathrm{B}_1$	(36,1)	1	$W(\mathrm{E}_6) \sim \mathrm{Sp}_4(3) : 2$	$(q+1)^6$
$[5,4,5,69,3,4,5]$	$2\mathrm{D}_{12}4\mathrm{B}_{24}$	(15,1)	2	$2 \times \mathrm{Sym}(6)$	$(q^2-1) \times (q+1)^4$
Id_W	$1\mathrm{A}_1 2\mathrm{C}_{12} 3\mathrm{C}_{32}$	(0,1)	8	$W(\mathrm{F}_4) \sim 2_+^{1+4} : \mathrm{Sym}(3)^2$	$(q-1)^2 \times (q^2-1)^2$
$[18,6,48,5]$	$6\mathrm{A}_{16}\mathrm{C}_{64}$	(12,80)	21	$3_+^{1+2} : \mathrm{SL}_2(3)$	$(q^2-q+1)^3$
$[5,2,4,69,3,18]$	$6\mathrm{C}_{32}\mathrm{E}_{64}\mathrm{F}_{144}$	(16,20)	4	$3 \times (\mathrm{Sym}(3) \wr 2)$	$(q+1)^3 \times (q^3+1)$
$[69,4]$	$2\mathrm{B}_{12}\mathrm{C}_{18}4\mathrm{D}_{144}6\mathrm{E}_{96}$	(6,1)	3	$\mathrm{Dih}(8) \times \mathrm{Sym}(4)$	$(q+1)^2 \times (q^2-1)^2$
$[2,1,3,48]$	$2\mathrm{B}_{32}4\mathrm{D}_{288}6\mathrm{A}_{128}\mathrm{E}_{32}$	(14,146)	10	$3 \times \mathrm{Sym}(3)^2$	Table 38
$[1]$	$2\mathrm{A}_{24}4\mathrm{C}_{36}6\mathrm{B}_{192}\mathrm{G}_{288}$	(1,4)	16	$4 \times \mathrm{Sym}(4)$	$(q-1)^2 \times (q^4-1)$
$[6,2,4,69]$	$4\mathrm{A}_{12}\mathrm{D}_{144}12\mathrm{A}_{384}$	(18,120)	14	$\mathrm{GU}_2(3) \sim 4{}^{\cdot}\mathrm{Sym}(4)$	$(q^3+q^2+q+1)^2$
$[69]$	$2\mathrm{A}_{12}\mathrm{D}_{72}4\mathrm{C}_{72}6\mathrm{D}_{96}\mathrm{G}_{288}$	(1,2)	5	$2^2 \times \mathrm{Sym}(4)$	$(q^2-1)^3$
$[5,6,2,69]$	$3\mathrm{A}_{16}\mathrm{C}_{128}6\mathrm{F}_{192}9\mathrm{A}_{384}$	(16,180)	25	$3 \times \mathrm{SL}_2(3)$	$(q^2+q+1) \times (q^4+q^2+1)$
$[4,2]$	$3\mathrm{A}_{64}\mathrm{B}_{32}6\mathrm{C}_{288}\mathrm{F}_{96}9\mathrm{A}_{384}12\mathrm{A}_{576}$	(2,2)	13	$6 \times \mathrm{Sym}(3)$	$(q^2-1) \times (q^4+q^3-q-1)$
$[6,5]$	$2\mathrm{C}_{96}3\mathrm{B}_{64}\mathrm{C}_{32}4\mathrm{D}_{288}5\mathrm{A}_{576}9\mathrm{A}_{384}$	(2,6)	22		$(q-1) \times (q^5-q^4+q^3-q^2+q-1)$
$[3,18,6]$	$2\mathrm{D}_{96}4\mathrm{C}_{288}6\mathrm{G}_{96}8\mathrm{A}_{576}12\mathrm{B}_{384}$	(5,22)	15		$(q^2-q+1) \times (q^4-q^3+q-1)$
$[69,4,3]$	$4\mathrm{B}_{192}6\mathrm{B}_{288}\mathrm{G}_{192}10\mathrm{A}_{576}12\mathrm{B}_{192}$	(7,8)	6		$(q+1)^2 \times (q^4-q^3+q-1)$
$[2,69,4]$	$2\mathrm{D}_{72}4\mathrm{B}_{36}\mathrm{C}_{72}6\mathrm{D}_{288}8\mathrm{A}_{576}10\mathrm{A}_{576}$	(7,6)	7	$4 \times \mathrm{Dih}(8)$	$(q+1)^2 \times (q^4-1)$
$[3,4]$	$3\mathrm{B}_{144}4\mathrm{D}_{576}5\mathrm{A}_{576}6\mathrm{A}_{192}\mathrm{C}_{96}12\mathrm{A}_{576}$	(2,4)	9	$3 \times \mathrm{Dih}(8)$	$(q^2-1) \times (q^4-q^3+q-1)$
$[6,4]$	$2\mathrm{C}_{144}3\mathrm{C}_{288}4\mathrm{A}_{144}\mathrm{D}_{72}5\mathrm{A}_{576}6\mathrm{E}_{288}\mathrm{F}_{576}9\mathrm{A}_{1152}$	(2,6)	11	$2^2 \times 4$	$(q^2-1) \times (q^4-1)$
$[6,2,3]$	$2\mathrm{D}_{288}4\mathrm{B}_{288}6\mathrm{B}_{192}\mathrm{D}_{96}\mathrm{G}_{1152}8\mathrm{A}_{576}10\mathrm{A}_{576}12\mathrm{B}_{1152}$	(3,6)	18	$2^2 \times 3$	Table 38
$[6,2,4]$	$6\mathrm{B}_{192}\mathrm{D}_{384}\mathrm{G}_{1152}8\mathrm{A}_{1152}10\mathrm{A}_{1152}12\mathrm{B}_{288}$	(3,4)	20	12	$q^6+q^5+q^4-q^2-q-1$
$[4,18,48,5]$	$4\mathrm{A}_{384}5\mathrm{A}_{1152}6\mathrm{C}_{576}\mathrm{F}_{576}9\mathrm{A}_{1152}12\mathrm{A}_{480}$	(6,56)	23		$q^6-q^5+q^3-q+1$
$[5,4,1]$	$4\mathrm{C}_{576}6\mathrm{B}_{576}\mathrm{G}_{576}8\mathrm{A}_{1152}10\mathrm{A}_{1152}12\mathrm{B}_{1152}$	(3,12)	17	10	q^6-q^5+q-1
$[2,69,3,18]$	$4\mathrm{D}_{576}5\mathrm{A}_{1152}6\mathrm{E}_{576}\mathrm{F}_{576}9\mathrm{A}_{1152}12\mathrm{A}_{1152}$	(8,40)	12		$(q+1) \times (q^5+1)$
$[2,18,6,3]$	$4\mathrm{D}_{1152}5\mathrm{A}_{1152}6\mathrm{A}_{384}\mathrm{C}_{384}\mathrm{E}_{384}9\mathrm{A}_{1152}12\mathrm{A}_{1152}$	(4,24)	24	9	q^6-q^3+1
$[5,2,4]$	$4\mathrm{C}_{576}6\mathrm{D}_{576}\mathrm{G}_{576}8\mathrm{A}_{2448}10\mathrm{A}_{1152}12\mathrm{B}_{1152}$	(3,8)	19	8	$q^6-q^4+q^2-1$

Table 38: The structure of various maximal tori in in $^2\mathrm{E}_6(q)$.

Class	Torsion coefficient decomposition	Uniform structure
$2\mathrm{B}_{32}4\mathrm{D}_{288}6\mathrm{A}_{128}\mathrm{E}_{32}$	$\begin{cases} 3 \times \frac{(q^3+1)}{3} \times (q^3+1), & q \equiv 2 \bmod 3 \\ (q^3+1)^2, & q \equiv 0, 1 \bmod 3 \end{cases}$	$(q+1) \times (q^2-q+1) \times (q^3+1)$
$2\mathrm{D}_{288}4\mathrm{B}_{288}6\mathrm{B}_{192}\mathrm{D}_{96}\mathrm{G}_{1152}$ $8\mathrm{A}_{576}10\mathrm{A}_{576}12\mathrm{B}_{1152}$	$\begin{cases} 3 \times \frac{(q^6-1)}{3}, & q \equiv 2 \bmod 3 \\ (q^6-1), & q \equiv 0, 1 \bmod 3 \end{cases}$	$(q^2-q+1) \times (q^4+q^3-q-1)$

7.6 Examples

As indicated in Section 7.2, the matrix $\sigma w - I$ need not always be diagonalizable in a uniform manner. That is, there may not exist a single collection of matrix operations which give a diagonal output at every prime power q. This is a consequence of the fact that, treating q as a variable, the polynomial ring $\mathbb{Z}[q]$ is not a principal ideal domain and hence there may not exist a Smith normal form for $\sigma w - \mathrm{I}$. An example in [26] is given of the torus corresponding to class 6G of $W(\mathrm{E}_6)$ (class name as in Table 29), where the operations performed to diagonalize $\sigma w - \mathrm{I}$ differ depending on the value of q modulo 3. Deriziotis and Fakiolas conclude by working backwards to write the torus structure as a direct product of cyclic groups in such a way that a single set of polynomials in q describe the torus structure uniformly. We remark that the methods demonstrated in this section can be applied to all the untwisted / twisted conjugacy classes presented in this work.

Consider the maximal torus in the adjoint group $\mathrm{E}_7(q)$ corresponding to the class 6M in $W(\mathrm{E}_7)$. We can obtain a class representative w from $[2, 5, 7][4, 6]$ in Table 30 by taking the product of the reflection matrices corresponding to roots $2, 4, 5, 6, 7$ in the given order. The matrix $\sigma w - I = q.w - 1$ is then as follows.

$$\begin{pmatrix} q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q-1 & 0 & -q & 0 & 0 & 0 \\ 0 & 0 & q-1 & q & 0 & 0 & 0 \\ 0 & q & 0 & q-1 & q & q & 0 \\ 0 & 0 & 0 & -q & -q-1 & -q & 0 \\ 0 & 0 & 0 & q & q & q-1 & q \\ 0 & 0 & 0 & 0 & 0 & -q & -q-1 \end{pmatrix}$$

We start diagonalizing by applying the operations: $C_2 \mapsto C_2 - C_4$, $C_4 \mapsto C_4 - qC_2$, $R_3 \mapsto R_3 - qR_2$, $R_4 \mapsto R_4 + R_2$, $R_5 \mapsto R_5 + qR_2$, $R_6 \mapsto R_6 - qR_2$, $R_3 \mapsto R_3 + R_5$, $R_6 \mapsto R_6 + R_5$, $C_5 \mapsto C_5 + qC_4$, $C_6 \mapsto C_6 + qC_4$, $R_5 \mapsto R_5 - (q^q + q)R_4$, $C_5 \mapsto C_5 - C_6$, $R_7 \mapsto R_7 + qR_5$, $C_6 \mapsto C_6 - (q^3 + q^2 + q)C_5$, $C_7 \mapsto C_7 + qC_6$, $R_7 \mapsto R_7 - (q^4 + q^3 + q^2 + q)R_6$, $R_3 \mapsto R_3 + (q^q + q)R_6$, $R_3 \mapsto R_3 - R_5$, $C_7 \mapsto C_7 - (q^3 + 2q^2 +$

$2q+2)C_3,\ R_3 \leftrightarrow R_6,\ C_3 \leftrightarrow C_6$. We are left with the matrix

$$\begin{pmatrix} q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q-1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q^5-q^4-q^3-q^2-q-1 \end{pmatrix}.$$

Now, if q is even we apply the operations $C_6 \to C_6 - \frac{q}{2}C_7, C_7 \to C_7 + 2C_6$ and $R_7 \to R_7 + \frac{(q^6+q^5+q^4+q^3+q^2+q)}{2}R_6$. If q is odd we apply operations $C_6 \to C_6 - \frac{(q-1)}{2}C_7,\ R_7 \mapsto R_7 + \frac{(q^5+q^4+q^3+q^2+q+1)}{2}R_6,\ R_6 \leftrightarrow R_7$. So a diagonal form of $q.w - I$ is

$$\begin{cases} \text{diag}(q-1,-1,-1,-1,-1,-1,q^6-1), & q \text{ even} \\ \text{diag}(q-1,-1,-1,-1,-1,2,(q^6-1)/2), & q \text{ odd} \end{cases}$$

and we can rearrange diagonal entries so that these matrices constitute a Smith normal form.

Now for the simply connected case. We obtain a class representative in the same way as before, however now we must specify the simply connected isogeny type. To do this, we call `RootDatum("E7":Isogeny:="SC")` and this will be the second argument in our function `makew`. The matrix $\sigma w - I = q.w - 1$ is then

$$\begin{pmatrix} q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & q & -q & q & 0 & 0 \\ 0 & 0 & q-1 & 0 & 0 & 0 & 0 \\ 0 & q & q & -q-1 & q & 0 & 0 \\ 0 & q & q & -q & q-1 & -q & q \\ 0 & 0 & 0 & 0 & q & -q-1 & q \\ 0 & 0 & 0 & 0 & q & -q & -1 \end{pmatrix}.$$

Now we apply the operations: $R_4 \to R_4 + qR_2, R_5 \to R_5 + qR_2, C_3 \to C_3 + qC_2, C_4 \to C_4 - qC_2, C_5 \to C_5 + qC_2, C_6 \to C_6 - qC_7, C_5 \to C_5 + qC_7, R_6 \to R_6 + qR_7, R_5 \to R_5 + qR_7, R_4 \to R_4 - R_5, R_5 \to R_5 - (q^2+q)R_4, C_5 \to C_5 - (q^2-1)C_4, C_6 \to C_6 + (q^2+q)C_4, C_6 \to C_6 + C_5, C_5 \to C_5 + (q^2+q)C_6, R_5 \to R_5 - (q^3+q^2+q+1)R_6, R_5 \to R_5 - (q+2)R_3, R_5 \leftrightarrow R_7, C_5 \leftrightarrow C_7, C_3 \leftrightarrow C_6, R_3 \leftrightarrow R_6$. We are left with the matrix

$$\begin{pmatrix} q-1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -q^5-q^4-q^3-q^2-q-1 \end{pmatrix}$$

which we see is the transpose of the matrix we had earlier. Hence by interchanging rows with columns and vice versa in the operations for cases q even and q odd from before, we arrive at the same diagonal matrices. Consequently, the torus structure is the same in both the adjoint and simply connected cases.

According to the Magma handbook, the function `TwistedTorusOrder` can be used to obtain the structures of the maximal tori in groups of Lie type. The function takes as arguments a root datum R and an element w of the Weyl group corresponding to R, and produces a list of polynomials in q which, upon evaluation, yield the orders of the cyclic factors of the maximal torus corresponding to w. The function was written by Haller, based on work from their thesis [38]. As noted in the remarks preceding Table 24, for groups of type E_6, E_7 and E_8, the output of `TwistedTorusOrder` does not always coincide with the cyclic factors given in [26].

We further illustrate the methods used to obtain the structures in Table 32 with another example. To reduce listing all column operations, we now signpost our calculations with the phrase kill row j. By kill row j we mean a_{jj} is the only non-zero entry in column j and divides all the entries in row j, so appropriate column operations will result in a_{jj} being the only non-zero entry in row j and not disturbing the rest of the matrix. Consider our representative w corresponding to the class 6W in $W(E_8)$. The matrix $q.w - I$ is

$$\begin{pmatrix} -q-1 & -q & -2q & -2q & -2q & -q & 0 & 0 \\ 0 & -q-1 & q & 2q & q & q & 0 & 0 \\ q & q & 2q-1 & 2q & q & 0 & 0 & 0 \\ 0 & -q & -q & -q-1 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -q & 0 & -q & -q & -1 & 0 & 0 \\ 0 & q & q & 2q & 2q & q & q-1 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q-1 \end{pmatrix}$$

As [26] remarks, this matrix is not diagonalizable over $\mathbb{Z}[q]$. Now we perform the following elementary row and column operations: $R_1 \mapsto R_1 + R_3,\ R_3 \mapsto R_3 + qR_1,\ C_3 \mapsto C_3 - C_1,\ C_5 \mapsto C_5 - qC_1,\ C_6 \mapsto C_6 - qC_1,\ R_2 \mapsto R_2 - R_3,\ R_2 \leftrightarrow R_1,\ C_2 \leftrightarrow C_1,\ R_3 \mapsto R_3 + qR_1,\ R_4 \mapsto R_4 - qR_1,\ R_5 \mapsto R_5 + qR_1,\ R_6 \mapsto R_6 - qR_1,\ R_7 \mapsto R_7 + qR_1,\ C_3 \mapsto C_3 + C_1,\ C_5 \mapsto C_5 + q^2C_1,\ C_6 \mapsto C_6 + (q^2+q)C_1,\ C_3 \mapsto C_3 - C_4,\ R_3 \leftrightarrow R_1,\ C_3 \leftrightarrow C_1,\ R_4 \mapsto R_4 + (-q+1)R_1,\ R_5 \mapsto R_5 + qR_1,\ C_4 \mapsto C_4 + 2qC_1,\ C_5 \mapsto C_5 + (q^3 - q^2 + q)C_1,\ C_6 \mapsto C_6 + q^3C_1,\ R_4 \mapsto R_4 + R_5,\ R_4 \mapsto R_4 + R_6,\ R_4 \leftrightarrow R_1,\ C_4 \leftrightarrow C_1,\ R_5 \mapsto R_5 + 2q^2R_1,\ R_6 \mapsto R_6 - qR_1,\ R_7 \mapsto R_7 + 2qR_1,\ C_5 \mapsto C_5 + (-q^2 - 1)C_1,\ C_6 \mapsto C_6 + (-q^2 - 1)C_1,\ R_7 \mapsto R_7 + R_8,\ R_7 \leftrightarrow R_1,\ C_8 \leftrightarrow C_1,\ R_8 \mapsto R_8 + (-q-1)R_1,\ C_5 \mapsto C_5 - q^3C_1,\ C_6 \mapsto C_6 + (-q^3 + q^2 - q)C_1,\ R_8 \mapsto R_8 + (-q-1)R_1,\ C_5 \mapsto C_5 - q^3C_1,\ C_6 \mapsto C_6 + (-q^3 + q^2 - q)C_1,\ C_7 \mapsto C_7 + (q-1)C_1,\ R_5 \mapsto R_5 - q^2R_6,\ R_8 \mapsto R_8 - (-q^2 - q)R_6,\ C_5 \mapsto C_5 - (-q^2 - q - 1)C_7,\ C_5 \leftrightarrow C_8,\ R_6 \leftrightarrow R_7,\ R_5 \leftrightarrow R_6$. We now have the matrix:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^2+q-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^4-q^2-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q^2+1 & q+1 \end{pmatrix} \tag{1}$$

If $q \equiv 0 \bmod 3$ then we may apply $R_8 \mapsto R_8 + \frac{q}{3}R_7$ and $R_7 \mapsto R_7 + 3R_8$. We may then kill row 8 to obtain the matrix $\mathrm{diag}(-1, -1, -1, -1, -1, -q^2 + q - 1, -q^6 + 1, 1)$.

If $q \equiv 1 \bmod 3$ then we may apply $R_8 \mapsto R_8 + \frac{q+2}{3}R_7$ and $R_7 \mapsto R_7 - 3R_8$. We may then kill row 8 to obtain $\mathrm{diag}(-1,-1,-1,-1,-1,-q^2+q-1,-q^6+1,1)$ once again.

If $q \equiv 2 \bmod 3$, then we apply $R_8 \mapsto R_8 + \frac{q+1}{3}R_7$, $C_7 \mapsto C_7 - (q^5 - q^4 + q^3 - q^2 + q - 1)C_8$ and $C_7 \leftrightarrow C_8$ to obtain $\mathrm{diag}(-1,-1,-1,-1,-1,-q^2+q-1,-3,-q^6+1)$.

Hence we see that the maximal torus corresponding to class 6W in $W(\mathrm{E}_8)$ has structure

$$T_w(q) \cong \begin{cases} 3 \times (q^2-q+1) \times \frac{(q^6-1)}{3} & q \equiv 2 \bmod 3 \\ (q^2-q+1) \times (q^6-1) & q \equiv 0,\ 1 \bmod 3 \end{cases}$$

and from this we can read off the torsion coefficients.

We now have a further example involving a maximal torus of E_8 that is an extension (by a cyclic factor of $q-1$) of a maximal torus from E_7. As we shall see, when adding on this additional cyclic factor, the resulting matrix may no longer be in Smith normal form. Thus, in these cases, there is additional work to be done in order to give the torsion coefficient decomposition of the corresponding torus.

Consider the class 6S in $W(\mathrm{E}_8)$ with corresponding torus index $22'$. We remark that the admissible graph for this class, namely $\mathrm{A}_1 + \mathrm{A}_5$, can consist of roots that are orthogonal to r_8 as one would expect. The matrix $q.w - I$ is

$$\begin{pmatrix} -q-1 & 0 & -q & 0 & 0 & 0 & 0 & 0 \\ -q & -q-1 & -q & -2q & -2q & -q & 0 & 0 \\ q & q & q-1 & q & 0 & 0 & 0 & 0 \\ 0 & -q & -q & -q-1 & 0 & 0 & 0 & 0 \\ 0 & q & q & q & q-1 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q-1 & 0 & 0 \\ q & q & 2q & 3q & 2q & 2q & q-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 \end{pmatrix}$$

This matrix can be decomposed into a 7×7 matrix that can be extended to an 8×8 matrix by adding an additional row and column consisting of all 0 entries apart from $q-1$ in the new diagonal entry. This 7×7 sub-matrix corresponds to the matrix $q.w - I$ where w is a representative of class 6O in $W(\mathrm{E}_7)$ (the class with torus index 22). We omit the details for the diagonalization of this 7×7 sub-matrix, but from Table 32 we see that

$$T_w(q) \cong \begin{cases} (q+1) \times (q^6-1), & q \text{ even} \\ 2 \times (q+1) \times \frac{(q^6-1)}{2}, & q \text{ odd.} \end{cases}$$

The operations used to diagonalize this sub-matrix are independent of the last row and column, hence in E_8 we have that

$$T_w(q) \cong \begin{cases} (q-1) \times (q+1) \times (q^6-1), & q \text{ even} \\ 2 \times (q-1) \times (q+1) \times \frac{(q^6-1)}{2}, & q \text{ odd.} \end{cases}$$

However, these structures are not in Smith normal form. When q is even, $q-1$ and $q+1$ are coprime. Indeed, let d be a common divisor. Then d divides $(q+1)-(q-1) = 2$. Hence $d = 1$ or 2. As $q-1$ and $q+1$ are both odd, we cannot have $d = 2$ and so $d = 1$ as required. Hence, using the fact $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ when n, m are coprime, we get that $T_w(q) \cong (q^2-1) \times (q^6-1)$ for q even.

Now suppose q is odd. By our previous remarks, we know that we may diagonalize $q.w - I$ to obtain the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(q^6-1)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q-1 \end{pmatrix} \tag{2}$$

Consider the 3×3 diagonal sub-matrix of (2) given by

$$\begin{pmatrix} q+1 & 0 & 0 \\ 0 & \frac{(q^6-1)}{2} & 0 \\ 0 & 0 & q-1 \end{pmatrix}$$

To this matrix, we may apply the following row and column operations to obtain the Smith normal form: $R_3 \mapsto R_3 + R_1,\ C_3 \mapsto C_3 - C_1,\ R_1 \mapsto R_1 - \frac{q+1}{2}R_3,\ C_1 \mapsto C_1 + \frac{q+1}{2}C_3$. This results in the matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{(q^2-1)}{2} & 0 \\ 0 & 0 & \frac{(q^6-1)}{2} \end{pmatrix}$$

and hence we have arrived at the result shown in Table 32.

The above examples were part of a collaborative effort by myself and Joe Parkin. In all, every single conjugacy class of the Weyl groups considered in this section were examined individually by myself or Joe. Shown below are few examples that I considered.

We consider the class 2F in $W(F_4)$ with corresponding admissible graph $2\mathrm{A}_1/2\tilde{\mathrm{A}}_1$. If w is the matrix representative of this class corresponding to the product of reflection matrices [1,20], then the matrix $q.w - I$ is:

$$\begin{pmatrix} -q-1 & 0 & 0 & 0 \\ 0 & -q-1 & -2q & -2q \\ q & 2q & 3q-1 & 2q \\ -q & -2q & -2q & -q-1 \end{pmatrix}.$$

We can fully diagonalise this matrix to get a uniform torus structure using the following operations: $R_1 \mapsto R_1 + R_3, R_3 \mapsto R_3 + qR_1, R_4 \mapsto R_4 - qR_1, C_2 \mapsto C_2 + 2qC_1, C_3 \mapsto C_3 + (3q-1)C_1, C_4 \mapsto C_4 + 2qC_1, C_3 \mapsto C_3 - C_4, R_3 \mapsto R_3 + 2qR_2, R_4 \mapsto R_4 - 2qR_2, R_4 \mapsto R_4 + R_3, R_3 \mapsto R_3 + 2qR_4, R_2 \mapsto R_2 + R_4, C_4 \mapsto C_4 - C_2$. After these operations, and multiplying terms by -1 when necessary, we are left with the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q-1 & 0 & 0 \\ 0 & 0 & q+1 & 0 \\ 0 & 0 & 0 & q^2-1 \end{pmatrix}.$$

Hence, we have that $T_w \cong (q-1)\times(q+1)\times(q^2-1)$ for all q. However, this is not the torsion coefficient decomposition of T_w. As with many of the previous examples, we must consider the cases of q odd and q even in order to find this decomposition. Suppose that q is even. Then we know from the example involving class 6S in $W(\mathrm{E}_8)$ that $q-1$ and $q+1$ are coprime. Hence, by the fact $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ when n, m are coprime, we have that $T_w \cong (q^2-1)^2$.

Now suppose q is odd. Into the above diagonal matrix, substitute $2q+1$ for q. After performing the following operations: $R_2 \mapsto R_2 + R_4, C_2 \mapsto C_2 + C_4, R_4 \mapsto R_4 + qR_2, C_4 \mapsto C_4 + qC_2$, and substituting back in $\frac{q-1}{2}$ for q, we obtain the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{q^2-1}{2} & 0 \\ 0 & 0 & 0 & q^2-1 \end{pmatrix}.$$

Hence, when q odd, the torsion coefficient decomposition is $T_w \cong 2 \times \frac{q^2-1}{2} \times (q^2-1)$.

We now consider a further example in $W(F_4)$. Consider the class 4D with representative w constructed using the product of reflection matrices [6,21][13]. The matrix $q.w - I$ is

$$\begin{pmatrix} q-1 & 2q & 2q & 0 \\ 0 & q-1 & 2q & 2q \\ 0 & -q & -q-1 & -q \\ -q & -q & -2q & -q-1 \end{pmatrix}.$$

After performing the operations: $R_1 \mapsto R_1 + R_4, R_4 \mapsto R_4 - qR_1, C_2 \mapsto C_2 + qC_1, C_4 \mapsto C_4 - (q+1)C_1, R_2 \mapsto R_2 + R_3, R_3 \mapsto R_3 - qR_2, R_4 \mapsto R_4 - (q^2+q)R_2, C_3 \mapsto C_3 + (q-1)C_2, C_4 \mapsto C_4 + qC_2, C_3 \mapsto C_3 - C_4, R_4 \mapsto R_4 + R_3$, we get the matrix

$$k = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & q-1 & -q^2-q \\ 0 & 0 & 0 & -q^3-q^2-q-1 \end{pmatrix}.$$

We now split into two cases. First suppose that q is even. Into the above matrix substitute in $2q$ for q. After performing the following operations: $C_4 \mapsto C_4 + (2q+2)C_3, C_3 \mapsto C_3 + qC_4, R_4 \mapsto R_4 + (-8q^4 - 4q^3 - 2q^2 - q)R_3, C_4 \mapsto C_4 - 2C_3$ and substituting back in $\frac{q}{2}$ for q, we are left with the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q^4-1 \end{pmatrix}.$$

Hence when q is even the torus T_w is cyclic and thus the torsion coefficient decomposition is $T_w \cong q^4 - 1$.

Now suppose q is odd. After substituting in $2q+1$ for q in the matrix k, performing the following operations: $C_4 \mapsto C_4 + (2q+3)C_3,\ C_3 \mapsto C_3 + qC_4,\ R_4 \mapsto R_4 - (4q^3 + 8q^2 + 6q + 2)R_3, R_4 \leftrightarrow R_3$, and then substituting back in $\frac{q-1}{2}$ for q, we are left with the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{q^4-1}{2} \end{pmatrix}.$$

Hence when q is odd, the torsion coefficient decomposition is $T_w \cong 2 \times \frac{q^4-1}{2}$ and we have proved the information shown in Table 32.

Finally, we consider an example from $W(\mathrm{E}_8)$. We consider the class 6E with corresponding representative w. The matrix $q.w - I$ is

$$\begin{pmatrix} -q-1 & -q & -2q & -2q & -2q & -q & 0 & 0 \\ q & 2q-1 & 2q & 3q & 2q & 2q & q & 0 \\ q & q & 2q-1 & 2q & q & 0 & 0 & 0 \\ 0 & -q & -q & -q-1 & 0 & 0 & 0 & 0 \\ -q & 0 & -q & -q & -q-1 & -q & -q & 0 \\ 0 & -q & 0 & -q & -q & -1 & 0 & 0 \\ 0 & q & q & 2q & 2q & q & q-1 & 0 \\ -q & -2q & -3q & -5q & -4q & -3q & -2q & -q-1 \end{pmatrix}.$$

This matrix can be diagonalised using the following operations: $R_1 \mapsto R_1 + R_2,\ R_2 \mapsto R_2 + qR_1,\ R_3 \mapsto R_3 + qR_1,\ R_5 \mapsto R_5 - qR_1,\ R_8 \mapsto R_8 - qR_1,\ C_2 \mapsto C_2 + (q-1)C_1,\ C_4 \mapsto C_4 + qC_1,\ C_6 \mapsto C_6 + qC_1,\ C_7 \mapsto C_7 + qC_1,\ R_2 \mapsto R_2 - R_3,\ R_2 \leftrightarrow R_1,\ C_3 \leftrightarrow C_1,\ R_3 \mapsto R_3 - (2q-1)R_1,\ R_4 \mapsto R_4 + qR_1,\ R_5 \mapsto R_5 + qR_1,\ R_7 \mapsto R_7 - qR_1,\ R_8 \mapsto R_8 + 3qR_1,\ C_2 \mapsto C_2 - (q-1)C_1,\ C_4 \mapsto C_4 - qC_1,\ C_5 \mapsto C_5 - qC_1,\ C_6 \mapsto C_6 - 2qC_1,\ C_7 \mapsto C_7 - qC_1,\ R_4 \mapsto R_4 + (q-1)R_5,\ R_4 \leftrightarrow R_1,\ C_4 \leftrightarrow C_1,\ R_3 \mapsto R_3 + (-q^2+3q)R_1,\ R_5 \mapsto R_5 - qR_1,\ R_6 \mapsto R_6 - qR_1,\ R_7 \mapsto R_7 + (-q^2+2q)R_1,\ R_8 \mapsto R_8 + (2q^2-5q)R_1,\ C_2 \mapsto C_2 + (q^2-2q)C_1,\ C_5 \mapsto C_5 + (q^3-q^2+1)C_1,\ C_6 \mapsto C_6 + (q^3+q)C_1,\ C_7 \mapsto C_7 + qC_1,\ R_5 \mapsto R_5 - R_6,\ C_5 \mapsto C_5 - qC_2,\ R_5 \leftrightarrow R_1,\ C_5 \leftrightarrow C_1,\ C_2 \mapsto C_2 + qC_1,\ C_6 \mapsto C_6 + (q^2-q+1)C_1,\ C_7 \mapsto C_7 - qC_1,\ R_3 \mapsto R_3 + (-q^4+4q^3-6q^2+6q)R_1,\ R_6 \mapsto R_6 + (-q^3+q^2-2q)R_1,\ R_7 \mapsto R_7 + (-q^4+3q^3-4q^2+4q)R_1,\ R_8 \mapsto R_8 + (2q^4-7q^3+9q^2-9q)R_1,\ R_8 \mapsto R_8 + 2R_7,\ C_7 \mapsto C_7 - (-q^3+2q^2-2q+2)C_8,\ R_3 \mapsto R_3 - (q-3)R_6,\ R_6 \mapsto R_6 - qR_3,\ R_7 \mapsto R_7 - (q^4 - 2q^3+3q^2-5q)R_6,\ C_7 \mapsto C_7 + C_2,\ C_7 \mapsto C_7 - (q^3-q^2)C_8,\ C_2 \mapsto C_2 - (q^3-q^2)C_8,\ R_8 \mapsto R_8 - qR_3,\ C_6 \mapsto C_6 - C_7,\ R_7 \mapsto R_7 - (-q+2)R_8,\ R_7 \mapsto R_7 - (-q^2+2q-1)R_6,\ R_3 \mapsto R_3 - (q^2+2)R_6,\ C_2 \mapsto C_2 - (-q^3+q^2-q)C_8,\ R_7 \mapsto R_7 - (q^2-2q)R_3,\ C_2 \mapsto C_2 - (q+1)C_7,\ R_7 \mapsto R_7 - (-q^2+2q+1)R_3,\ R_8 \mapsto R_8 + qR_3, R_7 \mapsto R_7 - (q-2)R_8,\ C_2 \mapsto C_2 - (q^3-q^2+q)C_8,\ R_2 \leftrightarrow R_7,\ R_3 \leftrightarrow R_7$. After multiplying diagonal entries by -1 when necessary, we are left with the diagonal matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2-q+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^2-q+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^3+1 \end{pmatrix}$$

and so we see that $T_w \cong (q+1)\times(q^2-q+1)^2\times(q^3+1)$ for all q. However, as $q+1$ does not divide q^2-q+1, this is not the torsion coefficient decomposition. Much like the previous examples, we now have several cases to consider.

Firstly, suppose $q \equiv 0 \bmod 3$. We shall show that $q+1$ and q^2-q+1 are coprime. Let d be a common divisor of $q+1$ and q^2-q+1. Then d must divide $q^2-q+1-(q-2)(q+1)=3$. So we get d is either 1 or 3. But 3 does not divide $q+1$ when $q\equiv 0 \bmod 3$, hence $d=1$ as desired. Therefore, we may rewrite the torus structure as

$$T_w \cong (q^2-q+1)\times(q+1)(q^2-q+1)\times(q^3+1)\cong(q^2-q+1)\times(q^3+1)^2$$

which does give the torsion coefficient decomposition.

The case $q\equiv 1 \bmod 3$ follows a very similar method. As before, let d be a common divisor of $q+1$ and q^2-q+1. Then d is either 1 or 3. But 3 does not divide $q+1=3t+2$ for some integer t. Hence $d=1$ and again we find that $T_w\cong(q^2-q+1)\times(q^3+1)^2$.

Finally, suppose $q\equiv 2 \bmod 3$. This time we find the torsion coefficient decomposition by diagonalization. In the above diagonal matrix, substitute in $3q+2$ for q and perform the following operations: $R_7\mapsto R_7+R_8,\ C_7\mapsto C_7-3qC_8,\ R_8\mapsto R_8+(3q^2+3q)R_7,\ C_8\mapsto C_8-(q+1)C_7$. After substituting back in $\frac{q-2}{3}$ for q and swapping rows and columns if necessary, we are left with the diagonal matrix

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&3&0&0&0\\0&0&0&0&0&q^2-q+1&0&0\\0&0&0&0&0&0&\frac{q^3+1}{3}&0\\0&0&0&0&0&0&0&q^3+1\end{pmatrix}.$$

Hence in this case, the torsion coefficient decomposition is $T_w\cong 3\times(q^2-q+1)\times\frac{(q^3+1)}{3}\times(q^3+1)$. We remark that the divisibility of coefficients is not obvious here, so we briefly discuss why the consecutive coefficients divide each other. As $q\equiv 2 \bmod 3$, we can write $q=3t+2$ for some integer t. Then $q^2-q+1=9t^2+3t+3$ which is clearly divisible by 3. Furthermore, we have

$$q^3+1=(q^2-q+1)(q+1)=(3t+3)(9t^2+3t+3)=3(t+1)(9t^2+3t+3).$$

Hence $(q^3+1)/3=(t+1)(9t^2+3t+3)$ is divisible by $9t^2+3t+3=q^2-q+1$. Moreover, from this we see that $(q^3+1)/3$ divides q^3+1.

We return to say one more thing about the Magma function `TwistedTorusOrder(R,w)` which takes as arguments an untwisted root datum `R` and Weyl group element `w` and, according to the handbook [6], produces a list of cyclic direct factor orders (as polynomials in q) for the maximal torus corresponding to `w` in a group with root datum `R`. As noted earlier, the function was written by Haller, whose thesis [38] explains that the torus structure can be obtained from the Smith normal form of $qw-I$ computed over the ring $\mathbb{Q}[q]/(q^{|w|}-1)$, but gives no details on how. Upon inspection of the Magma code, the function first computes the Smith normal form of $qw-I$ over $\mathbb{Q}[q]/(q^{|w|}-1)$ and compares the torsion coefficients of the abelian group corresponding to this Smith normal form and the additive groups of the eigenspaces of $qw - I$, at every prime power $q < 20$. If the torsion coefficients agree, then the function returns a list of the diagonal entries of the Smith normal form. Otherwise, the components of the Smith normal form are successively factored until the torsion coefficients of the corresponding abelian group agree with those from the eigenspaces at each $q < 20$. While we have been unable to find an example where `TwistedTorusOrder` returns an inaccurate torus structure, there is no guarantee that one can be sure the structure is accurate at every q by checking only against prime powers less than 20.

We close our discussion of maximal tori with a remark on something to be observed in the tables shown in Section 7.4. Suppose w_1 and w_2 are representatives from two distinct conjugacy classes of $W(E_n)$ (n=6,7,8) such that $\Gamma_{w_1} = \Gamma_{w_2}$. Then $T_{w_1}(q) \cong T_{w_2}(q)$ when q is even.

8 Brauer Character Tables and Feasible Decompositions

8.1 Notation

In this section, we present Brauer character tables and feasible decompositions for various groups listed in Tables 1 and 2. These tables largely follow Atlas [20] notation and are very similar to those shown in [62] and [41]. As in Table 3, we denote slave classes by adjoining letters to each other. For example, if we consider the Brauer character table for $L_2(27)$, we see a column headed 7AC. This implies that $L_2(27)$ has three classes of elements order 7, each having the same column in the Brauer character table, hence we only include it once. Moreover, we see columns headed 13AC and τ(13AC). This implies that there are six conjugacy classes of elements order 13, three of which give column 13AC and the other three have column τ(13AC), where τ is a map (defined in the relevant section) such that all the entries in column τ(13AC) may be obtained from those in 13AC by applying τ. Throughout this section we shall denote the n^{th} root of unity by ω_n.

The feasible decompositions are presented as sums of irreducible modules over K of the respective group. We denote these modules by φ_i for some integer i, and the Brauer character data for each φ_i is to be found in the corresponding row within the Brauer character table. For each φ_i we have an associated cohomological dimension; this is defined to be $\dim(H^1(G, \varphi_i)) = d$ and we denote this by $\varphi_i = d$. If the cohomological dimension $\dim(H^1(G, \varphi_i^*))$ of the dual module φ_i^* is not given, then the modules φ_i are all self-dual. Considering $L_2(27)$ again, all irreducible modules φ_i are self-dual, so we need not consider the cohomological dimensions of the corresponding dual modules. In some cases, due to limitations in computational power, we were not able to calculate the cohomological dimensions.

Along with each feasible decomposition is a fusion pattern. This tells us how the conjugacy classes of the group fuse into G. For example, in decomposition 1 for $L_2(27)$, we have 7AC $\to$ 7B. This implies that the three conjugacy classes of elements order 7 in $L_2(27)$ must all fuse into the class 7B in G. The labelling we use for conjugacy classes is found in Table 3.

For various groups in this section, also given are the socle layer structures of the projective indecomposable modules corresponding to each φ_i. Suppose that P_i denotes the projective cover of φ_i (for φ_i some irreducible KH-module) with socle series $1 = S_0 < S_1 < \cdots < S_n = P_i$. For $i > 1$, define $V_i = S_i/S_{i-1}$. Then we write $P_i = V_1|V_2|\ldots|V_n$. Each socle layer V_i can be expressed as some combination of the irreducible KH-modules.

8.2 $L_2(27)$

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$. We define τ to be the map $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Brauer Character Table

Table 39: Brauer character table for $L_2(27)$.

$L_2(27)$	1A	2A	7AC	13AC	τ(13AC)	14AC
φ_1	1	1	1	1	1	1
φ_2	9	-3	2	$-\alpha + 2$	$\alpha + 3$	4
φ_3	12	0	-2	-1	-1	0
φ_4	27	3	-1	-2α	$2\alpha + 2$	3
φ_5	27	-1	-1	1	1	-1
φ_6	36	0	1	$2\alpha - 2$	$-2\alpha - 4$	-7

Feasible Decompositions

1. $11\varphi_1 + 1\varphi_2 + 1\varphi_3 + 6\varphi_4 + 2\varphi_5 + 0\varphi_6$ (2A $\to$ 2A, 7AC $\to$ 7B, 13AC $\to$ 13CD, τ(13AC) $\to$ 13CD, 14AC $\to$ 14C)
2. $5\varphi_1 + 2\varphi_2 + 0\varphi_3 + 0\varphi_4 + 7\varphi_5 + 1\varphi_6$ (2A $\to$ $2B$, 7AC $\to$ 7B, 13AC $\to$ 13E, τ(13AC) $\to$ 13E, 14AC $\to$ 14E)
3. $2\varphi_1 + 4\varphi_2 + 4\varphi_3 + 1\varphi_4 + 1\varphi_5 + 3\varphi_6$ (2A $\to$ $2B$, 7AC $\to$ 7B, 13AC $\to$ 13F, τ(13AC) $\to$ 13F, 14AC $\to$ 14E)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 3,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 0.$

8.3 $L_2(81)$

Let $\alpha = \omega_{40}^{15} - 2\omega_{40}^{9} + 2\omega_{40}^{7} + \omega_{40}^{5} - 2\omega_{40}^{3} - 2\omega_{40}$ and suppose $\tau \in \Gamma(\mathbb{Q}(\alpha) : \mathbb{Q})$ is defined by $\tau(\alpha) = -\alpha$. Due to the long expressions of each entry, the columns corresponding to the 20 conjugacy classes of elements order 41 are omitted from Table 40.

Brauer Character Table

Table 40: Brauer character table for $L_2(81)$.

$L_2(81)$	1A	2A	4A	5AB	8AB	10AB	20AD	40AD	τ(40AD)
φ_1	1	1	1	1	1	1	1	1	1
φ_2	8	0	4	3	-4	-5	-1	1	1
φ_3	12	-4	4	2	4	6	4	$\alpha + 4$	$-\alpha + 4$
φ_4	16	0	-8	-4	0	0	2	α	$-\alpha$
φ_5	16	0	4	1	2	5	-1	$-\alpha - 3$	$\alpha - 3$
φ_6	18	2	2	3	6	7	-3	$\alpha + 1$	$-\alpha + 1$
φ_7	36	4	4	-4	-4	4	4	$2\alpha + 6$	$-2\alpha + 6$
φ_8	48	0	8	-2	0	-10	-2	0	0
φ_9	48	0	-8	-2	-8	-10	2	2	2
φ_{10}	48	0	-8	-2	8	10	-8	$-\alpha - 2$	$\alpha - 2$
φ_{11}	72	0	4	2	-4	-10	4	$-\alpha - 4$	$\alpha - 4$
φ_{12}	81	1	1	1	1	1	1	1	1
φ_{13}	108	-4	4	-2	-4	6	-6	$2\alpha + 6$	$-2\alpha + 6$
φ_{14}	144	0	-8	4	0	0	2	$-3\alpha - 10$	$3\alpha - 10$

Feasible Decompositions

1. $28\varphi_1 + 8\varphi_2 + 1\varphi_3 + 8\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2A, 4A $\rightarrow$ 4E, 5AB $\rightarrow$ 5A, 8AB $\rightarrow$ 8J, 10AB $\rightarrow$ 10E, 20AD $\rightarrow$ 20A, 40AD $\rightarrow$ 40DE, τ(40AD) $\rightarrow$ 40DE, 41AT $\rightarrow$ 41AE)

2. $6\varphi_1 + 0\varphi_2 + 5\varphi_3 + 0\varphi_4 + 8\varphi_5 + 1\varphi_6 + 1\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2B, 4A $\rightarrow$ 4B, 5AB $\rightarrow$ 5A, 8AB $\rightarrow$ 8B, 10AB $\rightarrow$ 10A, 20AD $\rightarrow$ 20C, 40AD $\rightarrow$ 40F, τ(40AD) $\rightarrow$ 40F, 41AT $\rightarrow$ 41AE)

3. $2\varphi_1 + 4\varphi_2 + 3\varphi_3 + 0\varphi_4 + 4\varphi_5 + 1\varphi_6 + 0\varphi_7 + 2\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2B, 4A $\rightarrow$ 4B, 5AB $\rightarrow$ 5A, 8AB $\rightarrow$ 8G, 10AB $\rightarrow$ 10C, 20AD $\rightarrow$ 20L, 40AD $\rightarrow$ 40C′, τ(40AD) $\rightarrow$ 40C′, 41AT $\rightarrow$ 41FJ)

4. $0\varphi_1 + 2\varphi_2 + 2\varphi_3 + 2\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7 + 1\varphi_8 + 1\varphi_9 + 1\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2B, 4A $\rightarrow$ 4F, 5AB $\rightarrow$ 5B, 8AB $\rightarrow$ 8I, 10AB $\rightarrow$ 10D, 20AD $\rightarrow$ 20K, 40AD $\rightarrow$ 40X′Y′, τ(40AD) $\rightarrow$ 40X′Y′, 41AT $\rightarrow$ 41KO)

5. $0\varphi_1 + 0\varphi_2 + 1\varphi_3 + 1\varphi_4 + 4\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 1\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 1\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2B, 4A $\rightarrow$ 4D, 5AB $\rightarrow$ 5B, 8AB $\rightarrow$ 8K, 10AB $\rightarrow$ 10B, 20AD $\rightarrow$ 20M, 40AD $\rightarrow$ 40W′, τ(40AD) $\rightarrow$ 40W′, 41AT $\rightarrow$ 41KO)

6. $4\varphi_1 + 0\varphi_2 + 1\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 2\varphi_{13} + 0\varphi_{14}$ (2A $\rightarrow$ 2B, 4A $\rightarrow$ 4D, 5AB $\rightarrow$ 5B, 8AB $\rightarrow$ 8K, 10AB $\rightarrow$ 10B, 20AD $\rightarrow$ 20M, 40AD $\rightarrow$ 40ZA′, τ(40AD) $\rightarrow$ 40ZA′, 41AT $\rightarrow$ 41FJ)

Cohomological Dimensions

$\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 4$, $\varphi_5 = 0$, $\varphi_6 = 0$, $\varphi_7 = 0$, $\varphi_8 = 0$, $\varphi_9 = 0$, $\varphi_{10} = 0$, $\varphi_{11} = 0$, $\varphi_{12} = 0$, $\varphi_{13} = 0$, $\varphi_{14} = 0$.

8.4 $L_3(5)$

Brauer Character Table

Table 41: Brauer character table for $L_3(5)$.

$L_3(5)$	1A	2A	4AB	4C	5A	5B	8AB	10A	20AB	31AJ
φ_1	1	1	1	1	1	1	1	1	1	1
φ_2	30	6	6	2	5	0	0	1	1	-1
φ_3	31	7	-5	-1	6	1	-1	2	0	0
φ_4	62	-10	-2	2	12	2	0	0	-2	0
φ_5	124	4	4	0	-1	-1	-2	-1	-1	0
φ_6	124	4	4	0	-1	-1	2	-1	-1	0
φ_7	186	-6	6	-2	11	1	0	-1	1	0
φ_8	248	-8	0	0	-2	-2	0	2	0	0
φ_9	248	8	-8	0	-2	-2	0	-2	2	0
φ_{10}	960	0	0	0	-40	10	0	0	0	-1

Feasible Decompositions

1. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 1\varphi_8 + 0\varphi_9$ (2A → 2B, 4AB → 4F, 4C → 4F, 5A → 5B, 5B → 5B, 8AB → 8K, 10A → 10D, 20AB → 20K, 31AJ → 31A)

Cohomological Dimensions

$\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, $\varphi_5 = 1$, $\varphi_6 = 0$, $\varphi_7 = 0$, $\varphi_8 = 0$, $\varphi_9 = 0$.

8.5 $\Omega_8^+(2)$

Brauer Character Table

Table 42: Brauer character table for $\Omega_8^+(2)$.

$\Omega_8^+(2)$	1A	2A	2B	2C	2D	2E	4A	4B	4C	4D	4E	4F	5A	5B	5C	7A	8A	8B	10A	10B	10C
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	28	-4	4	4	4	-4	8	0	0	0	0	0	3	3	3	0	2	-2	-1	-1	-1
φ_3	35	3	11	-5	-5	3	7	-1	-1	3	-1	-1	5	0	0	0	1	1	1	0	0
φ_4	35	3	-5	11	-5	3	7	-1	3	-1	-1	-1	0	5	0	0	1	1	0	1	0
φ_5	35	3	-5	-5	11	3	7	-1	-1	-1	3	-1	0	0	5	0	1	1	0	0	1
φ_6	48	16	8	8	8	0	-4	4	0	0	0	0	-2	-2	-2	-1	-2	-2	-2	-2	-2
φ_7	147	-13	11	11	11	3	-9	-1	-1	-1	-1	3	-3	-3	-3	0	-3	1	1	1	1
φ_8	322	2	-14	-14	-14	2	18	2	-2	-2	-2	2	-3	-3	-3	0	-2	2	1	1	1
φ_9	497	49	1	1	1	-15	-15	1	-3	-3	-3	1	-3	-3	-3	0	-3	1	1	1	1
φ_{10}	518	6	-26	38	-26	6	-2	-2	2	2	2	-2	3	-2	3	0	2	-2	-1	-2	-1
φ_{11}	518	6	38	-26	-26	6	-2	-2	2	2	2	-2	-2	3	3	0	2	-2	-2	-1	-1
φ_{12}	518	6	-26	-26	38	6	-2	-2	2	2	2	-2	3	3	-2	0	2	-2	-1	-1	-2
φ_{13}	567	-9	-9	-9	39	-9	15	-1	3	3	-1	-1	-3	-3	7	0	-1	-1	1	1	-1
φ_{14}	567	-9	39	-9	-9	-9	15	-1	3	-1	3	-1	7	-3	-3	0	-1	-1	-1	1	1
φ_{15}	567	-9	-9	39	-9	-9	15	-1	-1	3	3	-1	-3	7	-3	0	-1	-1	1	-1	1
φ_{16}	972	108	36	36	36	12	0	8	0	0	0	0	-3	-3	-3	-1	2	-2	1	1	1
φ_{17}	1197	-51	21	21	21	-3	-23	1	1	1	1	-3	-3	-3	-3	0	-1	3	1	1	1
φ_{18}	2268	-36	-36	-36	12	12	-12	4	0	0	-4	0	3	3	-2	0	0	0	-1	-1	2
φ_{19}	2268	-36	12	-36	-36	12	-12	4	0	-4	0	0	-2	3	3	0	0	0	2	-1	-1
φ_{20}	2268	-36	-36	12	-36	12	-12	4	-4	0	0	0	3	-2	3	0	0	0	-1	2	-1
φ_{21}	6075	27	-45	-45	-45	-21	-9	-1	3	3	3	3	0	0	0	-1	1	1	0	0	0

Feasible Decompositions

1. $3\varphi_1 + 5\varphi_2 + 1\varphi_3 + 1\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2B, 2B → 2A, 2C → 2A, 2D → 2A, 2E → 2B, 4A → 4B, 4B → 4G, 4C → 4D, 4D → 4D, 4E → 4D, 4F → 4G, 5A → 5A, 5B → 5A, 5C → 5A, 7A → 7B, 8A → 8K, 8B → 8H, 10A → 10C, 10B → 10C, 10C → 10C)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 2,\ \varphi_7 = 0.$

By Lemma 2.7 and Lemma 2.5, all feasible decompositions listed here exhibit a fixed vector, thus implying any subgroup $H < G$ with $\Omega_8^+(2) \cong F^*(H)$ is not maximal.

8.6 $\mathrm{L}_2(32)$

Brauer Character Table

Table 43: Brauer character table for $\mathrm{L}_2(32)$.

$\mathrm{L}_2(32)$	1A	2A	11AE	31AN
φ_1	1	1	1	1
φ_2	31	-1	-2	0
φ_3	155	-5	1	0
φ_4	495	15	0	-1

Feasible Decompositions

None.

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 1,\ \varphi_3 = 0,\ \varphi_4 = 0.$

8.7 $L_2(41)$

Brauer Character Table

Table 44: Brauer character table for $L_2(41)$.

$L_2(41)$	1A	2A	4A	5AB	7AC	10AB	20AD	41AB
φ_1	1	1	1	1	1	1	1	1
φ_2	40	0	0	0	-2	0	0	-1
φ_3	42	2	-2	2	0	2	-2	1
φ_4	42	-2	0	2	0	-2	0	1
φ_5	84	4	-4	-1	0	-1	1	2
φ_6	84	4	4	-1	0	-1	-1	2
φ_7	120	0	0	0	1	0	0	-3
φ_8	168	-8	0	-2	0	2	0	4

Feasible Decompositions

None.

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 1,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 0,\ \varphi_7 = 0\ \varphi_8 = 0.$

8.8 $L_2(13)$

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$ and let τ be a map defined by $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Brauer Character Table

Table 45: Brauer character table for $L_2(13)$.

$L_2(13)$	1A	2A	7AC	13A	τ(13A)
φ_1	1	1	1	1	1
φ_2	7	-1	0	α	$-\alpha + 1$
φ_3	7	-1	0	$-\alpha + 1$	α
φ_4	13	1	-1	0	0
φ_5	36	0	1	-3	-3

Feasible Decompositions

1. $7\varphi_1 + 10\varphi_2 + 10\varphi_3 + 5\varphi_4 + 1\varphi_5$ (2A → 2B, 7AC → 7B, 13A → 13E, τ(13A) → 13E)
2. $15\varphi_1 + 2\varphi_2 + 2\varphi_3 + 13\varphi_4 + 1\varphi_5$ (2A → 2A, 7AC → 7B, 13A → 13E, τ(13A) → 13E)

3. $3\varphi_1 + 7\varphi_2 + 7\varphi_3 + 3\varphi_4 + 3\varphi_5$ (2A → 2B, 7AC → 7B, 13A → 13F, τ(13A) → 13F)
4. $52\varphi_1 + 27\varphi_2 + 1\varphi_3 + 0\varphi_4 + 0\varphi_5$ (2A → 2A, 7AC → 7A, 13A → 13AB, τ(13A) → 13AB)
5. $52\varphi_1 + 1\varphi_2 + 27\varphi_3 + 0\varphi_4 + 0\varphi_5$ (2A → 2A, 7AC → 7A, 13A → 13AB, τ(13A) → 13AB)
6. $9\varphi_1 + 18\varphi_2 + 5\varphi_3 + 6\varphi_4 + 0\varphi_5$ (2A → 2B, 7AC → 7B, 13A → 13CD, τ(13A) → 13CD)
7. $9\varphi_1 + 5\varphi_2 + 18\varphi_3 + 6\varphi_4 + 0\varphi_5$ (2A → 2B, 7AC → 7B, 13A → 13CD, τ(13A) → 13CD)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1,\ \varphi_5 = 0.$

By Lemma 2.7 and Lemma 2.5, all feasible decompositions listed here exhibit a fixed vector, thus implying any subgroup $H < G$ with $L_2(13) \cong F^*(H)$ is not maximal.

8.9 $U_4(3)$

Brauer Character Table

Table 46: Brauer character table for $U_4(3)$

$U_4(3)$	1A	2A	4A	4B	5A	7AB	8A
φ_1	1	1	1	1	1	1	1
φ_2	15	-1	3	-1	0	1	1
φ_3	19	3	-1	-1	-1	-2	-3
φ_4	69	5	-3	1	-1	-1	1
φ_5	90	-6	2	2	0	-1	-2
φ_6	156	-4	-4	0	1	2	4
φ_7	729	9	-3	1	-1	1	-1

Feasible Decompositions

None.

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 3,\ \varphi_4 = 1,\ \varphi_5 = 0,\ \varphi_6 = 0,\ \varphi_7 = 0.$

8.10 $Sp_6(3)$

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$ and define the map τ by $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Brauer Character Table

Table 47: Brauer character table for $\mathrm{Sp}_6(3)$.

$\mathrm{Sp}_6(3)$	1A	2A	2B	4A	4B	4C	5A	7A	8A	8B	10A	13A	τ(13A)	14A	20A
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	13	-3	1	5	-3	1	-2	-1	-1	-1	2	0	0	1	0
φ_3	21	5	-3	9	1	1	1	0	3	-1	5	$-\alpha+1$	$\alpha+2$	4	-1
φ_4	57	-7	-3	5	5	1	2	1	-3	1	-2	$\alpha-1$	$-\alpha-2$	-3	0
φ_5	84	20	-4	-4	-4	4	-1	0	0	0	-5	α	$-\alpha-1$	-4	1
φ_6	90	10	6	10	2	-2	0	-1	0	0	0	-1	-1	-1	0
φ_7	189	-3	-3	29	-3	-3	-1	0	1	1	7	$-\alpha$	$\alpha+1$	4	-1
φ_8	358	-10	6	-6	2	-6	3	1	4	0	-5	$-\alpha$	$\alpha+1$	-1	-1
φ_9	594	-30	-6	-34	6	6	-1	-1	4	0	-5	$-\alpha+2$	$\alpha+3$	1	1
φ_{10}	623	63	7	-37	3	3	-2	0	-3	1	-2	-1	-1	0	-2
φ_{11}	903	23	-9	27	3	-5	-2	0	-3	1	-2	α	$-\alpha-1$	-2	2
φ_{12}	1302	-58	6	-62	-6	2	2	0	4	0	2	$-2\alpha+1$	$2\alpha+3$	6	-2
φ_{13}	2457	105	-15	-43	-3	-3	2	0	-5	-1	1 0	0	0	6	2
φ_{14}	4655	-81	-5	-93	3	-9	0	0	3	-1	4	1	1	2	2
φ_{15}	19683	243	27	-81	-9	-9	3	-1	-3	1	3	1	1	-1	-1

Feasible Decompositions

None.

Cohomological Dimensions

Not calculated.

8.11 $\mathrm{Sp}_6(2)$

Brauer Character Table

Table 48: Brauer character table for $\mathrm{Sp}_6(2)$.

$\mathrm{Sp}_6(2)$	1A	2A	2B	2C	2D	4A	4B	4C	4D	4E	5A	7A	8A	8B	10A
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	7	-5	-1	3	-1	3	-3	1	-1	1	2	0	-1	1	0
φ_3	14	-6	6	2	-2	-2	0	-4	2	0	-1	0	0	-2	-1
φ_4	21	9	-3	1	-3	5	3	-1	1	-1	1	0	-1	1	-1
φ_5	27	15	3	7	3	3	5	1	-1	1	2	-1	1	-1	0
φ_6	34	14	10	6	2	-2	0	4	2	0	-1	-1	-2	0	-1
φ_7	35	-5	3	-5	3	7	-1	-1	-1	-1	0	0	1	1	0
φ_8	49	-19	-7	5	1	-3	-1	3	1	-1	-1	0	1	-1	1
φ_9	91	11	11	-5	-5	-1	-1	7	-1	-1	1	0	1	1	1
φ_{10}	98	30	-6	6	2	-6	0	-4	-2	0	-2	0	0	-2	0
φ_{11}	189	21	-3	-11	-3	9	1	1	1	1	-1	0	-1	-1	1
φ_{12}	189	-51	-3	13	-3	-3	1	1	-3	1	-1	0	1	1	-1
φ_{13}	189	-39	21	1	-3	-3	-1	-5	1	-1	-1	0	-1	1	1
φ_{14}	196	-16	-4	-8	4	-4	2	-2	0	2	1	0	-2	0	-1
φ_{15}	405	45	-27	-3	-3	-3	-3	-3	5	1	0	-1	1	1	0

Feasible Decompositions

1. $4\varphi_1 + 6\varphi_2 + 0\varphi_3 + 5\varphi_4 + 1\varphi_5 + 0\varphi_6 + 2\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15}$ (2A $\rightarrow$ 2A, 2B $\rightarrow$ 2B, 2C $\rightarrow$ 2A, 2D $\rightarrow$ 2B, 4A $\rightarrow$ 4B, 4B $\rightarrow$ 4D, 4C $\rightarrow$ 4D, 4D $\rightarrow$ 4G, 4E $\rightarrow$ 4D, 5A $\rightarrow$ 5A, 7A $\rightarrow$ 7B, 8A $\rightarrow$ 8H, 8B $\rightarrow$ 8K, 10A $\rightarrow$ 10C)

Cohomological Dimensions

$\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 1$, $\varphi_4 = 0$, $\varphi_5 = 0$, $\varphi_6 = 1$, $\varphi_7 = 0$, $\varphi_8 = 0$, $\varphi_9 = 0$, $\varphi_{10} = 1$, $\varphi_{11} = 0$, $\varphi_{12} = 0$, $\varphi_{13} = 0$, $\varphi_{14} = 0$, $\varphi_{15} = 0$.

By Lemma 2.7 and Lemma 2.5, all feasible decompositions listed here exhibit a fixed vector, thus implying any subgroup $H < G$ with $\mathrm{Sp}_6(2) \cong F^*(H)$ is not maximal.

8.12 $L_2(11)$

Let $\alpha = \omega_{11}^9 + \omega_{11}^5 + \omega_{11}^4 + \omega_{11}^3 + \omega_{11}$ and let τ denote complex conjugation.

Brauer Character Table

Table 49: Brauer character table for $L_2(11)$.

$L_2(11)$	1A	2A	5AB	11A	τ(11A)
φ_1	1	1	1	1	1
φ_2	5	1	0	α	$-\alpha - 1$
φ_3	5	1	0	$-\alpha - 1$	α
φ_4	10	-2	0	-1	-1
φ_5	24	0	-1	2	2

Feasible Decompositions

1. $24\varphi_1 + 10\varphi_2 + 10\varphi_3 + 10\varphi_4 + 1\varphi_5$ (2A → 2A, 5AB → 5A, 11A → 11A, τ(11A) → 11A)
2. $24\varphi_1 + 2\varphi_2 + 2\varphi_3 + 18\varphi_4 + 1\varphi_5$ (2A → 2B, 5AB → 5A, 11A → 11A, τ(11A) → 11A)
3. $4\varphi_1 + 10\varphi_2 + 10\varphi_3 + 0\varphi_4 + 6\varphi_5$ (2A → 2A, 5AB → 5B, 11A → 11A, τ(11A) → 11A)
4. $4\varphi_1 + 2\varphi_2 + 2\varphi_3 + 8\varphi_4 + 6\varphi_5$ (2A → 2B, 5AB → 5B, 11A → 11A, τ(11A) → 11A)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1,\ \varphi_5 = 0.$

8.13 M_{11}

Let $\alpha = \omega_{11}^9 + \omega_{11}^5 + \omega_{11}^4 + \omega_{11}^3 + \omega_{11}$ and let τ denote complex conjugation.

Brauer Character Table

Table 50: Brauer character table for M_{11}.

M_{11}	1A	2A	4A	5A	8A	τ(8A)	11A	τ(11A)
φ_1	1	1	1	1	1	1	1	1
φ_2	5	1	-1	0	$-1-\sqrt{2}i$	$-1+\sqrt{2}i$	α	$-\alpha - 1$
φ_3	5	1	-1	0	$-1+\sqrt{2}i$	$-1-\sqrt{2}i$	$-\alpha - 1$	α
φ_4	10	2	2	0	0	0	-1	-1
φ_5	10	-2	0	0	$\sqrt{2}i$	$-\sqrt{2}i$	-1	-1
φ_6	10	-2	0	0	$-\sqrt{2}i$	$\sqrt{2}i$	-1	-1
φ_7	24	0	0	-1	2	2	2	2
φ_8	45	-3	1	0	-1	-1	1	1

Feasible Decompositions

1. $24\varphi_1 + 10\varphi_2 + 10\varphi_3 + 0\varphi_4 + 5\varphi_5 + 5\varphi_6 + 1\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4D, 5A → 5A, 8A → 8J, τ(8A) → 8J, 11A → 11A, τ(11A) → 11A)
2. $24\varphi_1 + 0\varphi_2 + 0\varphi_3 + 2\varphi_4 + 9\varphi_5 + 9\varphi_6 + 1\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4C, 5A → 5A, 8A → 8B, τ(8A) → 8B, 11A → 11A, τ(11A) → 11A)
3. $4\varphi_1 + 7\varphi_2 + 7\varphi_3 + 3\varphi_4 + 0\varphi_5 + 0\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4E, 5A → 5B, 8A → 8N, τ(8A) → 8N, 11A → 11A, τ(11A) → 11A)
4. $4\varphi_1 + 6\varphi_2 + 6\varphi_3 + 4\varphi_4 + 0\varphi_5 + 0\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4G, 5A → 5B, 8A → 8P, τ(8A) → 8P, 11A → 11A, τ(11A) → 11A)
5. $4\varphi_1 + 5\varphi_2 + 5\varphi_3 + 5\varphi_4 + 0\varphi_5 + 0\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4D, 5A → 5B, 8A → 8J, τ(8A) → 8J, 11A → 11A, τ(11A) → 11A)
6. $4\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 4\varphi_5 + 4\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4G, 5A → 5B, 8A → 8E, τ(8A) → 8E, 11A → 11A, τ(11A) → 11A)
7. $4\varphi_1 + 1\varphi_2 + 1\varphi_3 + 1\varphi_4 + 4\varphi_5 + 4\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4D, 5A → 5B, 8A → 8G, τ(8A) → 8G, 11A → 11A, τ(11A) → 11A)
8. $4\varphi_1 + 0\varphi_2 + 0\varphi_3 + 2\varphi_4 + 4\varphi_5 + 4\varphi_6 + 6\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4F, 5A → 5B, 8A → 8K, τ(8A) → 8K, 11A → 11A, τ(11A) → 11A)
9. $23\varphi_1 + 10\varphi_2 + 10\varphi_3 + 0\varphi_4 + 4\varphi_5 + 4\varphi_6 + 0\varphi_7 + 1\varphi_8$ (2A → 2A, 4A → 4D, 5A → 5A, 8A → 8L, τ(8A) → 8L, 11A → 11A, τ(11A) → 11A)
10. $23\varphi_1 + 9\varphi_2 + 9\varphi_3 + 1\varphi_4 + 4\varphi_5 + 4\varphi_6 + 0\varphi_7 + 1\varphi_8$ (2A → 2A, 4A → 4F, 5A → 5A, 8A → 8P, τ(8A) → 8P, 11A → 11A, τ(11A) → 11A)
11. $23\varphi_1 + 4\varphi_2 + 4\varphi_3 + 6\varphi_4 + 4\varphi_5 + 4\varphi_6 + 0\varphi_7 + 1\varphi_8$ (2A → 2A, 4A → 4C, 5A → 5A, 8A → 8G, τ(8A) → 8G, 11A → 11A, τ(11A) → 11A)
12. $3\varphi_1 + 0\varphi_2 + 0\varphi_3 + 2\varphi_4 + 3\varphi_5 + 3\varphi_6 + 5\varphi_7 + 1\varphi_8$ (2A → 2B, 4A → 4F, 5A → 5B, 8A → 8E, τ(8A) → 8E, 11A → 11A, τ(11A) → 11A)
13. $2\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 2\varphi_5 + 2\varphi_6 + 4\varphi_7 + 2\varphi_8$ (2A → 2B, 4A → 4G, 5A → 5B, 8A → 8P, τ(8A) → 8P, 11A → 11A, τ(11A) → 11A)
14. $2\varphi_1 + 1\varphi_2 + 1\varphi_3 + 1\varphi_4 + 2\varphi_5 + 2\varphi_6 + 4\varphi_7 + 2\varphi_8$ (2A → 2B, 4A → 4D, 5A → 5B, 8A → 8J, τ(8A) → 8J, 11A → 11A, τ(11A) → 11A)
15. $1\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 1\varphi_5 + 1\varphi_6 + 3\varphi_7 + 3\varphi_8$ (2A → 2B, 4A → 4G, 5A → 5B, 8A → 8Q, τ(8A) → 8Q, 11A → 11A, τ(11A) → 11A)
16. $1\varphi_1 + 1\varphi_2 + 1\varphi_3 + 1\varphi_4 + 1\varphi_5 + 1\varphi_6 + 3\varphi_7 + 3\varphi_8$ (2A → 2B, 4A → 4D, 5A → 5B, 8A → 8L/N, τ(8A) → 8L/N, 11A → 11A, τ(11A) → 11A)
17. $1\varphi_1 + 0\varphi_2 + 0\varphi_3 + 2\varphi_4 + 1\varphi_5 + 1\varphi_6 + 3\varphi_7 + 3\varphi_8$ (2A → 2B, 4A → 4F, 5A → 5B, 8A → 8P, τ(8A) → 8P, 11A → 11A, τ(11A) → 11A)
18. $0\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 0\varphi_5 + 0\varphi_6 + 2\varphi_7 + 4\varphi_8$ (2A → 2B, 4A → 4G, 5A → 5B, 8A → 8H, τ(8A) → 8H, 11A → 11A, τ(11A) → 11A)

19. $0\varphi_1 + 1\varphi_2 + 1\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6 + 2\varphi_7 + 4\varphi_8$ (2A → 2B, 4A → 4D, 5A → 5B, 8A → 8I/M, τ(8A) → 8I/M, 11A → 11A, τ(11A) → 11A)
20. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 2\varphi_4 + 0\varphi_5 + 0\varphi_6 + 2\varphi_7 + 4\varphi_8$ (2A → 2B, 4A → 4F, 5A → 5B, 8A → 8O, τ(8A) → 8O, 11A → 11A, τ(11A) → 11A)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 1,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 1,\ \varphi_7 = 0,\ \varphi_8 = 0$. These modules are not self-dual; the corresponding cohomological dimensions of the dual spaces are: $\varphi_1^* = 0,\ \varphi_2^* = 0,\ \varphi_3^* = 1,\ \varphi_4^* = 0,\ \varphi_5^* = 1,\ \varphi_6^* = 0,\ \varphi_7^* = 0,\ \varphi_8^* = 0$.

Feasible decompositions 2,4,7,9,10,14,16 and 19 all exhibit fusion patterns that contradict the information given in Table 3, hence these are not valid. Furthermore, by Lemma 2.7 and Lemma 2.5, any subgroup $H < G$ with $M_{11} \cong F^*(H)$ that is associated to feasible decompositions 1 and 11 is not maximal in G.

8.14 $L_2(7)$

Brauer Character Table

Table 51: Brauer character table for $L_2(7)$.

$L_2(7)$	1A	2A	4A	7AB
φ_1	1	1	1	1
φ_2	6	-2	2	-1
φ_3	6	2	0	-1
φ_4	7	-1	-1	0

Feasible Decompositions

1. $79\varphi_1 + 27\varphi_2 + 0\varphi_3 + 1\varphi_4$ (2A → 2A, 4A → 4A, 7AB → 7A)
2. $34\varphi_1 + 17\varphi_2 + 14\varphi_3 + 4\varphi_4$ (2A → 2A, 4A → 4B, 7AB → 7B)
3. $30\varphi_1 + 21\varphi_2 + 6\varphi_3 + 8\varphi_4$ (2A → 2B, 4A → 4B, 7AB → 7B)
4. $25\varphi_1 + 8\varphi_2 + 14\varphi_3 + 13\varphi_4$ (2A → 2A, 4A → 4C, 7AB → 7B)
5. $21\varphi_1 + 12\varphi_2 + 6\varphi_3 + 17\varphi_4$ (2A → 2B, 4A → 4C, 7AB → 7B)
6. $62\varphi_1 + 10\varphi_2 + 0\varphi_3 + 18\varphi_4$ (2A → 2A, 4A → 4B, 7AB → 7A)
7. $20\varphi_1 + 3\varphi_2 + 14\varphi_3 + 18\varphi_4$ (2A → 2A, 4A → 4F, 7AB → 7B)
8. $19\varphi_1 + 2\varphi_2 + 14\varphi_3 + 19\varphi_4$ (2A → 2A, 4A → 4D, 7AB → 7B)
9. $18\varphi_1 + 1\varphi_2 + 14\varphi_3 + 20\varphi_4$ (2A → 2A, 4A → 4G, 7AB → 7B)
10. $17\varphi_1 + 0\varphi_2 + 14\varphi_3 + 21\varphi_4$ (2A → 2A, 4A → 4E, 7AB → 7B)
11. $16\varphi_1 + 7\varphi_2 + 6\varphi_3 + 22\varphi_4$ (2A → 2B, 4A → 4F, 7AB → 7B)

12. $15\varphi_1 + 6\ \varphi_2 + 6\varphi_3 + 23\varphi_4$ (2A → 2B, 4A → 4D, 7AB → 7B)
13. $14\varphi_1 + 5\varphi_2 + 6\varphi_3 + 24\varphi_4$ (2A → 2B, 4A → 4G, 7AB → 7B)
14. $13\varphi_1 + 4\varphi_2 + 6\varphi_3 + 25\varphi_4$ (2A → 2B, 4A → 4E, 7AB → 7B)
15. $53\varphi_1 + 1\varphi_2 + 0\varphi_3 + 27\varphi_4$ (2A → 2A, 4A → 4C, 7AB → 7A)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1.$

Feasible decompositions 2,5,6,7,9,12 and 14 exhibit fusion patterns which contradict the information shown in Table 3 and so are not valid. Furthermore, by Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $L_2(7) \cong F^*(H)$ that are associated to feasible decompositions 1,3,4,8 and 15 are not maximal.

Projective Covers

$$P(\varphi_2) = \varphi_2$$

$$P(\varphi_3) = \varphi_3, \qquad P(\varphi_4) = \varphi_4|\varphi_1|\varphi_4$$

8.15 $L_2(9)$

Brauer Character Table

Table 52: Brauer character table for $L_2(9)$.

$L_2(9)$	1A	2A	4A	5AB
φ_1	1	1	1	1
φ_2	4	0	-2	-1
φ_3	6	-2	2	1
φ_4	9	1	1	-1

Feasible Decompositions

1. $44\varphi_1 + 30\varphi_2 + 11\varphi_3 + 2\varphi_4$ (2A → 2A, 4A → 4F, 5AB → 5A)
2. $30\varphi_1 + 2\varphi_2 + 11\varphi_3 + 16\varphi_4$ (2A → 2A, 4A → 4B, 5AB → 5A)
3. $46\varphi_1 + 34\varphi_2 + 11\varphi_3 + 0\varphi_4$ (2A → 2A, 4A → 4G, 5AB → 5A)
4. $39\varphi_1 + 20\varphi_2 + 11\varphi_3 + 7\varphi_4$ (2A → 2A, 4A → 4C, 5AB → 5A)
5. $45\varphi_1 + 32\varphi_2 + 11\varphi_3 + 1\varphi_4$ (2A → 2A, 4A → 4D, 5AB → 5A)
6. $18\varphi_1 + 2\varphi_2 + 19\varphi_3 + 12\varphi_4$ (2A → 2B, 4A → 4B, 5AB → 5A)
7. $27\varphi_1 + 20\varphi_2 + 19\varphi_3 + 3\varphi_4$ (2A → 2B, 4A → 4C, 5AB → 5A)
8. $24\varphi_1 + 20\varphi_2 + 6\varphi_3 + 12\varphi_4$ (2A → 2A, 4A → 4F, 5AB → 5B)
9. $26\varphi_1 + 24\varphi_2 + 6\varphi_3 + 10\varphi_4$ (2A → 2A, 4A → 4G, 5AB → 5B)

10. $19\varphi_1 + 10\varphi_2 + 6\varphi_3 + 17\varphi_4$ (2A → 2A, 4A → 4C, 5AB → 5B)
11. $27\varphi_1 + 26\varphi_2 + 6\varphi_3 + 9\varphi_4$ (2A → 2A, 4A → 4E, 5AB → 5B)
12. $25\varphi_1 + 22\varphi_2 + 6\varphi_3 + 11\varphi_4$ (2A → 2A, 4A → 4D, 5AB → 5B)
13. $12\varphi_1 + 20\varphi_2 + 14\varphi_3 + 8\varphi_4$ (2A → 2B, 4A → 4F, 5AB → 5B)
14. $14\varphi_1 + 24\varphi_2 + 14\varphi_3 + 6\varphi_4$ (2A → 2B, 4A → 4G, 5AB → 5B)
15. $7\varphi_1 + 10\varphi_2 + 14\varphi_3 + 13\varphi_4$ (2A → 2B, 4A → 4C, 5AB → 5B)
16. $15\varphi_1 + 26\varphi_2 + 14\varphi_3 + 5\varphi_4$ (2A → 2B, 4A → 4E, 5AB → 5B)
17. $13\varphi_1 + 22\varphi_2 + 14\varphi_3 + 7\varphi_4$ (2A → 2B, 4A → 4D, 5AB → 5B)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 2,\ \varphi_3 = 0,\ \varphi_4 = 0.$

Feasible decompositions 1,2,7,8,9,15,16 and 17 exhibit fusion patterns which contradict the information shown in Table 3 and so are not valid. Furthermore, by Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $L_2(9) \cong F^*(H)$ that are associated to feasible decompositions 3 and 10 are not maximal.

Projective Covers

$$P(\varphi_2) = \varphi_2|(\varphi_1 \oplus \varphi_1 \oplus \varphi_3)|(\varphi_2 \oplus \varphi_2 \oplus \varphi_2)|(\varphi_1 \oplus \varphi_1 \oplus \varphi_3)|\varphi_2$$
$$P(\varphi_3) = \varphi_3|(\varphi_2 \oplus \varphi_2)|(\varphi_1 \oplus \varphi_1 \oplus \varphi_3)|(\varphi_2 \oplus \varphi_2)|\varphi_3$$
$$P(\varphi_4) = \varphi_4$$

8.16 $L_2(19)$

Brauer Character Table

Table 53: Brauer character table for $L_2(19)$.

$L_2(19)$	1A	2A	5AB	10AB	19AB
φ_1	1	1	1	1	1
φ_2	18	2	-2	2	-1
φ_3	19	-1	-1	-1	0
φ_4	36	4	1	-1	-2
φ_5	36	-4	1	1	-2

Feasible Decompositions

1. $6\varphi_1 + 1\varphi_2 + 8\varphi_3 + 0\varphi_4 + 2\varphi_5$ (2A → 2B, 5AB → 5B, 10AB → 10F, 19AB → 19A)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 1,\ \varphi_4 = 0,\ \varphi_5 = 0.$

Projective Covers

$$P(\varphi_2) = \varphi_2, \qquad P(\varphi_3) = \varphi_3|\varphi_1|\varphi_3|\varphi_1|\varphi_3|\varphi_1|\varphi_3|\varphi_1|\varphi_3$$

$$P(\varphi_4) = \varphi_4, \qquad P(\varphi_5) = \varphi_5$$

8.17 $L_2(31)$

Brauer Character Table

Table 54: Brauer character table for $L_2(31)$.

$L_2(31)$	1A	2A	4A	5AB	8AB	16AD	31AB
φ_1	1	1	1	1	1	1	1
φ_2	30	-2	-2	0	-2	2	-1
φ_3	30	-2	-2	0	2	0	-1
φ_4	31	-1	-1	1	-1	-1	0
φ_5	60	-4	4	0	0	0	-2
φ_6	64	0	0	-1	0	0	2
φ_7	120	8	0	0	0	0	-4

Feasible Decompositions

1. $0\varphi_1 + 2\varphi_2 + 0\varphi_3 + 0\varphi_4 + 1\varphi_5 + 2\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4G, 5AB → 5B, 8AB → 8H, 16AD → 16E, 31AB → 31A)
2. $0\varphi_1 + 1\varphi_2 + 1\varphi_3 + 0\varphi_4 + 1\varphi_5 + 2\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4G, 5AB → 5B, 8AB → 8Q, 16AD → 16J, 31AB → 31A)
3. $0\varphi_1 + 0\varphi_2 + 2\varphi_3 + 0\varphi_4 + 1\varphi_5 + 2\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4G, 5AB → 5B, 8AB → 8P, 16AD → 16I/K (I if correct), 31AB → 31A)
4. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0\varphi_4 + 2\varphi_5 + 2\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4F, 5AB → 5B, 8AB → 8O, 16AD → 16I/K (K if correct), 31AB → 31A)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1,\ \varphi_5 = 0,\ \varphi_6 = 0,\ \varphi_7 = 0.$

Feasible decomposition 1 exhibits a fusion pattern which contradicts the information shown in Table 3 and so is not valid. Decompositions 3 and 4 show rare situations where the conjugacy class fusion cannot be completely determined. In both cases, we find that all elements of order 16 fuse into the conjugacy class of order 16 elements in G which have a fixed space dimension of 16 and a Brauer character value of 0. There are two classes with these properties in G, namely 16I and 16K. The only noticeable difference between these classes is that 16I squares into 8P, whilst 16K squares into 8O. As such, it could be true that both decompositions 3 and 4 are not actually valid (if it turns out that the pattern says 16I squares into 8O for example). However (and more likely), the fusion patterns could be correct and thus these cases would still need to be considered.

Projective Covers

$$P(\varphi_2) = \varphi_2, \quad P(\varphi_3) = \varphi_3$$
$$P(\varphi_4) = \varphi_4|\varphi_1|\varphi_4, \quad P(\varphi_5) = \varphi_5$$
$$P(\varphi_6) = \varphi_6|\varphi_6|\varphi_6, \quad P(\varphi_7) = \varphi_7$$

8.18 $L_2(61)$

Brauer Character Table

Let $\alpha = \omega_{61}^{59} + \omega_{61}^{55} + \omega_{61}^{54} + \omega_{61}^{53} + \omega_{61}^{51} + \omega_{61}^{50} + \omega_{61}^{44} + \omega_{61}^{43} + \omega_{61}^{40} + \omega_{61}^{38} + \omega_{61}^{37} + \omega_{61}^{35} + \omega_{61}^{33} + \omega_{61}^{32} + \omega_{61}^{31} + \omega_{61}^{30} + \omega_{61}^{29} + \omega_{61}^{28} + \omega_{61}^{26} + \omega_{61}^{24} + \omega_{61}^{23} + \omega_{61}^{21} + \omega_{61}^{18} + \omega_{61}^{17} + \omega_{61}^{11} + \omega_{61}^{10} + \omega_{61}^{8} + \omega_{61}^{7} + \omega_{61}^{6} + \omega_{61}^{2}$ and define $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Table 55: Brauer character table for $L_2(61)$.

$L_2(61)$	1A	2A	5AB	10AB	31AO	61A	τ(61A)
φ_1	1	1	1	1	1	1	1
φ_2	31	-1	1	-1	0	$\alpha + 1$	$-\alpha$
φ_3	31	-1	1	-1	0	$-\alpha$	$\alpha + 1$
φ_4	61	1	1	1	-1	0	0
φ_5	124	-4	-1	1	0	2	2
φ_6	124	4	-1	-1	0	2	2
φ_7	900	0	0	0	1	-15	-15

Feasible Decompositions

1. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0\varphi_4 + 2\varphi_5 + 0\varphi_6$ (2A $\to$ 2B, 5AB $\to$ 5B, 10AB $\to$ 10F, 31AO $\to$ 31A, 61A $\to$ 61GK, $\tau(61A) \to$ 61GK)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1,\ \varphi_5 = 0,\ \varphi_6 = 0.$

8.19 $G_2(3)$

Brauer Character Table

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$ and define $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Table 56: Brauer character table for $G_2(3)$.

$G_2(3)$	1A	2A	4A	4B	7A	8A	8B	13A	τ(13A)
φ_1	1	1	1	1	1	1	1	1	1
φ_2	7	-1	3	-1	0	-1	1	$\alpha+1$	$-\alpha$
φ_3	7	-1	-1	3	0	1	-1	$-\alpha$	$\alpha+1$
φ_4	27	3	-1	3	-1	-1	1	1	1
φ_5	27	3	3	-1	-1	1	-1	1	1
φ_6	49	1	-3	-3	0	-1	-1	-3	-3
φ_7	189	-3	-3	-3	0	1	1	$\alpha+1$	$-\alpha$
φ_8	189	-3	-3	-3	0	1	1	$-\alpha$	$\alpha+1$
φ_9	729	9	-3	-3	1	-1	-1	1	1

Feasible Decompositions

1. $52\varphi_1 + 27\varphi_2 + 1\varphi_3 + 0\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4A, 4B → 4C, 7A → 7A, 8A → 8B, 8B → 8A, 13A → 13AB, τ(13A) → 13AB)
2. $52\varphi_1 + 1\varphi_2 + 27\varphi_3 + 0\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4C, 4B → 4A, 7A → 7A, 8A → 8A, 8B → 8B, 13A → 13AB, τ(13A) → 13AB)
3. $6\varphi_1 + 5\varphi_2 + 18\varphi_3 + 3\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4G, 4B → 4B, 7A → 7B, 8A → 8K, 8B → 8H, 13A → 13CD, τ(13A) → 13CD)
4. $6\varphi_1 + 18\varphi_2 + 5\varphi_3 + 0\varphi_4 + 3\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4B, 4B → 4G, 7A → 7B, 8A → 8H, 8B → 8K, 13A → 13CD, τ(13A) → 13CD)
5. $9\varphi_1 + 2\varphi_2 + 2\varphi_3 + 6\varphi_4 + 0\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4D, 4B → 4C, 7A → 7B, 8A → 8L, 8B → 8G, 13A → 13E, τ(13A) → 13E)
6. $9\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 6\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2A, 4A → 4C, 4B → 4D, 7A → 7B, 8A → 8G, 8B → 8L, 13A → 13E, τ(13A) → 13E)
7. $3\varphi_1 + 7\varphi_2 + 7\varphi_3 + 0\varphi_4 + 0\varphi_5 + 3\varphi_6 + 0\varphi_7 + 0\varphi_8$ (2A → 2B, 4A → 4F, 4B → 4F, 7A → 7B, 8A → 8O, 8B → 8O, 13A → 13F, τ(13A) → 13F)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 2,\ \varphi_7 = 0,\ \varphi_8 = 0.$

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $G_2(3) \cong F^*(H)$ that are associated to feasible decompositions 1,2,3,4,5 and 6 are not maximal.

8.20 $L_2(25)$

Brauer Character Table

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$ and define $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Table 57: Brauer character table for $L_2(25)$.

$L_2(25)$	1A	2A	4A	5A	5B	13AC	τ(13AC)
φ_1	1	1	1	1	1	1	1
φ_2	13	1	-1	-2	3	0	0
φ_3	13	1	-1	3	-2	0	0
φ_4	25	1	1	0	0	-1	-1
φ_5	26	-2	0	1	1	0	0
φ_6	72	0	0	-3	-3	$-\alpha$	$\alpha+1$
φ_7	72	0	0	-3	-3	$\alpha+1$	$-\alpha$

Feasible Decompositions

1. $14\varphi_1 + 7\varphi_2 + 7\varphi_3 + 0\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2A, 4A → 4G, 5A → 5A, 5B → 5A, 13A → 13E, τ(13A) → 13E)
2. $15\varphi_1 + 6\varphi_2 + 6\varphi_3 + 1\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2A, 4A → 4D, 5A → 5A, 5B → 5A, 13A → 13E, τ(13A) → 13E)
3. $16\varphi_1 + 5\varphi_2 + 5\varphi_3 + 2\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2A, 4A → 4F, 5A → 5A, 5B → 5A, 13A → 13E, τ(13A) → 13E)
4. $21\varphi_1 + 0\varphi_2 + 0\varphi_3 + 7\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2A, 4A → 4C, 5A → 5A, 5B → 5A, 13A → 13E, τ(13A) → 13E)
5. $1\varphi_1 + 5\varphi_2 + 0\varphi_3 + 0\varphi_4 + 7\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4E, 5A → 5B, 5B → 5A, 13A → 13F, τ(13A) → 13F)
6. $1\varphi_1 + 0\varphi_2 + 5\varphi_3 + 0\varphi_4 + 7\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2B, 4A → 4E, 5A → 5A, 5B → 5B, 13A → 13F, τ(13A) → 13F)
7. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0\varphi_4 + 4\varphi_5 + 1\varphi_6 + 1\varphi_7$ (2A → 2B, 4A → 4G, 5A → 5B, 5B → 5B, 13A → 13F, τ(13A) → 13F)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 1,\ \varphi_5 = 0,\ \varphi_6 = 0,\ \varphi_7 = 0.$

Feasible decompositions 1,3,5 and 6 exhibit fusion patterns which contradict the information shown in Table 3 and so are not valid. Furthermore, by Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $L_2(25) \cong F^*(H)$ that are associated to feasible decompositions 2 and 4 are not maximal.

Projective Covers

$$P(\varphi_2) = \varphi_2|\varphi_3|\varphi_2, \qquad P(\varphi_3) = \varphi_3|\varphi_2|\varphi_3$$
$$P(\varphi_4) = \varphi_4|\varphi_1|\varphi_4, \qquad P(\varphi_5) = \varphi_5|\varphi_5|\varphi_5$$
$$P(\varphi_6) = \varphi_6, \qquad P(\varphi_7) = \varphi_7$$

More work

We briefly outline a method for explicitly constructing the KH-module described by feasible decomposition 7. From Lemma 2.10, $V|_H$ is a quotient of some sum of projective covers. Using Lemma 2.11, we have that there are at most 2 $P(\varphi_5)$ in this sum. However, due to the number of φ_5 in the feasible decomposition for $V|_H$, there must be exactly 2. Moreover, as φ_6 and φ_7 are both projective, we get that there are 1 of each in this sum. Hence, the projective cover $P(V|_H)$ of $V|_H$ is precisely

$$P(V|_H) = 2P(\varphi_5) + P(\varphi_6) + P(\varphi_7).$$

Using Magma, we find this has dimension 300. We search for submodules of $P(V|_H)$ which have dimension 52. Using the Magma command `Submodules`, we find there are 13 such submodules. Hence there are 13 possible quotients for $V|_H$ - all of which are self-dual.

8.21 $^2\mathrm{F}_4(2)'$

Brauer Character Table

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$. We define σ to be the map $\sigma(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Table 58: Brauer character table for ${}^2\mathrm{F}_4(2)'$.

${}^2\mathrm{F}_4(2)'$	1A	2A	2B	4A	4B	4C	5A	8AB	8C	8D	10A	13A	$\sigma(13\mathrm{A})$	16A	$\tau(16\mathrm{A})$
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	26	-6	2	-2	-2	2	1	0	0	0	-1	0	0	$-\sqrt{2}i$	$\sqrt{2}i$
φ_3	26	-6	2	-2	-2	2	1	0	0	0	-1	0	0	$\sqrt{2}i$	$-\sqrt{2}i$
φ_4	54	-10	6	6	-2	-2	4	-2	-2	-2	0	2	2	0	0
φ_5	77	13	-3	1	-3	1	2	1	-1	-1	-2	-1	-1	-1	-1
φ_6	124	-4	-4	0	4	0	-1	0	2	-2	1	$-\alpha$	$\alpha+1$	0	0
φ_7	124	-4	-4	0	4	0	-1	0	-2	2	1	$\alpha+1$	$-\alpha$	0	0
φ_8	351	31	15	3	-1	3	1	-1	1	1	1	0	0	-1	-1
φ_9	572	-4	12	4	4	-4	-3	0	0	0	1	0	0	0	0
φ_{10}	675	35	3	3	3	3	0	-1	-1	-1	0	-1	-1	1	1
φ_{11}	702	-2	-18	6	6	-2	2	-2	2	2	-2	0	0	0	0
φ_{12}	1099	11	-5	-1	-5	-1	-1	-1	1	1	1	$\alpha+1$	$-\alpha$	1	1
φ_{13}	1099	11	-5	-1	-5	-1	-1	-1	1	1	1	$-\alpha$	$\alpha+1$	1	1
φ_{14}	1404	60	12	-12	4	-4	4	0	0	0	0	0	0	0	0
φ_{15}	1728	-64	0	0	0	0	3	0	0	0	1	-1	-1	0	0

Feasible Decompositions

1. $9\varphi_1 + 0\varphi_2 + 0\varphi_3 + 3\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7$ (2A → 2B, 2B → 2A, 4A → 4C, 4B → 4G, 4C → 4D, 5A → 5A, 8AB → 8P, 8C → 8L, 8D → 8L, 10A → 10B, 13A → 13E, σ(13A) → 13E, 16A → 16F, τ(16A) → 16F)
2. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 0\varphi_4 + 0\varphi_5 + 1\varphi_6 + 1\varphi_7$ (2A → 2B, 2B → 2B, 4A → 4G, 4B → 4F, 4C → 4G, 5A → 5B, 8AB → 8O, 8C → 8Q, 8D → 8Q, 10A → 10F, 13A → 13F, σ(13A) → 13F, 16A → 16I/K, τ(16A) → 16I/K)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 1,\ \varphi_6 = 0,\ \varphi_7 = 0.$

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with ${}^2\mathrm{F}_4(2)' \cong F^*(H)$ that are associated to feasible decomposition 1 are not maximal. We remark that from [51], we have that ${}^2\mathrm{F}_4(2)'$ is a subgroup of G. Moreover, as with $\mathrm{L}_2(31)$, the fusion for classes of order 16 elements cannot be completely determined in decomposition 2.

8.22 $\mathrm{U}_3(9)$

Brauer Character Table

Let $\alpha = \omega_{40}^{15} - 2\omega_{40}^{9} + 2\omega_{40}^{7} + \omega_{40}^{5} - 2\omega_{40}^{3} - 2\omega_{40}$ and suppose $\tau \in \Gamma(\mathbb{Q}(\alpha) : \mathbb{Q})$ is defined by $\tau(\alpha) = -\alpha$. To save space, the columns and fusion possibilities for the 24 conjugacy classes of elements order 73 and the 16 conjugacy classes of elements order 80 are omitted.

Table 59: Brauer character table for $U_3(9)$.

$U_3(9)$	1A	2A	4A	5AD	5EF	8AB	10AD	10EH	10IJ	10KN	16AD	20AD	40AD	τ(40AD)
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	12	-4	4	-3	2	4	1	1	6	-4	-4	-1	$\alpha - 1$	$-\alpha - 1$
φ_3	14	-2	-2	4	-1	2	8	-2	3	3	2	-2	$-\alpha + 2$	$\alpha + 1$
φ_4	24	8	0	-6	4	8	-2	-2	8	-2	0	0	$\alpha - 2$	$-\alpha - 2$
φ_5	36	4	4	1	-4	-4	-11	-1	4	4	4	-1	$-\alpha + 1$	$\alpha + 1$
φ_6	49	1	1	-1	-1	-7	11	1	-9	1	-1	1	$-\alpha + 3$	$\alpha + 3$
φ_7	54	6	-2	9	-1	6	21	1	11	1	2	3	$-\alpha + 1$	$\alpha + 1$
φ_8	60	-4	-4	5	0	12	-9	1	16	-4	-4	1	$\alpha - 3$	$-\alpha - 3$
φ_9	72	-8	0	-8	2	0	-8	2	2	2	0	0	0	0
φ_{10}	72	-8	0	17	2	0	-23	-3	2	2	0	5	$-2\alpha + 5$	$2\alpha + 5$
φ_{11}	84	4	-4	-1	-6	-12	19	-1	-6	-6	-4	1	$\alpha - 7$	$-\alpha - 7$
φ_{12}	144	16	-4	9	4	8	-19	1	-4	-4	-8	-5	$-\alpha + 3$	$\alpha + 3$
φ_{13}	168	-8	-4	-12	-2	-8	12	2	-18	2	8	0	$2\alpha - 8$	$-2\alpha - 8$
φ_{14}	180	4	0	10	0	-4	-26	4	4	-6	-4	-4	$-\alpha + 6$	$\alpha + 6$
φ_{15}	180	4	0	-15	0	-4	-31	-1	4	14	12	1	1	1

Feasible Decompositions

1. $18\varphi_1 + 9\varphi_2 + 1\varphi_3 + 0\varphi_4 + 3\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15}$ (2A → 2B, 4A → 4B, 5AD → 5B, 5EF → 5A, 8AB → 8C, 10AD → 10F, 10EH → 10E, 10IJ → 10A, 10KN → 10D, 16AD → 16A, 20AD → 20K, 40AD → 40RS, τ(40AD) → 40RS)

2. $5\varphi_1 + 1\varphi_2 + 6\varphi_3 + 0\varphi_4 + 0\varphi_5 + 3\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15}$ (2A → 2B, 4A → 4G, 5AD → 5A, 5EF → 5B, 8AB → 8Q, 10AD → 10A, 10EH → 10D, 10IJ → 10F, 10KN → 10E, 16AD → 16G, 20AD → 20E, 40AD → 40IJ, τ(40AD) → 40IJ)

3. $4\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 1\varphi_9 + 0\varphi_{10} + 1\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15}$ (2A → 2B, 4A → 4F, 5AD → 5B, 5EF → 5B, 8AB → 8O, 10AD → 10E, 10EH → 10F, 10IJ → 10E, 10KN → 10F, 16AD → 16I/K, 20AD → 20O, 40AD → 40W′, τ(40AD) → 40W′)

4. $4\varphi_1 + 2\varphi_2 + 2\varphi_3 + 0\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 1\varphi_{10} + 1\varphi_{11} + 0\varphi_{12} + 0\varphi_{13} + 0\varphi_{14} + 0\varphi_{15}$ (2A → 2B, 4A → 4F, 5AD → AB, 5EF → 5B, 8AB → 8O, 10AD → 10B, 10EH → 10D, 10IJ → 10E, 10KN → 10F, 16AD → 16I/K, 20AD → 20J, 40AD → 40P′Q′, τ(40AD) → 40P′Q′)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 2,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 0,\ \varphi_7 = 0,\ \varphi_8 = 0,\ \varphi_9 = 0,\ \varphi_{10} = 0,\ \varphi_{11} = 4,\ \varphi_{12} = 0,\ \varphi_{13} = 0,\ \varphi_{14} = 0,\ \varphi_{15} = 0.$

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $\mathrm{U}_3(9) \cong F^*(H)$ that are associated to feasible decomposition 1 are not maximal. The rest of the feasible decompositions remain open problems. As with $\mathrm{L}_2(31)$, there are some cases here where the fusion for classes of order 16 elements cannot be completely determined.

8.23 $\mathrm{L}_4(3)$

Brauer Character Table

Let $\alpha = \omega_{13}^{11}+\omega_{13}^{8}+\omega_{13}^{7}$, $\beta = \omega_{13}^{9}+\omega_{13}^{3}+\omega_{13}$, $\gamma = \omega_{13}^{6}+\omega_{13}^{5}+\omega_{13}^{2}$ and let τ denote complex conjugation.

Table 60: Brauer character table for $L_4(3)$.

$L_4(3)$	1A	2A	2B	4A	4B	4C	5A	8A	10A	13A	B^*	C^*	D^*	20A	τ(20A)
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	6	2	-2	-4	2	0	1	-2	-3	$-\alpha-\gamma-1$	$\alpha+\gamma$	$-\alpha-\gamma-1$	$\alpha+\gamma$	1	1
φ_3	10	-2	2	-4	2	0	0	0	-2	$-\gamma$	$\alpha+\beta+\gamma+1$	$-\alpha$	$-\beta$	$1-\sqrt{5}i$	$1+\sqrt{5}i$
φ_4	10	-2	2	-4	2	0	0	0	-2	$-\alpha$	$-\beta$	$-\gamma$	$\alpha+\beta+\gamma+1$	$1+\sqrt{5}i$	$1-\sqrt{5}i$
φ_5	15	-1	-1	7	3	-1	0	1	4	2	2	2	2	2	2
φ_6	19	3	3	7	-1	-1	-1	1	3	$\alpha+\gamma$	$-\alpha-\gamma-1$	$\alpha+\gamma$	$-\alpha-\gamma-1$	-3	-3
φ_7	44	4	-4	-8	-4	0	-1	0	-1	$\alpha+\gamma-1$	$-\alpha-\gamma-2$	$\alpha+\gamma-1$	$-\alpha-\gamma-2$	-3	-3
φ_8	45	-3	-3	9	1	1	0	-1	2	$\beta+\gamma$	$-\alpha-\beta-1$	$-\beta-\gamma-1$	$\alpha+\beta$	$-1+\sqrt{5}i$	$-1-\sqrt{5}i$
φ_9	45	-3	-3	9	1	1	0	-1	2	$-\beta-\gamma-1$	$\alpha+\beta$	$\beta+\gamma$	$-\alpha-\beta-1$	$-1-\sqrt{5}i$	$-1+\sqrt{5}i$
φ_{10}	69	5	5	1	-3	1	-1	-3	-5	$-\alpha-\gamma-3$	$\alpha+\gamma-2$	$-\alpha-\gamma-3$	$\alpha+\gamma-2$	1	1
φ_{11}	126	-6	6	-12	-2	0	1	0	-1	$\alpha+\beta+\gamma$	$-\alpha-1$	$-\beta-1$	$-\gamma-1$	$-2+\sqrt{5}i$	$-2-\sqrt{5}i$
φ_{12}	126	-6	6	-12	-2	0	1	0	-1	$-\beta-1$	$-\gamma-1$	$\alpha+\beta+\gamma$	$-\alpha-1$	$-2-\sqrt{5}i$	$-2+\sqrt{5}i$
φ_{13}	156	-4	-4	16	-4	0	1	0	1	0	0	0	0	1	1
φ_{14}	294	2	-2	-4	-6	0	-1	2	7	$\alpha+\gamma+2$	$-\alpha-\gamma+1$	$\alpha+\gamma+2$	$-\alpha-\gamma+1$	1	1
φ_{15}	729	9	9	9	-3	1	-1	-1	-1	1	1	1	1	-1	-1

Feasible Decompositions

1. $12\varphi_1 + 12\varphi_2 + 2\varphi_3 + 2\varphi_4 + 7\varphi_5 + 1\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13}$ (2A → 2A, 2B → 2B, 4A → 4D, 4B → 4B, 4C → 4D, 5A → 5A, 8A → 8H, 10A → 10C, 13A → 13CD, 13B* → 13CD, 13C* → 13CD, 13D* → 13CD, 20A → 20D, τ(20A) → 20D)

2. $0\varphi_1 + 0\varphi_2 + 2\varphi_3 + 2\varphi_4 + 2\varphi_5 + 0\varphi_6 + 2\varphi_7 + 1\varphi_8 + 1\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13}$ (2A → 2B, 2B → 2B, 4A → 4G, 4B → 4F, 4C → 4G, 5A → 5B, 8A → 8O, 10A → 10F, 13A → 13F, 13B* → 13F, 13C* → 13F, 13D* → 13F, 20A → 20P, τ(20A) → 20P)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \varphi_4 = 0,\ \varphi_5 = 0,\ \varphi_6 = 1,\ \varphi_7 = 2,\ \varphi_8 = 0,\ \varphi_9 = 0,\ \varphi_{10} = 1,\ \varphi_{11} = 0,\ \varphi_{12} = 0,\ \varphi_{13} = 0.$

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $L_4(3) \cong F^*(H)$ that are associated to feasible decomposition 1 are not maximal.

8.24 $Sp_4(9)$

The Brauer character table for this group is omitted due to its size. However, this information is obtainable in Magma.

Feasible Decompositions

1. $15\varphi_1 + 6\varphi_2 + 8\varphi_3 + 1\varphi_4 + 1\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13}$ (2A → 2B, 2B → 2A, 4A → 4D, 4B → 4B, 4C → 4D, 5AB → 5A, 5CD → 5A, 5E → 5A, 8AB → 8C, 8CD → 8L, 8E → 8E, 10AB → 10A, 10CD → 10B, 10EF → 10C, 10G → 10B, 20AB → 20A, 20CD → 20D, 40AB → 40F, τ(40AB) → 40F, 40CD → 40DE, τ(40CD) → 40DE)

2. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 1\varphi_4 + 0\varphi_5 + 1\varphi_6 + 0\varphi_7 + 2\varphi_8 + 0\varphi_9 + 0\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13}$ (2A → 2B, 2B → 2B, 4A → 4B, 4B → 4F, 4C → 4G, 5AB → 5B, 5CD → 5A, 5E → 5B, 8AB → 8O, 8CD → 8E, 8E → 8O, 10AB → 10E, 10CD → 10B, 10EF → 10B, 10G → 10E, 20AB → 20O, 20CD → 20F, 40AB → 40ZA′, τ(40AB) → 40ZA′, 40CD → 40C′, τ(40CD) → 40C′)

3. $0\varphi_1 + 0\varphi_2 + 0\varphi_3 + 1\varphi_4 + 0\varphi_5 + 0\varphi_6 + 0\varphi_7 + 1\varphi_8 + 0\varphi_9 + 1\varphi_{10} + 0\varphi_{11} + 0\varphi_{12} + 0\varphi_{13}$ (2A → 2B, 2B → 2B, 4A → 4G, 4B → 4F, 4C → 4G, 5AB → 5B, 5CD → 5B, 5E → 5B, 8AB → 8O, 8CD → 8P, 8E → 8O, 10AB → 10E, 10CD → 10F, 10EF → 10F, 10G → 10F, 20AB → 20O, 20CD → 20P, 40AB → 40W′, τ(40AB) → 40W′, 40CD → 40X′Y′, τ(40CD) → 40X′Y′)

Cohomological Dimensions

Not calculated.

8.25 $^3\mathrm{D}_4(2)$

Brauer Character Table

Table 61: Brauer character table for $^3\mathrm{D}_4(2)$.

$^3\mathrm{D}_4(2)$	1A	2A	2B	4A	4B	4C	7AC	7D	8A	8B	13AC	14AC	28AC
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	25	-7	1	5	-3	1	4	-3	-1	-1	-1	0	-2
φ_3	52	20	-4	8	0	0	3	3	2	-2	0	-1	1
φ_4	196	-28	-4	0	8	0	0	0	-2	2	1	0	0
φ_5	324	36	12	8	0	0	9	2	-2	2	-1	1	1
φ_6	441	-7	9	-7	9	-3	0	0	1	-3	-1	0	0
φ_7	1053	189	21	33	9	-3	-4	3	3	3	0	0	-2
φ_8	1053	-99	-3	29	-3	-3	3	-4	1	1	0	-1	1
φ_9	1222	134	-10	6	-10	2	-3	-3	-2	2	0	1	-1
φ_{10}	1963	-85	11	-13	-13	-1	3	3	3	-1	0	-1	1
φ_{11}	6318	270	54	-6	18	6	-3	-3	0	0	0	-3	1
φ_{12}	7371	-405	3	15	-9	3	-7	0	-3	-3	0	1	1
φ_{13}	11907	-189	-45	-21	27	3	0	0	3	3	-1	0	0

Feasible Decompositions

1. $21\varphi_1 + 7\varphi_2 + 1\varphi_3 + 0\varphi_4$ (2A → 2B, 2B → 2A, 4A → 4B, 4B → 4G, 4C → 4C, 7AC → 7A, 7D → 7B, 8A → 8K, 8B → 8E, 13AC → 13E, 14AC → 14A, 28AC → 28G)
2. $0\varphi_1 + 0\varphi_2 + 1\varphi_3 + 1\varphi_4$ (2A → 2B, 2B → 2B, 4A → 4F, 4B → 4F, 4C → 4G, 7AC → 7B, 7D → 7B, 8A → 8O, 8B → 8O, 13AC → 13F, 14AC → 14E, 28AC → 28A′)

Cohomological Dimensions

$\varphi_1 = 0,\ \varphi_2 = 1,\ \varphi_3 = 0,\ \varphi_4 = 0.$

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $^3\mathrm{D}_4(2) \cong F^*(H)$ that are associated to feasible decomposition 1 are not maximal.

8.26 $\Omega_7(3)$

Brauer Character Table

Let $\alpha = \omega_{13}^{11} + \omega_{13}^{8} + \omega_{13}^{7} + \omega_{13}^{6} + \omega_{13}^{5} + \omega_{13}^{2}$. We define τ to be the map $\tau(\lambda\alpha + \mu) = -\lambda\alpha + (\mu - \lambda)$ for $\lambda, \mu \in \mathbb{Z}$.

Table 62: Brauer character table for $\Omega_7(3)$.

$\Omega_7(3)$	1A	2A	2B	2C	4A	4B	4C	4D	5A	7A	8A	8B	10A	10B	13A	τ(13A)	14A	20A
φ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
φ_2	7	-5	3	-1	-3	3	-1	1	2	0	1	-1	-2	0	$-\alpha$	$\alpha + 1$	2	2
φ_3	21	9	1	-3	3	5	1	-1	1	0	1	-1	1	-1	$\alpha + 2$	$-\alpha + 1$	2	3
φ_4	27	15	7	3	5	3	-1	1	2	-1	-1	1	2	0	1	1	1	0
φ_5	35	-5	-5	3	-1	7	-1	-1	0	0	1	1	0	0	$\alpha + 3$	$-\alpha + 2$	2	4
φ_6	63	-25	7	-1	-1	-5	3	-1	-2	0	-3	1	2	0	$-2\alpha - 3$	$2\alpha - 1$	-4	-6
φ_7	132	44	12	4	0	-8	0	0	-3	-1	-2	-2	-3	-1	$-\alpha - 5$	$\alpha - 4$	-5	-5
φ_8	189	21	-11	-3	1	9	1	1	-1	0	-1	-1	-1	1	$\alpha + 1$	$-\alpha$	0	1
φ_9	189	-51	13	-3	1	-3	-3	1	-1	0	1	1	3	-1	$-\alpha$	$\alpha + 1$	-2	1
φ_{10}	309	81	9	-3	3	-7	-3	-1	-1	1	-1	1	-1	1	$-2\alpha - 4$	$2\alpha - 2$	-3	-7
φ_{11}	518	-26	-26	6	2	-2	-2	2	3	0	2	-2	-1	-1	-2	-2	2	-3
φ_{12}	672	-144	16	0	0	-16	0	0	-3	0	0	0	1	1	$-2\alpha - 5$	$2\alpha - 3$	-4	-5
φ_{13}	797	121	1	-11	-5	-19	1	-1	-3	-1	-3	3	1	1	$-3\alpha - 4$	$3\alpha - 1$	-5	-5
φ_{14}	1602	234	26	18	-6	-6	2	2	-3	-1	2	-2	1	-1	$2\alpha + 4$	$-\alpha + 2$	3	9
φ_{15}	2052	-120	-32	12	2	-12	0	-2	2	1	0	-2	-2	0	-2	-2	-1	2
φ_{16}	2855	-341	19	-1	13	-29	-1	1	-5	-1	-3	-1	-1	-1	$-2\alpha - 6$	$2\alpha - 4$	-5	-7
φ_{17}	5670	450	-6	-18	-12	-6	-2	0	0	0	0	2	4	0	2	2	2	-2
φ_{18}	19683	-729	-81	27	9	-9	3	-3	3	-1	1	-1	-1	1	1	1	-1	-1

Feasible Decompositions

1. $4\varphi_1 + 6\varphi_2 + 5\varphi_3 + 1\varphi_4 + 2\varphi_5 + 0\varphi_6 + 0\varphi_7 + 0\varphi_8 + 0\varphi_9$ (2A → 2A, 2B → 2A, 2C → 2B, 4A → 4D, 4B → 4B, 4C → 4G, 4D → 4D, 5A → 5A, 7A → 7B, 8A → 8K, 8B → 8H, 10A → 10C, 10B → 10C, 13A → 13CD, τ(13A) → 13CD, 14A → 14C, 20A → 20D)

Cohomological Dimensions

Not calculated.

8.27 $\mathrm{L}_3(9)$

The Brauer character table is omitted in this case due to its large size. However, this is easily obtainable in Magma. We remark that there are 38 irreducible GF(3) $\mathrm{L}_3(9)$-modules of dimension ≤ 248 and only those which have a non-zero multiplicity are given in the feasible decompositions below. Moreover, the fusion information is omitted for all conjugacy classes of semisimple elements of order > 20.

Feasible Decompositions

1. $18\varphi_1 + 9\varphi_2 + 9\varphi_3 + 3\varphi_4 + 3\varphi_5 + 1\varphi_8 + 3\varphi_9$ (2A → 2B, 4AB → 4G, 4C → 4B, 5AB → 5B, 7AB → 7B, 8AD → 8O/Q, 8EF → 8K, 8GH → 8C, 8IJ → 8H, 10AB → 10A, 13AD → 13CD, 16AD → 16G, 20AD → 20E)
2. $5\varphi_1 + 1\varphi_2 + 1\varphi_3 + 6\varphi_8 + 3\varphi_{20}$ (2A → 2B, 4AB → 4B, 4C → 4G, 5AB → 5B, 7AB → 7B, 8AD → 8C, 8EF → 8H, 8GH → 8O/Q, 8IJ → 8K, 10AB → 10F, 13AD → 13CD, 16AD → 16A, 20AD → 20L)

Cohomological Dimensions

The only non-zero cohomological dimensions for the 38 irreducible GF(3)-modules of dimension ≤ 248 are $\varphi_8 = 2,\ \varphi_{18} = 2,\ \varphi_{19} = 2$.

By Lemma 2.7 and Lemma 2.5, all subgroups $H < G$ with $\mathrm{L}_3(9) \cong F^*(H)$ that are associated to feasible decomposition 1 are not maximal.

9 Appendix

9.1 Centralisers of Semisimple Elements in $\mathrm{E}_8(3)$

In this section, we discuss the work which supports the results shown in Table 3. For a finite group $G(q)$ defined over the field GF(q) ($q = p^a$), we say $g \in G(q)$ is semisimple if the order of g is not divisible by p. In particular, the semi-simple elements of $G \cong \mathrm{E}_8(3)$ are those whose order are not divisible by 3. It is well known that G has 3^8 conjugacy classes of semi-simple elements, so there are as many centralisers to determine. Thanks to Frank Lübeck and his database [58], we were able to obtain a complete list of all possible orders of such centralisers. Moreover, the database gives a general description of the structure of each centraliser. However, it is not known which elements of G these centraliser orders correspond too, and it is this correspondence that we have attempted to establish. More information on how Lübeck obtained this information can be found in his paper [57] and surrounding background theory can be found in both Carter's paper [19] and Fleischmann and Janiszczak's paper [30]. Furthermore, details of how the centralisers of semisimple elements in $E_8(2)$ were determined can be found in [3]. The use of MAGMA was essential in this work. Moreover, having a complete list of how many conjugacy classes we have for each element order inside G (see [64]) was invaluable.

9.1.1 Main Table

Throughout this section we will be considering the conjugacy classes of semisimple elements of G; we shall introduce some notation to aid future explanations. Let X be a conjugacy class of G and assume $g, h \in X$. We know that $C_G(g) \cong C_G(h)$ and $\dim(C_V(g)) = \dim(C_V(h))$. Hence, it will often be more convenient to refer to $C_G(g)$ and $C_V(g)$ as $C_G(X)$ and $C_V(X)$ respectively. In Table 3, for each conjugacy class X of G, we list the centraliser sizes $|C_G(X)|$ for all elements of order ≤ 44 and all elements of prime-power order, the dimension of the fixed space $C_V(X)$ and also the power maps of elements in X into other conjugacy classes.

The first column of Table 3 gives a class label for each class X of G considered. In the majority of cases, these are assigned according to ATLAS [20] conventions. In particular, let $g \in G$ have order n and let X denote the conjugacy class of G containing g. Suppose that X is the smallest class of elements of order n in G. Then we assign X the label nA. We then assign the second smallest class of elements order n the label nB, the third smallest nC and so on. However, as some orders in G have more than 26 conjugacy classes, we must extend this notation slightly. If we reach Z and still have classes to label, we use a prime. For example, the labelling would be nA, nB, ... , nZ, nA′, nB′, ... and so on. This only occurs for elements of order 40 in Table 3.

We also change the notation used in the ATLAS regarding slave classes. Suppose $g \in A$ and $h \in B$ where A and B are slave classes in G. Then g and h are the same order, have isomorphic centralisers, $\dim(C_V(g)) = \dim(C_V(h))$ and the power maps of g and h are the same. Hence in these cases, we only include one row in Table 3 and change the label according to how slave classes there are. For example, consider the class with label 26AB in Table 3. This means that there are 2 conjugacy classes of G of elements order 26 with the given centraliser size, dimension of fixed space and power map. A further example would be the class with label 41AE. This means there are 5 classes of elements order 41 in G with the given information. In a small minority of cases, we find classes that have the same centraliser sizes and dimensions of fixed spaces but different power maps; see classes 40K′ and 40L′ for example. In these cases, we assign the labelling according to the power maps of the classes in question. For example, 40K′ precedes 40L′ as 20F precedes 20H; we do not consider the fusion into the classes of order 8 elements. If these classes had fused into the same class of order 20 elements, then we would look at the next element order in each power map until we find different fusion and decide the labelling in the same way as just explained. You will also see some classes have a '−' followed by a number in brackets in their label, like 73M−(15). This implies that the remainder of the classes of elements order 73, of which there are 15, have the information given in that row. As another example, the class label 547A−(39) implies that all conjugacy classes of elements order 547, of which there are 39, have the properties given in that row.

The second column of Table 3 lists the corresponding centraliser size $|C_G(X)|$ factored into a product of primes and the third column gives the value $\dim(C_V(X))$. The final column gives the power maps associated to the elements of X. For example, consider $g \in$ 8P. The final column lists the class 4G, meaning that $g^2 \in$ 4G. We can then find the remaining fusion information associated to g by looking at the row for 4G. In particular, we see that $g^4 \in$ 2B.

Having a list of centraliser orders is very useful, especially when using `FindCent`. Depending on the given parameters, `FindCent` may stop before the entire centraliser is constructed. Knowing the order allows us to keep `FindCent` running until we have found the entire centraliser. A variety of methods have been employed here. A simple yet key tactic used was to explicitly calculate the centralisers of certain elements in sufficiently large subgroups of G. The function `FindCent` was used frequently, although many changes to this had to be made in order to yield success. Such changes are discussed in Section 9.2.9 alongside the code. Having obtained the centraliser of a given element in some suitably large subgroup of G, we calculate its order and then search inside the list of centraliser sizes in G for all orders that are divisible by the order of the smaller centraliser. In some cases, this was enough to determine the order inside the whole group G or at least it would give a small list of possibilities for us to work with. However, in many cases, much more work had to be done.

Another method that was used frequently to determine centraliser orders for prime-order elements was the use of fusion of elements of a larger composite order. This is best illustrated with an example. There exists an element $g \in G$ of order $949 = 13 \cdot 73$. We know that g is centralised by its powers, hence in particular we have that g^{13} and g^{73} commute. So we must have that an element of order 73 commutes with an element of order 13. Moreover, through the use of Brauer characters and the dimensions of fixed spaces, we can determine which conjugacy classes such elements belong to. This allows us to not only restrict the number of possible orders for a centraliser of some element, but also allows us to determine which specific conjugacy class the centraliser must be associated to.

We briefly remark that throughout this section, we exclude the centraliser of the identity element from any of the searching that we do. In particular, we are considering the $3^8 - 1$ centraliser orders of non-trivial semisimple elements.

9.1.2 Elements of Prime-Power Order

We begin by looking at the centraliser orders of prime-power order elements. In some cases, we were able to explicitly construct the centraliser inside G and thus determine its structure, more detail is given as and when we do this. Moreover, in certain cases, the Sylow p-subgroup of G was constructed inside the 248-dimensional representation.

$p = 2^n$

Let P be a Sylow 2-subgroup of G, then $|P| = 2^{30}$ and $|Z(P)| = 2$. Let $g \in Z(P)$ have order 2, then $\dim(C_V(g)) = 120$. We must have that P is also a Sylow 2-subgroup of $C_G(g)$ and there is only one centraliser order divisible by 2^{30}; this gives us the centraliser order for our class 2B. Moreover, this centraliser is actually the group of Lie type $D_8(3)$ and has shape $2.\Omega_{16}^{+}(3).2$ (see [48]). We have one more class of involutions to consider and this class has $\dim(C_V(g)) = 136$. Inside P, we find random involutions contained in this class and calculate their centraliser orders inside a permutation representation of P. This is extremely quick and so is an efficient method in determining a large power of 2 that must divide $|C_G(g)|$. We find that there exists an involution g such that $\dim(C_V(g)) = 136$ and $|C_P(g)| = 2^{27}$. Searching through the list of all possible orders, we find only one that is divisible 2^{27}, this gives us our centraliser order for the class 2A.

We remark that constructing the centralisers of involutions in large groups is actually extremely quick thanks to the brilliant Bray method [13]. A simpler method to that detailed above in obtaining the centraliser sizes of involutions is to simply use the Bray method in Magma by calling `CentraliserOfInvolution(G,g)` and then use `LMGFactoredOrder` to calculate the order. This will work, but despite the centraliser taking only seconds to construct, calculating the order afterwards takes hours and LMG functions may return incorrect outputs.

There are 7 conjugacy classes of elements of order 4 to consider. We do not give all the tedious details here, however these centraliser sizes are easily found using the methods described above. The orders here are so large and contain so many different prime divisors that there were only a few possibilities for each class of order 4 elements. In the cases where we had multiple possibilities, calculating centraliser sizes inside P was sufficient to distinguish the orders.

The 17 conjugacy classes of elements of order 8 put up much more of a fight. As was the case with most of the element orders considered, the largest centralisers for elements of order 8 where easily found by using the methods described above; the centralisers for classes 8A, 8B, 8C, 8D, 8E, 8F where all obtained in this way. For all other cases, there were multiple possibilities and other methods had to be used. The fusion of conjugacy classes (and their centralisers) was a useful thing to consider here. For example, consider the class 8N. We find that $|C_{G(\Phi)}(8\text{N})| = 2^{17} \cdot 3^9 \cdot 5 \cdot 7^2$ where $\Phi = \{2, 3, 4, 5, 6, 7, 8, 120\} \cong \text{D}_8$, hence $|C_G(8\text{N})|$ must be divisible by this. Moreover, there exists an element $g \in G$ of order $488 = 8 \cdot 61$ such that $g^{61} \in 8\text{N}$. Hence 61 is also a divisor of $|C_G(8\text{N})|$. From this, we get a small list of possibilities. Included in this list are the orders $2^{20} \cdot 3^{30} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41 \cdot 61 \cdot 73$ and $2^{19} \cdot 3^{20} \cdot 5^3 \cdot 7 \cdot 13 \cdot 41 \cdot 61$. However, we know that 8N fuses into 4E (and at this point we know $|C_G(4\text{E})|$). As 5^3 does not divide $|C_G(4\text{E})|$, we know that neither of these orders are applicable. This method allowed us to restrict the possibilities for $|C_G(8\text{N})|$ further until we could finally deduce its true value.

A modified version of FindCent was also implemented here in order to find large divisors of our desired centraliser orders. Consider the class 8H. We find that there exists an element $g \in 8\text{H} \cap P$ such that $|C_P(8\text{H})| = 2^{21}$ and also we find that 7, 13 and 41 must also divide $|C_G(8\text{H})|$ by the fusion of elements of order 56, 104 and 328 respectively. From this we find only 2 possibilities, namely $2^{21} \cdot 3^{18} \cdot 5^2 \cdot 7 \cdot 13^2 \cdot 41$ and $2^{22} \cdot 3^{20} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 61$. Using the modified version of FindCent, we start the process of building up our centraliser with the hope that we are able to find a distinction between our two possibilities. This method was often more efficient than attempting to construct the entire centraliser in a large subgroup as we are able to stop the process of constructing the centraliser at any point after we find a divisor dividing only one of our possibilities. Indeed, with this case we found that 13^2 must divide $|C_G(8\text{H})|$ as this divisor must appear in the order of $C_{G(\Phi)}(8\text{H})$ where $\Phi \cong \text{D}_8$ is as defined before.

All the centraliser sizes for classes of order 8 elements that fused into 2B were easily obtainable. In fact, any semi-simple element of even order that powers into 2B can be obtained using the following method. By picking any element of order 8 from the desired conjugacy class and then calculating $C_G(g^4)$ using the Bray method, we are able to use the command LMGCentraliser to explicitly construct $C_G(g) \leq C_G(g^4) = C_G(2\text{B}) \cong 2.\Omega^+_{16}(3).2$. Although easy, we tried to avoid this method as calculating such centralisers took upwards of a day of computation time. However, this would be necessary in order to determine the centraliser sizes for elements of order 16.

For all of the remaining cases, we had several possibilities for the order of the centraliser. Despite our best efforts, we were not able to distinguish many of these without first considering elements of different orders. As mentioned previously, the largest centralisers for an element of a given order are usually easily found. Hence in order to finish the elements of order 8, we started to look at other elements of small order and attempted to find centraliser sizes for these with the hope of removing some possibilities for the remaining classes of order 8 elements.

We now consider elements of order 16. Here we have 11 conjugacy classes to contend with. At this point, most of the centraliser sizes are small enough for there to be far too many possibilities outputted by simply finding large divisors. However, the centraliser orders for classes 16A, 16B and 16C are all found in this way. The remaining 8 cases must be explicitly constructed. These classes all fuse into 2B and so we are able to utilise `LMGCentraliser` in order to construct $C_G(g)$ inside $C_G(g^8)$.

Alternatively and preferably, we construct $C_G(g^2)$ first (using `LMGCentraliser`) and then construct $C_G(g)$ inside this instead. This takes a similar amount of computation time with the added advantage of obtaining the centraliser of the order 8 element g^2 as well. This is useful as by examining the conjugacy classes of this centraliser, we are able to deal with all centralisers of order 16 elements that fuse into it very quickly. This is the main method of attack for all classes that have a 'small' centraliser size - to construct a larger centraliser that our desired centraliser lives inside and then use `LMGCentraliser`. The function `LMGClasses` was often used on these larger centralisers to get a complete list of conjugacy class representatives and their orbit sizes. Using this list, we were able to determine all centraliser sizes of elements that fuse into the associated class.

Finally, we consider the single class of elements order 32. This fuses into 2B, so we are able to use `LMGCentraliser(C,g)` where $C = C_G(g^{16})$ and g is any element of order 32 to determine the centraliser order.

$p = 5$

We have 2 classes to consider here. Let $P \in \mathrm{Syl}_5(G)$, so $|P| = 5^5$. We are able to construct P inside the 248-dimensional representation using the following method. Let $\Phi_1 = \{2,3,4,6,7,8\} \cong 2\mathrm{A}_3$ and let $g \in G(\Phi_1) \cap 5\mathrm{B}$. Then set $\Phi_2 = \Phi_1 \cup \{24, 106\} \cong \mathrm{A}_8$, $\Phi_3 = \Phi_1 \cup \{r_5, r_{120}\} \cong \mathrm{D}_8$ and let $C_1 = C_{G(\Phi_2)}(g) \sim 80.(\mathrm{L}_2(81).2)$, $C_2 = C_{G(\Phi_3)}(g) \sim 10.(Alt(6).U_3(9))$. Then $P \leq H = \langle C_1, C_2 \rangle$ and P can be obtained through the use of `LMGSylow(H,5)`. From our construction, we have $g \in Z(H)$ so $H \leq C_G(g)$. But there is only 1 order in our list divisible by $|H|$, namely $|H|$ itself. Hence $H = C_G(g) \cong \mathrm{SU}_5(9)$.

Now we consider the other class of elements order 5. Searching inside P, we find an element $g \in P$ such that $\dim(C_V(g)) = 68$ and $C_P(g) \cong 5^4$. Moreover, there exists an element of order 5 $g \in G(\Phi)$ where $\Phi = \{1,3,4,5,6,7,8,120\} \cong \mathrm{A}_8$ such that $\dim(C_V(g)) = 68$ and $|C_{G(\Phi)}(g)| = 2^{13} \cdot 3^{10} \cdot 5^2 \cdot 11^2 \cdot 13$. Using these divisors, we are able to determine the order of the centraliser for the other class of elements order 5. In Section 9.1.3, as part of our work on elements of order 10, we construct a copy of $C_G(5\mathrm{A})$.

$p = 7$

Here we have 2 classes. Let $P \in \mathrm{Syl}_7(G)$ and suppose $g \in G$ is order 7. By [48], there exists a subgroup $H \in G$ such that $H \cong \mathrm{U}_3(3)^4$ and $P \in \mathrm{Syl}_7(H)$. By calculation inside H, we find that $P \cong 7^4$. Hence, we have $|C_G(g)|$ is divisible by 7^4. This leaves us with only 2 possible orders and these must be our centraliser orders. Now there exists $g \in G(\Phi)$ where $\Phi = \{1,2,3,4,5\}$ such that $\dim(C_V(g)) = 80$ and $C_{G(\Phi)}(g) \sim 56.\mathrm{Alt}(6)$. In particular, g commutes with an element of order 5 (whilst we know from group orders that the other class of order 7 elements does not), allowing us to distinguish our centraliser orders.

We construct the centraliser for 7A as follows. Set $\Phi_1 = \{2,3,4,5\} \cong \mathrm{D}_4$ and let $g \in 7\mathrm{A} \cap G(\Phi_1)$. We build up $C_G(g)$ in larger subgroups containing $G(\Phi_1)$. Define $\Phi_2 = \Phi_1 \cup \{1\} \cong \mathrm{D}_5$, $\Phi_3 = \Phi_1 \cup \{69\} \cong \mathrm{D}_5$ and set $C_1 = C_{G(\Phi_2)}(g)$, $C_2 = C_{G(\Phi_3)}(g)$. We have that $C_1 \cong C_2 \sim 56.\mathrm{Alt}(6)$. Finally, set $\Phi_4 = \Phi_1 \cup \{8,74,101,104\} \cong 2\mathrm{D}_4$ and let $C_3 = C_{G(\Phi_4)}(g) \sim (4 \times 28).\Omega_8^+(3)$. Then $C_G(g) = \langle C_1, C_2, C_3 \rangle \sim 7.{}^2\mathrm{E}_6(3)$.

We are also able to construct the centraliser for class 7B. Define $\Phi_1 = \{2,4,5,6,7\} \cong \mathrm{A}_5$ and let $g \in 7\mathrm{B} \cap G(\Phi_1)$. Let $\Phi_2 = \Phi_1 \cup \{8,120\} \cong \mathrm{A}_7$ and set $C_1 = C_{G(\Phi_2)}(g) \sim 364.\mathrm{Sym}(4)$. Now let $\Phi_3 = \Phi_1 \cup \{97,1\} \cong \mathrm{A}_2 + \mathrm{A}_5$ and set $C_2 = C_{G(\Phi_3)}(g) \sim 364.\mathrm{L}_3(3)$. Now let $\Phi_4 = \Phi_1 \cup \{69\} \cong \mathrm{D}_5$ and set $C_3 = C_{G(\Phi_4)}(g) \sim (2\times 28).\mathrm{L}_2(27)$. Finally, set $\Phi_5 = \Phi_1 \cup \{65\} \cong \mathrm{E}_6$. We cannot use `LMGCentraliser` on $G(\Phi_5)$ due to its structure, so we must use `FindCent`. Luckily, we need not construct the entire centraliser $C_{G(\Phi_5)}(g)$ in order to finish the construction of $C_G(g)$. Indeed, we run `FindCent` until we obtain a subgroup $C_4 \leq C_{G(\Phi_5)}(g)$ such that $C_4 \cong 7 \times \mathrm{Aut}(3_-^{1+2})$. Then $C_G(g) = \langle C_1, C_2, C_3, C_4 \rangle$.

$p = 11^n$

We only have 1 class to consider here, so let $g \in G$ be order 11. We construct $P \in \mathrm{Syl}_{11}(G)$ inside the 248-dimensional representation of G as follows. Set $\Phi = \{1,3,4,5\} \dot\cup \{69,7,8,120\} \cong 2\mathrm{A}_4$. Then $P \in \mathrm{Syl}_{11}(G(\Phi))$. We find that $P \cong 121 \times 121$, hence P is abelian. Therefore, we must have that 11^4 divides $|C_G(g)|$. There are only two possibilities, either $2^9 \cdot 3^{10} \cdot 5 \cdot 11^4 \cdot 13$ or 11^4. But we have seen that g centralises an involution, hence $|C_G(g)|$ must be the former. Moreover, we have that $C_G(g) \cong C_{\mathrm{A}_4(3)^2}(g) \cong \mathrm{L}_5(3) \times 121$.

Now we consider the 33 conjugacy classes of elements order 121. As 11^4 must also divide these centraliser orders, we have that $|C_G(g)|$ is either $2^9 \cdot 3^{10} \cdot 5 \cdot 11^4 \cdot 13$ or 11^4. The first order appears again here as it appears in the list of centraliser sizes 12 times, meaning that we still have 11 of these to deal with. The second order appears 22 times. Upon gathering representatives for all 33 conjugacy classes of elements order 121, we find that 6 of these have a representative g such that $\dim(C_V(g)) = 28$ and the remainder have $\dim(C_V(g)) = 8$. A quick calculation inside $C_G(11\mathrm{A})$ shows that the classes with $\dim(C_V(g)) = 28$ must have the larger of these centraliser sizes and all the classes with $\dim(C_V(g)) = 8$ consequently have centraliser P.

$p = 13$

There are 6 classes of elements order 13 in G. As with $p = 11$, we construct $P \in \mathrm{Syl}_{13}(G)$ inside the 248-dimensional representation of G. Set $\Phi = \{1,3\} \dot\cup \{5,6\} \dot\cup \{8,120\} \dot\cup \{2,69\} \cong 4\mathrm{A}_2$. Then $P \in \mathrm{Syl}_{13}(G(\Phi))$. We find that $P \cong 13^4$ and so 13^4 must divide our 6 centraliser orders; there are only 6 orders in the list with this property. In order to distinguish our classes, we calculate centralisers in large subgroups to find divisors. For example, considering the group $G(\Phi)$ where $\Phi = \{2,3,4,5,6,7\} \cong \mathrm{D}_6$, there exists an element $g \in G(\Phi)$ such that g is order 13, $\dim(C_V(g)) = 80$ and $|C_{G(\Phi)}(g)| = 2^9 \cdot 3^6 \cdot 5 \cdot 13^2$. Hence g commutes with an element of order 5 and only one of our orders is divisible by 5, allowing us to decide the order for class 13AB. Similar methods were used to decide the rest.

The following process allows us to construct the centraliser for both 13AB and 13CD. Set $\Phi_1 = \{2,4\} \dot\cup \{6,7\} \cong 2\mathrm{A}_2$. Let $g \in C_1$ such that $g \in 13\mathrm{AB} \cup 13\mathrm{CD}$. Now set $\Phi_2 = \Phi_1 \dot\cup \{93,1\} \dot\cup \{106,65\} \cong 4\mathrm{A}_2$. Set $C_1 = C_{G(\Phi_2)}(g) \cong 13 \times \mathrm{L}_3(3)^3$. Finally, set $\Phi_3 = \Phi_1 \cup \{3,5,8,120\} \cong \mathrm{D}_8$. Using `LMGCentraliser`, we construct $C_2 = C_{G(\Phi_3)}(g) \cong 26 \times \Omega_{10}^+(3)$. Then $C_G(g) = \langle C_1, C_2 \rangle$. We find that $C_G(13\mathrm{AB}) \sim 13.\mathrm{E}_6(3)$ and $C_G(13\mathrm{CD}) \sim 13.{}^3\mathrm{D}_4(3).\mathrm{L}_3(3)$.

We use a slightly different construction for 13E as $G(\Phi_1)$ from the previous example does not contain any elements from this class. Instead, we set $\Phi_1 = \{2,4\} \dot\cup \{6,7\} \dot\cup \{93,1\} \cong 3\mathrm{A}_2$. Let $g \in 13\mathrm{E} \cap G(\Phi_1)$. Now set $\Phi_2 = \Phi_1 \cup \{65,106\} \cong 4\mathrm{A}_2$. We define $C_1 = C_{G(\Phi_2)}(g) \cong 13^3 \times \mathrm{L}_3(3)$. Finally, we set $\Phi_3 = \Phi_1 \cup \{3,8\} \cong \mathrm{A}_8$. Now set $C_2 = C_{G(\Phi_3)}(g) \sim (2\times 13^2).(\mathrm{L}_2(27).2)$. Then $C_G(g) = \langle C_1, C_2 \rangle \cong 13^2 \times {}^3\mathrm{D}_4(3)$. Upon further investigation, we find that no element from the class 13F commutes with an involution from 2B.

The centraliser for class 13F is much easier to construct. Indeed, set $\Phi_1 = \{2,4\} \dot\cup \{6,7\} \dot\cup \{93,1\} \dot\cup \{106,65\}$ and choose $g \in 13\mathrm{F} \cap G(\Phi_1)$. Then $C_G(g) = C_{G(\Phi_1)}(g) \cong 13^2 \times \mathrm{L}_3(3)^2$.

$p = 19$

We have only 1 class to consider here, hence it suffices to let $g \in G$ be any element of order 19. The Sylow 19-subgroup of G is $\langle g \rangle$. From [48], we have that there exists a subgroup $H \leq G$ such that $H \cong U_9(3)$. Without loss of generality, suppose $g \in H$. Then $C_H(g) \cong 7 \times 19 \times 37$. There are many orders in our list divisible by this, so we must find more divisors of $|C_G(g)|$. Having found the centraliser orders for elements of order 13 and 5, we know that g does not commute with any element of these orders. Moreover, we know from the centraliser order of class 2A that g must commute with an involution. This is enough to decide the order for $|C_G(g)|$.

We are able to deduce the centraliser sizes for elements of order $703 = 19 \cdot 37$ by our work here. Let $g \in G$ be order 703, then $\dim(C_V(g)) = 14$. We know that $|C_G(g)|$ must be divisible by $19 \cdot 37$. Also, we are able to find a complete set of representatives for all conjugacy classes inside the centralisers for classes 2A and 7A. Hence $|C_G(g)|$ must be divisible by $2 \cdot 7 \cdot 19 \cdot 37$ and there is only one remaining order that satisfies this, namely $2^5 \cdot 3^3 \cdot 7 \cdot 19 \cdot 37$.

$p = 31, 271$

Let $g \in G$ be any element of order 31 or 271. There are 9 classes of elements order 271 in G and only a single class of order 31 elements. In both cases, the Sylow p-subgroup of G is cyclic. Upon searching through the list of orders for centraliser sizes divisible by 31 and 271 respectively, we find only 1 possibility. Hence we have that $|C_G(g)| = 31 \cdot 271$. This order appears 280 times in the list, meaning 270 of these are still unaccounted for. However, we note that there are 270 conjugacy classes of elements order $8401 = 31 \cdot 271$. Hence we must have that for $h \in G$ of order 8401, $|C_G(h)| = 31 \cdot 271$.

$p = 37$

There are 2 classes in G of elements order 37 and both of these have $\dim(C_V(g)) = 14$. Let $P \in \mathrm{Syl}_{37}(G)$, then $P \cong 37$. By our previous work, we know no element of order 37 commutes with any element of order 5 or 13. Furthermore, elements of order 37 do commute with elements of order 2, 7 and 19. By Sylow's theorem, we know that this is true for both classes of elements order 37. These restrictions allow us to find the order of the centraliser.

$p = 41$

We have 15 classes to consider here with 5 classes of each fixed space dimension 32, 16 and 8. The Sylow 41-subgroup of G is elementary abelian of order 41^2; there are only 3 distinct orders remaining in our list divisible by this. These are $2^{12} \cdot 3^{12} \cdot 5 \cdot 7 \cdot 13 \cdot 41^2$, $2^6 \cdot 3^4 \cdot 5 \cdot 41^2$ and $2^2 \cdot 41^2$. Using Magma, we find the existence of an element $g \in G$ of order $287 = 7 \cdot 41$ such that $\dim(C_V(g^7)) = 32$. Similarly, there exists an element $g \in G$ of order $205 = 5 \cdot 41$ such that $\dim(C_V(g^5)) = 16$. This allows us to decide the centraliser orders for our conjugacy classes of elements order 41.

$p = 61$

There are 11 conjugacy classes in G of elements order 61, of which 6 have $\dim(C_V(g)) = 28$ whilst the remaining 5 have $\dim(C_V(g)) = 8$. The Sylow 61-subgroup of G is elementary abelian of order 61^2 - this was verified inside the subgroup $\mathrm{U}_5(3)^2 \leq G$ which appears in [48]. There are only 2 distinct orders in our list divisible by 61^2, these are $2^{11} \cdot 3^{10} \cdot 5 \cdot 7 \cdot 61^2$ and 61^2 which appear 6 and 5 times respectively. Let $\Phi = \{2, 3, 4, 5, 6, 7\} \cong \mathrm{D}_6$. Inside $G(\Phi) \sim 2^2.\Omega_{12}^+(3)$, we find an element g of order 61 such that $|C_{G(\Phi)}(g)| = 2^4 \cdot 61$. Hence we can distinguish our 2 possibilities using this and the dimensions of fixed spaces. Furthermore, by working inside $\mathrm{U}_5(3)^2$ and comparing orders, we are able to identify the structure $C_G(61\mathrm{AF}) \cong 61 \times \mathrm{U}_5(3)$.

$p = 73$

There are 27 conjugacy classes of elements order 73 and these must have a fixed space dimension of 32, 20 or 8. Moreover, the Sylow 73-subgroup of G is elementary abelian and has order 73^2. There are three distinct orders inside our list divisible by 73^2; counting multiplicities we have 27 in total. Hence if $g \in G$ is order 73, $|C_G(g)|$ must be one of: $2^6 \cdot 3^{12} \cdot 7^2 \cdot 13^2 \cdot 73^2$, $2^5 \cdot 3^6 \cdot 5^2 \cdot 73^2$, 73^2. There exists an element $h \in G$ of order $365 = 5 \cdot 73$ such that $\dim(C_V(h^5)) = 20$. Using the same method but with an element h of order $949 = 13 \cdot 73$, we find that $\dim(C_V(h^{13})) = 32$. This allows us to completely decide the centralisers for our classes of elements order 73.

By working inside $\mathrm{U}_3(9)^2$, which is a subgroup of a maximal rank subgroup, we can identify the structure of $C_G(73\mathrm{GL})$. We take an element of order 365 and then take its 5th power. This is an element of order 73 that commutes with an element of order 5, hence would belong to the 73GL class in G. Taking its centraliser in $\mathrm{U}_3(9)^2$ yields the full centraliser in G, and it has structure $C_G(73\mathrm{GL}) \cong 73 \times \mathrm{U}_3(9)$.

$p = 547$

There are 39 conjugacy classes to deal with here. The Sylow 547-subgroup of G is cyclic. By our previous work, we know no element of order 547 commutes with an element of order 5, 7 or 13. Moreover, there exists an element of order $1641 = 3 \cdot 547$ in G. Hence 3 must divide our centraliser order. Using these restrictions and Sylow's theorem, we can conclude that if $g \in G$ has order 547, then $|C_G(g)| = 2^5 \cdot 3 \cdot 547$.

$p = 757$

Here we have 42 conjugacy classes and as with all the large primes we consider, the Sylow 757-subgroup of G is cyclic. Let $g \in G$ be any element of order 757. We know no elements of order 757 commute with any elements of order 5 or 7. Moreover, by Sylow's theorem, we have that g must commute with an involution in 2A and an element in 13AB. These restrictions leave us with only one possibility.

$p = 1093$

There are 78 conjugacy classes of elements order 1093 in G and the Sylow 1093-subgroup of G is cyclic. By our previous work we know an element of order 1093 cannot commute with an element of order 5, 13 or 41. Moreover, there exists an element $h \in G$ of order $3279 = 3 \cdot 1093$. Hence 3 must divide our centraliser order. Using this and Sylow's theorem, this is sufficient to decide our centraliser orders here.

$p = 1181$

There are 59 classes here and the Sylow 1181-subgroup of G is cyclic. Excluding the centraliser order of 5B (which only appears once in the list), there is only 1 distinct order in our list divisible by 1181; namely $5 \cdot 1181$. This appears 295 times in the list, meaning that 236 of these are still unaccounted for. However, there are 236 conjugacy classes of elements order $5905 = 5 \cdot 1181$. Hence if $g \in G$ is order 1181 or 5905, then $|C_G(g)| = 5 \cdot 1181$.

$p = 4561, 6481$

Let $g \in G$ have order 4561 or 6481. There are 152 / 270 conjugacy classes of elements order 4561 / 6481 respectively. In both cases the Sylow p-subgroup is cyclic. Moreover, upon searching through the list for all orders divisible by 4561 and 6481, we find that the only possibility is $C_G(g) = \langle g \rangle$.

9.1.3 Elements of Composite Order

$n = 10$

We have 6 classes to deal with here. We find that the classes 10A, 10B, 10D, 10E and 10F all power into 2B, hence the corresponding centralisers can all be constructed inside $C_G(2\mathrm{B})$ using `LMGCentraliser`. We find that $|C_G(10\mathrm{A})| = |C_G(5\mathrm{A})|$, hence we can also construct $C_G(5\mathrm{A})$ using this method.

Suppose $g \in 10C$, then $g^5 \in 2\mathrm{A}$. Due to the size of the centraliser of 2A, we are unable to use `LMGCentraliser` here. Eventually, `FindCent` constructs a subgroup of order $2^{17} \cdot 3^{14} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$. Even using this, we still have several possibilities. We were able to deduce the order after slowly ruling out some of the other orders from our list as they corresponded to elements of a different order.

$n = 14$

There are 5 classes to consider here. In all cases, the standard methods were enough to be able to determine the centraliser orders. We remark that these centraliser sizes are still quite large and many were found to have 7^3 as a divisor which greatly reduced the possibilities. Constructing centralisers for classes 14A and 14B is can be done inside the centralisers of 2B and 2A respectively using `LMGCentraliser` and `FindCent`. Furthermore, the remaining classes all fuse into 7B and their centralisers can be constructed inside $C_G(7\mathrm{B})$ by using `LMGCentraliser`. However, we present an alternative method for constructing the centraliser for class 14C as using `LMGCentraliser` can take a long time. Set $\Phi_1 = \{5, 6, 7, 8, 120\} \cong \mathrm{A}_5$. Let $g \in G(\Phi_1) \cap 14\mathrm{C}$. Next, we set $\Phi_2 = \Phi_1 \cup \{1, 3\} \cong \mathrm{A}_2 + \mathrm{A}_5$ and let $C_1 = C_G(\Phi_2)(g)$. Finally, set $\Phi_3 = \Phi_1 \cup \{2, 3, 4\} \cong \mathrm{D}_8$ and define $C_2 = C_{G(\Phi_3)}(g)$. Then $C_G(g) = \langle C_1, C_2 \rangle$.

$n = 20$

There are 16 classes of elements order 20. The centralisers for classes 20A and 20B were obtained easily using the standard methods. To determine the orders of the remaining centralisers, we use the method described in Section 9.1.2 involving the conjugacy classes of a larger centraliser. We know that 5 of the 6 classes of order 10 elements fuse into 2B, meaning we can use `LMGCentraliser` to easily obtain the centraliser. As such, we can easily construct the centralisers of any order 20 elements that square into the classes 10A, 10B, 10D, 10E and 10F. Moreover, it turns out that $C_G(10\mathrm{C})$ can be constructed inside $C_G(5\mathrm{A})$. Hence we can obtain all centralisers of order 20 elements by working inside the centralisers of their squares.

$n = 22, 44$

First we consider the elements of order 22. There are only 2 classes to deal with here with fixed space dimensions 20 and 16. We follow the same approach used for elements of order 16. We have previously constructed the centraliser $C_G(11\text{A})$ and both centralisers for elements of order 22 must be contained as subgroups inside this. After using `LMGClasses` on $C_G(11\text{A})$ and looking at the orbit sizes for elements of order 22, we can deduce the sizes of their centralisers.

Now we consider the 3 classes of elements order 44. We use exactly the same approach as above, searching the conjugacy classes of $C_G(11\text{A})$ for elements of order 44 and then looking at the sizes of their orbits. From this we can quickly deduce the corresponding 3 centraliser sizes.

$n = 26$

We have 14 conjugacy classes to consider here. We follow the same process as with the elements of order 16. In Section 9.1.2, we detailed a method for constructing centralisers for all class of elements order 13. We remark that it is not actually necessary to construct the centraliser for 13AB in order to find the orders of centralisers for elements order 26 that fuse inside it. Indeed, these may be found using the standard methods used in the cases for prime-order elements.

After constructing the centraliser for class 13AB, the centralisers for 26AB and 26CD are now be easily obtained. Indeed, let C denote the centraliser for class 13AB. Then there exists an involution $t \in C \cup 2\text{B}$ such that $C_C(t) \cong C_G(26\text{AB})$. Similarly, there exists an involution $t \in C \cup 2\text{A}$ such that $C_C(t) \cong C_G(26\text{CD})$. Using `CentraliserOfInvolution` leads to a very quick calculation of these centralisers.

The centralisers for classes 13CD, 13E and 13F are small enough to use `LMGClasses` and employ the method described earlier to find the centraliser sizes for the remaining classes of elements order 26. In fact, we can easily construct all these centralisers inside the centralisers for elements order 13 through the use of `LMGCentraliser`.

$n = 28$

There are 27 classes of elements order 28 in G. The orders of all corresponding centralisers to these classes can be easily found by using `LMGClasses` on each centraliser for the conjugacy classes of elements order 14. Furthermore, we can construct all centralisers for elements of order 28 inside the centralisers of order 14 elements.

$n = 35$

There is only 1 class to consider here and it fuses into 5A. Suppose $g \in 10\text{A}$. Firstly, we construct $C = C_G(g) = C_G(g^2)$ inside $C_G(g^5)$ using `LMGCentraliser`. Then, let $z \in Z(C)$ be order 5. Now pick any element from C of order 7, we shall call this h. Then zh is order 35 and $(zh)^7 \in Z$. In particular, $C = C_G((zh)^7)$ and $C_G(zh) = C_C(zh)$. Finally, we use `LMGCentraliser(C,zh)`, to construct $C_G(zh)$ and so we are done.

$n = 38$

There is only 1 conjugacy class of elements order 38 in G. Let $g \in G$ be any element of order 38. We know that $C_G(g) \leq C_G(g^2)$, hence $|C_G(g)|$ divides $2^5 \cdot 3^3 \cdot 7 \cdot 19 \cdot 37$. Moreover, there exist elements of order $1406 = 38 \cdot 37$ and $114 = 38 \cdot 3$ in G, hence both 37 and 3 divide $|C_G(g)|$. Hence we have determined that $|C_G(g)|$ is divisible by $2 \cdot 3 \cdot 19 \cdot 37$ and there is only one remaining order (upto multiplicity) with this property, namely $2^5 \cdot 3 \cdot 19 \cdot 37$.

$n = 40$

There are 51 classes to contend with here. Luckily, most of them are easily dealt with by searching inside the conjugacy classes of centralisers of elements of order 10. However, all the classes that fuse into 10A must be dealt with more carefully as $C_G(10\text{A})$ is too large to obtain representatives and orbits sizes for every conjugacy class. In these cases, we were able to construct the centraliser corresponding to the class of order 20 elements in which our elements of order 40 fuse. Then, we could use `LMGClasses` and apply the same procedure as before. Using this method, the centraliser sizes for elements of order 40 are obtained without much hassle.

9.1.4 Future Work

It remains to determine the corresponding conjugacy classes of G for the orders not yet eliminated from the list (of which there are still over 5000). These are all composite-order elements and the following method will likely be a useful tool in determining their centralisers. Let p be a prime and suppose that $o = pm$, where $(p, m) = 1$ and $m \in \mathbb{N}$, is an order we have not yet considered. Suppose that elements of order p in G have centralisers of the form $p \times H$ for some subgroup H. Then by considering all such classes of p-elements in G and all classes of order m elements in H, we can fully determine the centralisers for elements of order pm in G.

For example, let $p = 61$ and $m = 2$. From Table 4, we have that $C_G(61\text{AF}) \cong 61 \times \text{U}_5(3)$. To determine the centralisers of elements order $122 = 61 \cdot 2$ in G, it suffices to look at the centralisers of involutions inside $H \cong \text{U}_5(3)$. We find that H has two classes of involutions with centralisers $C_1 = 4.(\text{U}_4(3) : 4)$ and $C_2 = 4.(\text{U}_3(3) \times \text{Sym}(4))$. Thus we have that there are 2 classes of elements order 122 in G with centralisers $61 \times C_1$ and $61 \times C_2$. We remark that this method hinges on the fact that the centralisers for elements of order pm must live inside the centralisers of p-elements, which we have already determined. An advantage of using this approach over the method used frequently in this report of searching through conjugacy classes of $C_G(g)$ is that we need not work inside G directly; it is sufficient to work inside H independently. However, for this to work, $C_G(g)$ must be of the form $p \times H$.

9.2 Supplementary Code

9.2.1 Root System Code

```
//This code can be used to find parabolic subgroups of various Dynkin
types inside E_8(3); it supports the work in Section 2.3.

G:=GroupOfLieType("E8",3);
W := WeylGroup(G);
C := Classes(W);
Rd := RootDatum("E8");
R := StandardRootSystem("E",8);
Z<q> := PolynomialRing(Integers());
```

```
nab := function(a,b);
    top := 2*(Root(R,a), Root(R,b));
    bottom := (Root(R,a), Root(R,a));
    return top/bottom;
end function;

weight := function(a,b);
    return nab(a,b)*nab(b,a);
end function;

//if weight(a,b)=1, then the roots a,b are joined by an edge.
Otherwise, a and b are orthogonal.

getedges := function(L);
//Given a set of root labels L, this function finds all edges connecting
the roots in L.
    edges := {@@};
    weights:={@@};
    for i in L do
        for j in L do
            if i ne j and weight(i,j) ne 0 then
                Include(~edges, {i,j});
                Include(~weights,[{i,j},{@weight(i,j)@}]);
            end if;
        end for;
    end for;
    return edges, weights;
end function;

makegraph := function(L);
//Given a set of root labels L, this function outputs the adjacency matrix and Dynkin graph
corresponding to the roots in L.
    edges,w := getedges(L);
    weights := [w[i][2][1] : i in [1..#w]];
    g := Graph<IndexedSet(L)|edges>;
    A:=AdjacencyMatrix(g);
    return g,A;
end function;
```

9.2.2 Constructing Parabolic Subgroups

```
//The function makesub below constructs the subgroup generated by the root subgroups
corresponding to the roots with labels given in the set L.

H:=GroupOfLieType("E8",GF(3));
f:=AdjointRepresentation(H); //map from H into GL(248,3)
```

```
Q:=Codomain(f); //GL(248,3)
makesub := function(L);
SGens:=[];
for i in L do;
Append(~SGens, elt<H|<i,GF(3)!2>>);//generators of root subgroup for roots in L
Append(~SGens, elt<H|<120+i,GF(3)!2>>);//generators of root subgroups for the
 negatives of the roots in L
end for;
Sgens:=[];
for h in SGens do
Append(~Sgens, f(h)); //map over the generators
end for;
S:=sub<Q|Sgens>; //generate the group as a subgroup of GL(248,3)
return S;
end function;

//The function below makeQ constructs Q(Phi) for some set Phi = L.

R:=Roots(RootDatum("E8")); //all roots in E8 root system
makeQ := function(L);
inds := [];
notL := {1,2,3,4,5,6,7,8} diff L; //fundamental roots not contained in L
for i in notL do
for j in [1..#R] do //loop through all roots to find roots with the correct positive components
if R[j][i] gt 0 then
Append(~inds, j);
end if;
end for;
end for;
inds := SetToSequence(Set(inds));
SGens:=[];
for i in inds do;
for el in GF(3) do
Append(~SGens, elt<H|<i,GF(3)!el>>); //generators of root subgroups for the roots we have found
end for;
end for;
Sgens:=[];
for h in SGens do
Append(~Sgens, f(h));//map over roots
end for;
S:=sub<Q|Sgens>;//generate the group as a subgroup of GL(248,3)
return S;
end function;
```

9.2.3 Constructing Initial Sets FinSub, BadSub, ActnGpDiff

```
// Below we have the code for the first part of the algorithm described in Section 3.1.
In this part of the code, we form our initial sets FinSub, BadSub and ActnGpDiff and
then do no further calculations. In particular, we deal with BadSub in the next stage
of the code.

//The code below is specific to L_2(2187), and so must be adapted if considering other
groups. We initialise O to be Q(Phi) for whichever Phi we are considering. For example,
if Phi = {1,3,4,5,6,7}, then we set O := makeQ({1,3,4,5,6,7}). We set x1093 to be our
element of order 1093 and this must be initialised before starting the below process.

GROUPS:=[**];
RESULTS:=[**];
FinSub:={@@};
BadSub:={@@};
SetSub2:={@O@};
ActnGpDiff:={@@}; //initialise our sets
count:=0;

repeat
 countt:=0;
 SetSub:=SetSub2;
 count+:=1;
 SetSub2:={@@};
 for x in SetSub do //begin breaking down the groups in SetSub
  countt+:=1;
  Sub1093:=sub<Q|x,x1093>;
  Fx1093:=FrattiniSubgroup(x);
  MNt5aa, phit5aa:=GModule(Sub1093,x,Fx1093); //create the <x>-module of the Frattini quotient
  if Order(ActionGroup(MNt5aa)) ne 1093 then //if <x> doesn't act faithfully on the action group
   Include(~ActnGpDiff,x);//save for later, we must search inside the Frattini in these cases
  else
   Com:=DirectSumDecomposition(MNt5aa); //find composition factors of Frattini quotient
   Dim:=[Dimension(Com[i]): i in [1..#Com]];
   CheckSet:= {@ 1 @};
   ModSet:= {@ Com[1] @};
   for i in [2..#Com] do
    check:=0;
    for j in CheckSet do
     if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
    end for;
    if check eq 0 then
     Include(~CheckSet,i); Include(~ModSet,Com[i]);
    end if;
   end for;
```

```
 if Order(Fx1093) eq 1 then //Frattini is trivial, we start to find our elementary abelian groups
  for m in ModSet do
   if Dimension(m) eq 7 then
    GenSet:={@@};
    for n in Com do
     if IsIsomorphic(n,m) then
      Include(~GenSet,n); //identify which module we need to take pre-image of
     end if;
    end for;
    IncMod:= sub<MNt5aa|GenSet>;
    IncGrp:= IncMod@@phit5aa; //pre-image of module, elementary abelian
    Include(~FinSub,IncGrp);
   end if;
  end for;
  else //Frattini not trivial
   if #ModSet eq 1 then
    if Dimension(Com[1]) eq 7 then //only 1 7-dim module in decomposition, so add to BadSub
     Include(~BadSub,x);
    end if;
   else
    for m in ModSet do
     if Dimension(m) eq 7 then
      GenSet:={@@};
      for n in Com do
       if IsIsomorphic(n,m) then
        Include(~GenSet,n); //collect the blocks of isomorphic 7-dim modules
       end if;
      end for;
      IncMod:= sub<MNt5aa|GenSet>;
      IncGrp:= IncMod@@phit5aa; //pre-image of block
      Include(~SetSub2,IncGrp); //add to SetSub2 for further consideration
     end if;
    end for;
   end if;
  end if;

   count,#SetSub,countt,Dim,#ModSet,#ActnGpDiff,"FinSub",#FinSub,\
   "BadSub",#BadSub,#SetSub2,#RESULTS;

  end if;
 end for;
until #SetSub2 eq 0; //repeat until we have broken down all the blocks of isomorphic modules
Append(~RESULTS, [*#FinSub,#BadSub,#ActnGpDiff*]);
Append(~GROUPS,BadSub);
```

9.2.4 Dealing with BadSub, $L_2(2187)$

```
//We now deal with the BadSub formed in stage 1. This is where the bulk of
the work is done in these constructions - running stage 1 can take only
15 minutes whilst breaking down BadSub could take months. We remark that
the below block of code was very rarely ran as a whole. The vast majority
of the time, smaller parts of the code where ran individually and then the
situation was re-assessed.

//As remarked in Section 3, for L_2(2187) we reduce the number of generators
for various groups we calculate here to speed up computation time. In
particular, Sub4aa has over 1000 generators which can be significantly
reduced. This process is not necessary for L_2(81) and L_2(27). The code
used to deal with BadSubs associated with these 2 groups can be found
later in the section titled Dealing with Badsub, L_2(27) & L_2(81).


BadSetNew:=BadSub;
  loopn:=0;
  bool:={@@};
  bool2:={@@};
  SetKeepZero:={@@};
  repeat
   BadSub:=BadSetNew; BadSetNew:={@@}; //reset BadSetNew and BadSub
   for k in [1..#BadSub] do
    b:=BadSub[k];
    Fb:=FrattiniSubgroup(b);
    Pb,pmap:=PCGroup(b);
    PFb:=pmap(Fb);

    "Reducing frattini gens";
    bla,bla2,exp1 := IsPrimePower(Order(PFb));
bla,bla2,exp2 := IsPrimePower(Order(FrattiniSubgroup(PFb)));
repeat
gens := {@@};
repeat
el := Random(PFb); Include(~gens, el);
until #gens eq exp1-exp2;
until Order(sub<PFb|gens>) eq Order(PFb);
PFb := sub<PFb|gens>;
Fbgens := {@ i@@pmap : i in gens@};
Fb := sub<Fb|Fbgens>;
"Frattini gens reduced successfully";

    C:=CommutatorSubgroup(Pb,PFb);
    QPFb,qPFb:=quo<PFb|C>;
```

```
    FQPFb:=FrattiniSubgroup(QPFb);
    A:=(FQPFb@@qPFb)@@pmap;
    orderA := LMGOrder(A);
    MNt,phit:= GModule(sub<Q|x1093,b>,b,Fb); //Frattini quotient as <x>-module
    actMNtstar:={@@};
    for g in ActionGroup(MNt) do
     if Order(g) ne 1 then
      Include(~actMNtstar,g);
     end if;
    end for;
    Include(~bool, forall{g : g in actMNtstar | \
    Dimension(Eigenspace(g,1)) eq 0});
    Com:= DirectSumDecomposition(MNt);
    IsLarge:=[Dimension(Com[i]): i in [1..#Com]];

//We now wish to start finding a non-zero vector from every irreducible <x>-submodule.

    SetKeep:= {@@};
    for i in [1..#Com-1] do
     repeat
      xm:= Random(Com[i]);
     until xm ne Zero(Com[i]);
    x:= xm@@phit;
    setym:={@@};
    for j in [i+1..#Com] do
     Include(~setym,Com[j]);
    end for;
    YM:= sub<MNt|setym>;
    countym:=0;
    for ym in YM do
     countym:= countym+1;
     y:= ym@@phit;
     t:= x*y;
     if t*t*t in A then //we only keep those that cube into A
      Include(~SetKeep,t);
     end if;
    end for;
   end for;

   repeat
    x:= Random(Com[#Com]); //we now only have the last module to consider
   until x ne Zero(Com[#Com]);
   t:=x@@phit;
   if t*t*t in A then
    Include(~SetKeep,t);
   end if;
```

```
Include(~bool2, #SetKeep ne 0);
if #SetKeep eq 0 then
 Include(~SetKeepZero,b);
end if;

//Having now collected all our vectors, we need to calculate the corresponding
groups H and start breaking them down.

for r in [1..#SetKeep] do
 start := Realtime(); //timer to track progress
 x:=Q!SetKeep[r]; //our non-zero vector
 set1093:={@@};
 for i in [1..1093] do
  Include(~set1093,x^(x1093^i)); //the action of x1093 on our vector, so t^<x>
 end for;
 Sub4aa:=sub<Q|Fb,set1093>; //the group labelled H in Section 3.1

 //we now reduce the number of generators of H. We remark that the values 42, 3^42 and 28
 are specific to the situation when Phi = {3,4,5,6,7,8} and may need to be modified if
 considering a different situation.

 "About to reduce H gens";
 repeat
  Hgens := {@Q!i : i in Generators(Fb)@};
  Include(~Hgens,Q!x);
  repeat Include(~Hgens, Random(Sub4aa)); until #Hgens eq 42;
  H := sub<Q|Hgens>;
  ord := Order(H);
 until ord eq 3^42;
 "H gens reduced to 42, now finding PC group of H to reduce further";
 PH,phmap := PCGroup(H);
 repeat
  gens := {@@};
  repeat el := Random(PH); Include(~gens, el); until #gens eq 28;
  ord := Order(sub<PH|gens>);
 until ord eq 3^42;
 Hgens := {@i@@phmap : i in gens@};
 Sub4aa := sub<H|Hgens>; //form new H with less generators
 Sub1093:=sub<Q|Fb,x,x1093,Hgens,A>; //form the overgroup, ensuring Magma knows H is a subgroup

 "Finding quotient module";
 MNt4aa,phit4aa:=GModule(Sub1093,Sub4aa,A);//find Frattini quotient as <x>-module
 "Found module";
 ends := Realtime();//end timer
 "time elapsed : ", ends-start;
```

```
 //we now form a new SetSub2 from the blocks of isomorphic 7-dim irreducible submodules.
 This is done in the same way as in stage 1.

 Com:=DirectSumDecomposition(MNt4aa);
 Dim:=[Dimension(Com[i]): i in [1..#Com]];
 CheckSet:= {@ 1 @};
 ModSet:= {@ Com[1] @};
 for i in [2..#Com] do
  check:=0;
  for j in CheckSet do
   if IsIsomorphic(Com[i],Com[j]) then
    check:=1;
   end if;
  end for;
 if check eq 0 then
  Include(~CheckSet,i);
  Include(~ModSet,Com[i]);
 end if;
end for;
if Order(A) eq 1 then
 for m in ModSet do
  if Dimension(m) eq 7 then
   GenSet:={@@};
   for n in Com do
    if IsIsomorphic(n,m) then
     Include(~GenSet,n);
    end if;
   end for;
   IncMod:= sub<MNt4aa|GenSet>;
   IncGrp:= IncMod@@phit4aa;
   Include(~FinSub,IncGrp);
  end if;
 end for;
else
 if #ModSet eq 1 then
  if Dimension(Com[1]) eq 7 then
   Include(~SetSub2,Sub4aa);
  end if;
 else
  for m in ModSet do
   if Dimension(m) eq 7 then
    GenSet:={@@};
    for n in Com do
     if IsIsomorphic(n,m) then
      Include(~GenSet,n);
     end if;
```

```
    end for;
    IncMod:= sub<MNt4aa|GenSet>;
    IncGrp:= IncMod@@phit4aa;
    Include(~SetSub2,IncGrp);
   end if;
  end for;
 end if;
end if;
IsLarge,"loopn",loopn,#BadSub,"k",k,bool,bool2,#SetKeep,r,Dim,#ModSet,\
"FinSub",#FinSub,"BadSetNew",#BadSetNew,#SetSub2;
end for;

//we have now got a SetSub2 which needs breaking down. This is done
in exactly the same way as in stage 1, apart from any groups we previously added
to BadSub are now added to BadSetNew.

count:=0;
repeat
 countt:=0;
 SetSub:=SetSub2;
 count+:=1;
 SetSub2:={@@};
 for x in SetSub do
 countt+:=1;
  Sub1093:=sub<Q|x,x1093>;
  Fx1093:=FrattiniSubgroup(x);
  MNt5aa,phit5aa:=GModule(Sub1093,x,Fx1093);
  if Order(ActionGroup(MNt5aa)) ne 1093 then
   Include(~ActnGpDiff,x);
  else
   Com:=DirectSumDecomposition(MNt5aa);
   Dim:=[Dimension(Com[i]): i in [1..#Com]];
   CheckSet:= {@ 1 @};
   ModSet:= {@ Com[1] @};
   for i in [2..#Com] do
    check:=0;
    for j in CheckSet do
     if IsIsomorphic(Com[i],Com[j]) then
      check:=1;
     end if;
    end for;
    if check eq 0 then
     Include(~CheckSet,i);
     Include(~ModSet,Com[i]);
    end if;
   end for;
```

```
     if Order(Fx1093) eq 1 then
      for m in ModSet do
       if Dimension(m) eq 7 then
        GenSet:={@@};
        for n in Com do
         if IsIsomorphic(n,m) then
          Include(~GenSet,n);
         end if;
        end for;
        IncMod:= sub<MNt5aa|GenSet>;
        IncGrp:= IncMod@@phit5aa;
        Include(~FinSub,IncGrp);
       end if;
      end for;
     else
      if #ModSet eq 1 then
       if Dimension(Com[1]) eq 7 then
        Include(~BadSetNew,x);
       end if;
      else
       for m in ModSet do
        if Dimension(m) eq 7 then
         GenSet:={@@};
         for n in Com do
          if IsIsomorphic(n,m) then
           Include(~GenSet,n);
          end if;
         end for;
         IncMod:= sub<MNt5aa|GenSet>;
         IncGrp:= IncMod@@phit5aa;
         Include(~SetSub2,IncGrp);
        end if;
       end for;
      end if;
     end if;
     "loopn",loopn,count,#SetSub,countt,Dim,#ModSet,#ActnGpDiff,\
     "FinSub",#FinSub,"BadSetNew",#BadSetNew,#SetSub2;
    end if;
   end for;
 until #SetSub2 eq 0; //repeat until we have broken down all the groups in SetSub
 end for;
 loopn+:=1;
 Append(~RESULTS, <loopn,#BadSetNew,#FinSub,#ActnGpDiff,\
  bool eq {@true@},bool2 eq {@true@},#SetKeepZero>);
 Append(~GROUPS,BadSetNew);
until #BadSetNew eq 0; //repeat until nothing more is added to BadSetNew, so we are done
```

```
BadSub:={@@};
Append(~GROUPS,FinSub);
Append(~GROUPS,ActnGpDiff);
Append(~GROUPS,SetKeepZero);
```

9.2.5 Checking BadSub for Conjugates

```
//The below code checks for conjugacy inside the centraliser C_P(x13) between groups
in a given BadSetNew. The centraliser is constructed using makesub, a function defined
later on in this section.

//Returned is 3 sets, keep, rels and no. The set keep contains the indexes of groups in
BadSetNew that we wish to keep, these groups act as our conjugacy representatives
for the whole of BadSetNew. So every group in BadSetNew is conjugate to a group in
BadSetNew with index in keep. The set no contains all groups we don't consider further
as they are conjugate to some group with index in keep. The set rels keeps track of the
conjugacy relations between groups with indexes in keep and no. We remark that saving
the indexes of the groups instead of the groups themselves is preferred as adding large
groups to sets can take some time and also uses more memory.

//Typically after running this code, the last group in BadSetNew will have not been considered.
So we add the index of this group into keep manually. Then to create our new, smaller BadSetNew,
we run the command: BadSetNew := {@BadSetNew[i] : i in keep@};

//We remark that this code can be ran multiple times and BadSetNew could be reduced each
time due to the randomness of choosing the element from the centraliser. When
it was used, I would run it until no conjugacy relations where found, which was typically after
2 or 3 iterations.

Cpx := LMGCentraliser(makesub({1,3,6,7}),x13); //construct centraliser
inds := [1..#BadSetNew];
keep := {@@}; //the groups we wish to keep, our representatives
rels := {@@}; //to keep track of our conjugacy relations
no := {@@};//the groups we don't need to consider
for i in [1..#inds-1] do
 if i in keep or i in no then continue; end if;
 cnt:=0;
 check := 0;
 for tick in [1..45] do //loop through 45 random elements of the centraliser
  el := Random(Cpx);
  cnt:=cnt+1;
  for j in [i+1..#inds] do
   if j in no then continue; end if;
   [i,cnt,j,999,#rels];
   if BadSetNew[i] eq BadSetNew[j]^el then //if we have equality, we have found conjugate groups
    Include(~keep,i); Include(~rels, {@i,j@}); Include(~no,j); check:=1; break;
```

```
   end if;
  end for;
  if check eq 1 then continue; end if;
 end for;
 Include(~keep,i);
end for;
```

9.2.6 Dealing with BadSub, $L_2(27)$ & $L_2(81)$

The code below was used extensively when dealing with sets BadSub associated to both groups $L_2(27)$ and $L_2(81)$. This follows the same algorithm as in Section 9.2.4; much of the code is identical. Small modifications have been made which relate to what is described in Procedure 3.11. The code below is specific to $L_2(27)$ where it was mainly used, however this can be easily changed to work with $L_2(81)$.

```
//The following code is used to deal with sets BadSub where we wish
to define Fb to be a preimage of a set number of irreducible modules
instead of leaving it to be defined as the Frattini subgroup of the
3-group we are considering. In particular, stage 1 of the algorithm
has already been executed here; dealing with the outputted BadSub is
the task of the code below.

//Moreover, this code is specific to L_2(27), thus meaning we are
searching for elementary abelian subgroups of order 3^3 that along
with an element of order 13, represented below by x13, generate a subgroup
of a potential L_2(27) inside E_8(3). As with L_2(2187), the 2 representatives
for x13 are saved and must be loaded in before using this code.

//Before starting the process of breaking down BadSub, which we do in the same
way as before, we now define an integer denoted CN. This was always
either 2 or 3. This corresponds to the number of irreducible that we take pre-images
of to define Fb. Note that if b is the group we are considering, and say b
is isomorphic to the direct sum of 6 irreducible isomorphic 3-dimensional modules,
then setting CN:=2 means we define Fb to be the pre-image of the direct
sum of the first 4 = 6-2 modules. If CN:=3, then Fb would be the pre-image
of the first 3 = 6-3 modules.

//We remark that if CN=#Com, then we would be taking the pre-image of 0 modules.
If this is the case, we are typically near the end of the process, and so
the code will proceed as before using b/Fb.

//After generating SetKeep, which is done exactly as before, the algorithm proceeds
as normal. The only difference here is that there are constant checks on the dimension
of the fixed space of <b,x13> where b is any 3-group we are considering in the process.
If this is greater than 1, the code skips over this case and works with the next group.
In particular, this is done with the group Sub4aa. This check is also conducted on any
3-group which could be added to either FinSub, BadSetNew, ActnGpDiff or SetSub2.
```

As we will be dealing with a lot of groups, we want to get rid of any we do not need as soon as we can.

```
CN := 3; //this will give a SetKeep of size 757 for all groups considered
in the L_2(27) case
CN := 2; //this will give a SetKeep of size 28 for all groups considered
in the L_2(27) case

repeat
BadSub := BadSetNew; BadSetNew := {@@};
Append(~sizes, #BadSub);
for bb in [1..#BadSub] do
b := sub<Q|BadSub[bb]>;


"initial calculations for b";
Fb:=FrattiniSubgroup(b);
Pb,pmap:=PCGroup(b);
PFb:=pmap(Fb);

C:=CommutatorSubgroup(Pb,PFb);
QPFb,qPFb:=quo<PFb|C>;
FQPFb:=FrattiniSubgroup(QPFb);
A:=(FQPFb@@qPFb)@@pmap;
MNt,phit:= GModule(sub<Q|x13,b,Fb>,b,Fb);
actMNtstar:={@@};
for g in ActionGroup(MNt) do
 if Order(g) ne 1 then
  Include(~actMNtstar,g);
 end if;
end for;
Include(~bool, forall{g : g in actMNtstar | Dimension(Eigenspace(g,1)) eq 0});
Com:= DirectSumDecomposition(MNt);
IsLarge:=[Dimension(Com[i]): i in [1..#Com]];
"complete";

gens := [Com[i] : i in [1..#Com-CN]]; //use comnumber to take appropriate
number of modules
if #gens gt 0 then
F := sub<MNt | gens>;  // generate submodule according to however many modules
we have taken
Fb := F@@phit; //take pre-image
PFb:=pmap(Fb); //redefine Fb
```

```
 C:=CommutatorSubgroup(Pb,PFb);
 QPFb,qPFb:=quo<PFb|C>;
 FQPFb:=FrattiniSubgroup(QPFb);
 A:=(FQPFb@@qPFb)@@pmap;
 MNt,phit:= GModule(sub<Q|x13,b,Fb>,b,Fb);
 actMNtstar:={@@};
 for g in ActionGroup(MNt) do
  if Order(g) ne 1 then
   Include(~actMNtstar,g);
  end if;
 end for;
 Include(~bool, forall{g : g in actMNtstar | Dimension(Eigenspace(g,1)) eq 0});
 Com:= DirectSumDecomposition(MNt);
 IsLarge:=[Dimension(Com[i]): i in [1..#Com]];
 end if;
 SetKeep:= {@@};
 for i in [1..#Com-1] do
  repeat
   xm:= Random(Com[i]);
  until xm ne Zero(Com[i]);
  x:= xm@@phit;
  setym:={@@};
  for j in [i+1..#Com] do
   Include(~setym,Com[j]);
  end for;
  YM:= sub<MNt|setym>;
  countym:=0;
  for ym in YM do
   countym:= countym+1;
   y:= ym@@phit;
   t:= x*y;
   if t*t*t in A then
    Include(~SetKeep,t);
    nosetkeep := #SetKeep;
    if nosetkeep mod 1000 eq 0 then nosetkeep; end if;
   end if;
  end for;
 end for;

repeat
 x:= Random(Com[#Com]);
until x ne Zero(Com[#Com]);
t:=x@@phit;
if t*t*t in A then
```

```
  Include(~SetKeep,t);
 end if;
"for badsub ", k, " we have SetKeep = ",#SetKeep;
 Include(~bool2, #SetKeep ne 0);
 if #SetKeep eq 0 then
  Include(~SetKeepZero,b);
 end if;



 for r in [1..#SetKeep] do
  if r mod 100 eq 0 then r; end if;
  start := Realtime();
  x:=Q!SetKeep[r];
  set13:={@@};
  for i in [1..13] do
   Include(~set13,x^(x13^i));
  end for;
  Sub4aa:=sub<Q|Fb,set13>;
  if dims(sub<Q|Sub4aa,x13>) gt 1 then "skipping this H as dims"; continue; end if;
  Sub13:=sub<Q|Fb,x,x13,Generators(Sub4aa),A>;
  "Finding quotient module";
  MNt4aa,phit4aa:=GModule(Sub13,Sub4aa,A);
  "Found module";
  ends := Realtime();
  "time elapsed : ", ends-start;

  Com:=DirectSumDecomposition(MNt4aa);
  Dim:=[Dimension(Com[i]): i in [1..#Com]];
  CheckSet:= {@ 1 @};
  ModSet:= {@ Com[1] @};
  for i in [2..#Com] do
   check:=0;
   for j in CheckSet do
    if IsIsomorphic(Com[i],Com[j]) then
     check:=1;
    end if;
   end for;
  if check eq 0 then
   Include(~CheckSet,i);
   Include(~ModSet,Com[i]);
  end if;
 end for;
 if Order(A) eq 1 then
  for m in ModSet do
   if Dimension(m) eq 3 then
```

```
   GenSet:={@@};
   for n in Com do
    if IsIsomorphic(n,m) then
     Include(~GenSet,n);
    end if;
   end for;
   IncMod:= sub<MNt4aa|GenSet>;
   IncGrp:= IncMod@@phit4aa;
   if dims(sub<Q|x13,IncGrp>) gt 1 then "no need for this one"; continue; end if;
   Include(~FinSub,IncGrp);
  end if;
 end for;
else
 if #ModSet eq 1 then
  if Dimension(Com[1]) eq 3 then
   Include(~SetSub2,{@Q!kk : kk in Generators(Sub4aa)@});
  end if;
 else
  for m in ModSet do
   if Dimension(m) eq 3 then
    GenSet:={@@};
    for n in Com do
     if IsIsomorphic(n,m) then
      Include(~GenSet,n);
     end if;
    end for;
    IncMod:= sub<MNt4aa|GenSet>;
    IncGrp:= IncMod@@phit4aa;
    if dims(sub<Q|x13,IncGrp>) gt 1 then "no need for this one"; continue; end if;
    Include(~SetSub2,{@Q!kk : kk in Generators(IncGrp)@});
   end if;
  end for;
 end if;
end if;
"-----------------------------------------------------------";
IsLarge;
"we are on badsub = ", bb, " out of ",#BadSub;
bool,bool2;
"Setkeep = ",#SetKeep," and we are on ",r;
Dim,#ModSet;
"FinSub = ",#FinSub;
"BadSetNew = ",#BadSetNew;
"SetSub2 = ",#SetSub2;
"ActnGpDiff = ",#ActnGpDiff;
"SetKeepZero = ",#SetKeepZero;
sizes;
```

```
"------------------------------------------------------------";
end for;

count:=0;
repeat
 SetSub:=SetSub2;
 count+:=1;
 SetSub2:={@@};
 cnt:=0;
 for xxx in SetSub do
 x := sub<Q|xxx>; cnt := cnt+1;
 start := Realtime();
  Fx13:=FrattiniSubgroup(x);
  Sub13:=sub<Q|x,x13,Fx13>;
  if dims(Sub13) gt 1 then "skipping as dims"; continue; end if;
  MNt5aa,phit5aa:=GModule(Sub13,x,Fx13);
  if Order(ActionGroup(MNt5aa)) ne 13 then
   Include(~ActnGpDiff,x);
  else
   Com:=DirectSumDecomposition(MNt5aa);
   Dim:=[Dimension(Com[i]): i in [1..#Com]];
   CheckSet:= {@ 1 @};
   ModSet:= {@ Com[1] @};
   for i in [2..#Com] do
    check:=0;
    for j in CheckSet do
     if IsIsomorphic(Com[i],Com[j]) then
      check:=1;
     end if;
    end for;
    if check eq 0 then
     Include(~CheckSet,i);
     Include(~ModSet,Com[i]);
    end if;
   end for;
  if Order(Fx13) eq 1 then
   for m in ModSet do
    if Dimension(m) eq 3 then
     GenSet:={@@};
     for n in Com do
      if IsIsomorphic(n,m) then
       Include(~GenSet,n);
      end if;
     end for;
     IncMod:= sub<MNt5aa|GenSet>;
     IncGrp:= IncMod@@phit5aa;
```

```
       if dims(sub<Q|x13,IncGrp>) gt 1 then "no need for this one"; continue; end if;
       Include(~FinSub,IncGrp);
      end if;
     end for;
    else
     if #ModSet eq 1 then
      if Dimension(Com[1]) eq 3 then
       Include(~BadSetNew,{@Q!kk : kk in Generators(x)@});
      end if;
     else
      for m in ModSet do
       if Dimension(m) eq 3 then
        GenSet:={@@};
        for n in Com do
         if IsIsomorphic(n,m) then
          Include(~GenSet,n);
         end if;
        end for;
        IncMod:= sub<MNt5aa|GenSet>;
        IncGrp:= IncMod@@phit5aa;
        if dims(sub<Q|x13,IncGrp>) gt 1 then "no need for this one"; continue; end if;
        Include(~SetSub2,{@Q!kk : kk in Generators(IncGrp)@});
       end if;
      end for;
     end if;
    end if;
    ends := Realtime();
    "-----------------------------------------------------------";
    "time elapsed this iteration = ", ends-start;
    "we are on badsub = ", bb, " out of ",#BadSub;
    "SetSub = ",#SetSub, "and we are on = ", cnt;
    Dim,#ModSet;
    "ActnGpDiff = ",#ActnGpDiff;
    "FinSub = ",#FinSub;
    "BadSetNew = ",#BadSetNew;
    "SetSub2 = ",#SetSub2;
    "SetKeepZero = ",#SetKeepZero;
    sizes;
    "------------------------------------------------------------";
   end if;
  end for;
until #SetSub2 eq 0;
end for;
until #BadSetNew eq 0;
Append(~sizes, #FinSub);
Append(~sizes, #ActnGpDiff);
```

```
Append(~sizes, #SetKeepZero);sizes;
```

9.2.7 Dealing with FinSub

Given a non-empty set FinSub, the following code uses the algorithm described in Section 3.2.1 to search for all elementary abelian subgroups that are acted on irreducibly by the element x. The code is designed for $L_2(2187)$, but can be easily adapted for $L_2(81)$, $L_2(27)$ and $L_2(8)$ in Section 6. We remark that Lemma 3.7 is used here to discard any elementary abelian group which will lead to non-maximal subgroups.

```
Elems := {@@}; //suitable elementary abelian groups will be stored here
I := sub<Q|Id(Q)>;
sum := 0;
for i in [1..#FinSub] do
 f := sub<Q|FinSub[i]>;
 T,map := GModule(sub<Q|f,x1093>,f,I);
 M := MinimalSubmodules(T);
 sum := sum+#M; "we are on ", i, "and number of min submods so far is = ",sum;
 for j in M do
  m := j@@map;
  t := sub<Q|m,x1093>;
  if dims(t) lt 1 then //if group does not fix a vector, steinberg trick
   Include(~Elems,f);
   "ELEMS = ", #Elems;
  end if;
 end for;
end for; "elems found = ", #Elems;
```

9.2.8 Constructing Copies of $L_2(81)$

The code given here corresponds to the work in at the end of Section 3.3.2 where overgroups are constructed for each of the 320 copies of $L_2(81)$.

```
//Several variables must be initialised before using the code below. We set x40
to be the element x of order 40 and H to be the set containing the associated 80
groups isomorphic to L_2(81) which were previously constructed.

//Also required is the function makesub which is defined earlier in this section.

C := LMGCentraliser(makesub({1,2,3,4,6,7,8,97}),x40);

PGLs := {@@}; //we will store the groups isomorphic to PGL_2(81) here
for h in H do \\loop through our copies of L_2(81)
repeat
repeat //repeatedly generate random elements of C until we find a potential PGL_2(81)
overgroup for h
o := Random(C); O := sub<Q|h,Q!o>; M := GModule(O); Cfs := CompositionFactors(M);
until #Cfs eq 9; //will have the same number of composition factors as h
```

```
"found potential PGL";
until IsIsomorphic(O,PGL(2,81)); //if it is isomorphic to PGL_2(81) add it to our set
Include(~PGLs, O); "success, so far we have found : ",#PGLs;
end for;

t := x40^20; //involution
G := makesub({1,2,3,4,5,6,7,8}); //E_8(3)
Ct := CentraliserOfInvolution(G,t);
Ct := sub<Q|Ct,CentraliserOfInvolution(G,t)>;
Ct := sub<Q|Ct,CentraliserOfInvolution(G,t)>;
Ct := sub<Q|Ct,CentraliserOfInvolution(G,t)>; //use Bray method several times to
ensure we get the full centraliser

time N := LMGNormaliser(Ct,sub<Q|x40>); //around 11 hours

//Having now obtained N, we shall search for elements which extend our PGLs to
the full automorphism group.

overs := []; //we shall store the elements which extend the PGLs here
cnt := 0;
for i in PGLs do
cnt := cnt+1;
repeat
el := Random(N); //random element of the normaliser
o:=1;
b := sub<Q|el,i>;
n:=#CompositionFactors(GModule(b));
if n eq 9 then //testing showed that the automorphism group would probably have
the same number of composition factors as H
 o := Order(b); //only calculate the order if the number of composition factors
                        is sufficiently large
end if;
until o eq 2125440; //if the order is equal to the order of the automorphism group, add
the element to our set
Append(~overs, el);
cnt,#overs;
end for;

//We now found overgroups for each automorphism group we have constructed.

overs2 := []; //we store the elements which generate the overgroups here
for i in [1..80] do
A := sub<Q|overs[i],PGLs[i]>; //the automorphism group
repeat
el := Random(N);
test := sub<Q|A,el>;
```

```
n := #CompositionFactors(GModule(test));
n;
until n in [2..8]; //repeat generating random elements until we have an overgroup
of A which is not all of E_8(3)
Append(~overs2,el);
i,#overs2;
end for;
```

9.2.9 FindCent

The following code was used to construct centralisers (or parts of) to aid the work in conducted in this report. This is a modified version of the full `FindCent` function which can be found in [62].

```
//Here G is the group you wish the centraliser to be constructed inside, g
is the element you are constructing the centraliser for and Q is a general linear group
of dimension n over the field GF(q). In our case, we always have n=248, q=3.
The parameter k is a small positive integer that must be as large as the dimension of
the smallest non-trivial irreducible <g>-module over GF(3). We find a value for k before
starting the procedure of constructing the centraliser. The variable factors
is also a positive integer and should be set by the user before running the code.
This should be chosen with some care, as if this is set too low the code
will likely never stop running. Typically, when I used this code I would initially set
factors to be around 40 which would usually return only a subgroup of the centraliser.
Then, by checking the value of #CFVCg, I could decide what to set factors as for the next
run through of the code. Usually, if I still required to keep constructing the centraliser
I would set factors to be #CFVCg-1 for the next running of the code.
Typically, larger centralisers have a smaller value of #CFVCg - this could end up
being less than 10. The code stops running when #CFVCg is less than or
equal to factors and outputs the centraliser (or part of) as a subgroup of Q.

//We remark that the centraliser under construction must be of a reasonable size
compared to the size of the group G. If the centraliser is too small, this will likely
not work.

 Q := GL(n,q);
 Cg:=sub<Q|Id(Q)>;
 count:=0;
 M := GModule(sub<Q|g>); D := DirectSumDecomposition(M); k := Max({Dimension(i) : i in D}); k;

 factors := 40;
 repeat
  repeat
   t:=Element(G,2);
   Y:=sub<Q|t,g>;
   U:=GModule(Y);
   CF:=CompositionFactors(U);
```

```
 until #CF ge 5;
 print "Found a suitable involution t";
 for i:=1 to 5 do
  flag:=0;
  counter:=0;
  repeat
   a := Element(Y,2);
    L:=sub<Q|a,g>;
    W:=GModule(L);
    CFW:=CompositionFactors(W);
    dims:={};
    for c in CFW do
      Include(~dims,Dimension(c));
    end for;
    counter:=counter+1;
    if Max(dims) le k then
      flag:=1;
    end if;
  until Max(dims) le k or counter eq 20;
  if flag eq 1 then
    l:=LMGOrder(L);
    Factorization(l);
     if l le 2^25 then
      CL:=Centralizer(L,g);
      Cg:=sub<Q|Cg,CL>;
     else
      print "Couldn't find small enough subgroup to use, trying again";
   end if;
  end if;
 end for;
 VCg:=GModule(Cg);
 count:=count+1;
 count;
 CFVCg:=CompositionFactors(VCg);
 dimsVCg:={};
 for c in CFVCg do
   Include(~dimsVCg,Dimension(c));
 end for;
 if #CFVCg le factors then
   print "Composition factors of GModule of centralizer subgroup are";
   CFVCg;
  else
   print "GModule of centralizer subgroup has more than 30 composition factors";
 end if;
until #CFVCg le factors;
```

9.2.10 Constructing Copies of $L_3(5)$

In this section we give all code used to construct the initial 30 copies of $L_3(5)$ inside $E_8(3)$.

```
//Prior to using the code in this section, we load in several groups which
are specified below:

S : Sylow 5 subgroup of E_8(3)
cubes : Our 25 subgroups isomorphic to P = 5^{1+2}, constructed in Section L_3(5)
C : Centraliser of 5B element inside E_8(3) (called C_G(z) in methodology)
E : Unique normal elementary abelian subgroup of order 5^4 in S
NE : Normaliser of E in G, as described in methodology
ZC : Center of C, labelled <z> in methodology
Q : GL(248,3)

//The following code was used for ALL P in the set cubes. Here we set P to
be the first group in the set. Firstly, we gather the 6 elementary abelian
subgroups of order 5^2 in P in a set called subs. Following this, we gather
the 15 possible pairs of these 6 subgroups in a set called pairs.

P := sub<Q|cubes[1]>;
NN := NormalSubgroups(P : OrderEqual := 25);
subs := {@i`subgroup : i in NN@};
N := LMGNormaliser(S,P);

pairs := {@@};
for i in subs do
for j in subs do
if i ne j then Include(~pairs, {@i,j@}); end if;
end for;
end for;

//We now check the conjugacy of these pairs inside the normaliser N_S(P) = N

cnt := 0;
rels := {@@};
keep := {@@};
no := {@@};
 for i in [1..15] do
  for j in [1..15] do
   if i ne j and i notin no and j notin no then
    cnt := 0;
    for el in N do
    cnt := cnt+1;
    p := pairs[i]; q := pairs[j];
    if p[1]^el eq q[1] and p[2]^el eq q[2] or p[1]^el eq q[2] and p[2]^el eq q[1] then
     Include(~rels, {@i,j@}); Include(~keep,i); Include(~no,j); break;
```

```
    end if;
    i, " checking ",j," we are on element: ",cnt, " / 625 and we got rid of: ", no;
    end for;
   end if;
  end for;
 end for;
pairs := {@pairs[i] : i in keep@};

//Now the set pairs contains our 3 pairs that we need to consider.
Before further calculations, we label the groups in these pairs as we did
in the methodology.

X := pairs[1] meet pairs[2];
X := X meet pairs[3];
A := pairs[1] diff X; A := A[1];
B := pairs[2] diff X; B := B[1];
D := pairs[3] diff X; D := D[1];
X := X[1];

//We remark that C has been used to label C_G(z), whereas in the write up
we used C to denote one of the elementary abelian groups. Hence in this code our
pairs have structure {X,A}, {X,B} and {X,D}.

//We now wish to find which of these groups contains elements that
are NE-conjugate to our central element z. The code below is set to only
consider the group A, however this can be easily altered to consider the
groups X, B and D.

cnt := 0;
for i in A do
 if i notin sub<Q|z> then
  cnt := cnt+1; kk,ell := LMGIsConjugate(NE,z,i);
  if kk eq true then y := i; "FOUND"; this := A; break; end if;
  " we are on ", cnt, " and ",kk;
 end if;
end for;

//Assuming that the group A does contain these conjugate elements
(indeed it actually did for 24 of the 25 cases) we then run the code below
to find the second centraliser Cy (labelled C_G(x_2) before) and then
the group <N_1, N_2>

Cy := C^ell;
N1 := LMGNormaliser(C,this); N2 := LMGNormaliser(Cy,this);
ONA := sub<Q|N1,N2>;
```

```
//To find <N_1, N_2> for the remaining groups X, B and D, the following
function is used. Comments on the code are given throughout to explain
how it works.

ONmaker := function(K);

repeat y := Element(K,5); until y notin sub<Q|z>; //firstly find the
element labelled x_2

//The next step is to find an involution inside C=C_G(z) that centralises y.
Constructing C_C(y) is possible however time consuming, instead we use the
first part of the FindCent code to construct a very small part of this centraliser.
This terminates when the centraliser has order >= 20.

G := C;
g := y;
k:=4;
factors:=40;
 Cg:=sub<Q|Id(Q)>;
 count:=0;

 repeat
  repeat
   t:=Element(G,2);
   Y:=sub<Q|t,g>;
   U:=GModule(Y);
   CF:=CompositionFactors(U);
  until #CF ge 5;
  print "Found a suitable involution t";
  for i:=1 to 5 do
   flag:=0;
   counter:=0;
   repeat
    a := Element(Y,2);
     L:=sub<Q|a,g>;
     W:=GModule(L);
     CFW:=CompositionFactors(W);
     dims:={};
     for c in CFW do
       Include(~dims,Dimension(c));
     end for;
     counter:=counter+1;
     if Max(dims) le k then
       flag:=1;
     end if;
   until Max(dims) le k or counter eq 20;
```

```
   if flag eq 1 then
     l:=LMGOrder(L);
     Factorization(l);
      if l le 2^30 then
       CL:=Centralizer(L,g);
       Cg:=sub<Q|Cg,CL>;
      else
       print "Couldn't find small enough subgroup to use, trying again";
    end if;
   end if;
  end for;
 until LMGOrder(Cg) ge 20;

//having now obtained a small piece of the centraliser C_C(y), we
choose t1 to be a centralising involution of y from here

t1 := Element(Cg,2);

//We now use the Bray method to find C_G(t1) and then we find the
centraliser of y inside this.

Ct1 := CentraliserOfInvolution(makesub({1,2,3,4,5,6,7,8}),t1);
Cy1 := LMGCentraliser(Ct1,y);

//We now wish to find a centralising involution of y, labelled t2,
such that C_G(t1) and C_G(t_2) are distinct. Following this, we
find the centraliser of y inside C_G(t_2) and then we can generate
all of Cy = C_G(y) using the 2 centralisers we have made.

repeat t2 := Element(Cy1,2); check := 0;
 for i in [1..10] do
  el := Random(Ct1);
  if t2^el ne t2 then check:=1; break; end if;
 end for;
until check eq 1;

Ct2 := CentraliserOfInvolution(makesub({1,2,3,4,5,6,7,8}),t2);
Cy2 := LMGCentraliser(Ct2,y);
Cy := sub<Q|Cy1,Cy2>;

//As a final check, we check that Cy 248-dimensional module has 2
composition factors - we know this should be true by the structure
of C_G(z). Should this be true, we proceed to calculate <N_1, N_2>

if #CompositionFactors(GModule(Cy)) eq 2 then
 N1 := LMGNormaliser(C,K); N2 := LMGNormaliser(Cy,K); ONK := sub<Q|N1,N2>;
```

```
end if;

return ONK;
end function;

//We now use the above function to find <N_1, N_2> for the remaining groups

ONX := ONmaker(X); ONB := ONmaker(B); OND := ONmaker(D);

//We now must look inside each of ONX, ONA, ONB and OND for subgroups
that could be a normaliser inside some L_3(5). From the methodology, we
know the groups we are looking for have order 2^5 * 3 * 5^3. Firstly, we
find a copy of this normaliser we want outside the setting of E_8(3) so
we can check for isomorphisms in the next step.

H := PSL(3,5);
HS5 := Sylow(H,5);
Hsubs := NormalSubgroups(HS5 : IsElementaryAbelian := true, OrderEqual := 25);
Hsubs := {@i`subgroup : i in Hsubs@};
Hnorms := {@Normaliser(H,i) : i in Hsubs | Order(Normaliser(H,i)) eq 2^5 * 3 * 5^3@};
want := Hnorms[1];

//The group labelled want is what we wish to find inside our groups ONX,
ONA, ONB and OND. To find these, we work inside a permutation representation
as this is far quicker than working with matrices.

px,PX := PermutationRepresentation(ONX);
pa,PA := PermutationRepresentation(ONA);
pb,PB := PermutationRepresentation(ONB);
pd,PD := PermutationRepresentation(OND);

overX := {@i`subgroup  : i in SubgroupClasses(PX : OrderEqual := 2^5 * 3 * 5^3) |
IsIsomorphic(i`subgroup,want)@};
overA := {@i`subgroup  : i in SubgroupClasses(PA : OrderEqual := 2^5 * 3 * 5^3) |
IsIsomorphic(i`subgroup,want)@};
overB := {@i`subgroup  : i in SubgroupClasses(PB : OrderEqual := 2^5 * 3 * 5^3) |
IsIsomorphic(i`subgroup,want)@};
overD := {@i`subgroup  : i in SubgroupClasses(PD : OrderEqual := 2^5 * 3 * 5^3) |
IsIsomorphic(i`subgroup,want)@};

//As we saw in the methodology, each of groups <N_1, N_2> (for all X,A,B,D)
contains 2 conjugacy classes of subgroups isomorphic to our group 'want'.
The sets above contain only these 2 representatives and not the entire classes
which is what we desire. To obtain the classes of length 400, we use the following
simple function.
```

```
getsubs := function(overK,K);
res := {@@};
for i in overK do
 cl := Class(K,i);
 for j in cl do
  Include(~res, j);
 end for;
end for;
return res;
end function;

Xsubs := getsubs(overX,PX);
Asubs := getsubs(overA,PA);
Bsubs := getsubs(overB,PB);
Dsubs := getsubs(overD,PD);

//Now the above 4 sets each contain 800 groups isomorphic to the
normaliser of order 2^5 * 3 * 5^3 that we were searching for. We now consider
the pairs of subgroups, by looping over the relevant sets of the 800 normalisers
and checking the order of one random element, as described in the methodology.
Recall the pairs we must consider are {X,A}, {X,B} and {X,D}.

ords := {@ 1, 2, 3, 4, 5, 6, 8, 10, 12, 20, 24, 31 @};

cnt := 0;
keepXA := {@@};
for i in [1..#Xsubs] do
 for j in [1..#Asubs] do
  cnt := cnt+1;
  Hposs := sub<Q|Xsubs[i]@@px, Asubs[j]@@pa>;
  el := Random(Hposs);
  if Order(el) in ords then Include(~keepXA,{@i,j@}); end if; //USING INDEXES,
  STILL COULD BE ALL OF E8(3) AND ADDING THIS TO A SET WOULD BE VERY BAD
  "for X,A we are on ", cnt, " / 640000 and we have found: ",#keepXA;
 end for;
end for;

cnt := 0;
keepXB := {@@};
for i in [1..#Xsubs] do
 for j in [1..#Bsubs] do
  cnt := cnt+1;
  Hposs := sub<Q|Xsubs[i]@@px, Bsubs[j]@@pb>;
  el := Random(Hposs);
  if Order(el) in ords then Include(~keepXB,{@i,j@}); end if; //USING INDEXES,
  STILL COULD BE ALL OF E8(3) AND ADDING THIS TO A SET WOULD BE VERY BAD
```

```
  "for X,B we are on ", cnt, " / 640000 and we have found: ",#keepXB;
 end for;
end for;

cnt := 0;
keepXD := {@@};
for i in [1..#Xsubs] do
 for j in [1..#Dsubs] do
  cnt := cnt+1;
  Hposs := sub<Q|Xsubs[i]@@px, Dsubs[j]@@pd>;
  el := Random(Hposs);
  if Order(el) in ords then Include(~keepXD,{@i,j@}); end if; //USING INDEXES,
  STILL COULD BE ALL OF E8(3) AND ADDING THIS TO A SET WOULD BE VERY BAD
  "for X,D we are on ", cnt, " / 640000 and we have found: ",#keepXD;
 end for;
end for;

//Now the sets keepXA, keepXB and keepXD contain approximately 100 possibilities
for H. We wish to trim these further, so we now check the orders of 100 random
elements from these groups.

keepXA2 := {@@};
for ij in keepXA do
 if #ij eq 2 then i := ij[1]; j := ij[2]; end if;
 if #ij eq 1 then i := ij[1]; j := ij[1]; end if;
 Hposs := sub<Q|Xsubs[i]@@px, Asubs[j]@@pa>;
 check := 0;
 for c in [1..100] do
  el := Random(Hposs);
  if Order(el) notin ords then check := 1; break; end if;
 end for;
 if check eq 0 then Include(~keepXA2, ij); #keepXA2; end if;
end for; #keepXA2;

keepXB2 := {@@};
for ij in keepXB do
 if #ij eq 2 then i := ij[1]; j := ij[2]; end if;
 if #ij eq 1 then i := ij[1]; j := ij[1]; end if;
 Hposs := sub<Q|Xsubs[i]@@px, Bsubs[j]@@pb>;
 check := 0;
 for c in [1..100] do
  el := Random(Hposs);
  if Order(el) notin ords then check := 1; break; end if;
 end for;
 if check eq 0 then Include(~keepXB2, ij); #keepXB2; end if;
end for; #keepXB2;
```

```
keepXD2 := {@@};
for ij in keepXD do
 i := ij[1];
 j := ij[2];
 Hposs := sub<Q|Xsubs[i]@@px, Dsubs[j]@@pd>;
 check := 0;
 for c in [1..100] do
  el := Random(Hposs);
  if Order(el) notin ords then check := 1; break; end if;
 end for;
 if check eq 0 then Include(~keepXD2, ij); #keepXD2; end if;
end for; #keepXD2;

//Now as was stated in the Table, the sets keepXA2, keepXB2, keepXD2 each contain
either 0 groups or exactly 2. If these sets are non-empty, we can proceed to
perform one final check on each of these groups by checking the orders of
10000 elements. This code is omitted from this section but is very similar to
the code used above. In any case, if the sets were non-empty then the groups
contained in them by this point were isomorphic to L_3(5) as stated in
the methodology. We collect these 6 groups in a set called MASTER, before
finally checking for an isomorphism between them and L_3(5).

MASTER := {@@};

for ij in keepXA2 do
 i := ij[1];
 j := ij[2];
 Hposs := sub<Q | Xsubs[i] @@ px , Asubs[j] @@ pa>;
 Include(~MASTER, Hposs);
end for;

for ij in keepXB2 do
 i := ij[1];
 j := ij[2];
 Hposs := sub<Q | Xsubs[i] @@ px , Bsubs[j] @@ pb>;
 Include(~MASTER, Hposs);
end for;

for ij in keepXD2 do
 i := ij[1];
 j := ij[2];
 Hposs := sub<Q | Xsubs[i] @@ px , Dsubs[j] @@ pd>;
 Include(~MASTER, Hposs);
end for;
```

```
for i in MASTER do
 t := IsIsomorphic(i,H); t; //t was always true here
end for;

//We now have 6 copies of L_3(5) inside E_8(3). The above was repeated
for all 25 P in cubes. Overall, we find 30 groups isomorphic to L_3(5).
As stated in the methodology, we can trim these 30 groups down to just 4
by looking for conjugacy inside C_NE(z). The code used to do this is given
below, however it is possible for this to not trim the 30 groups down to
4 after just one passing. Moreover, there is more work to be done
after this has finished running. The user must look through 'rels'
at the conjugacy relations in order to find further conjugacy relations.
For example, we could find 1 is conjugate to 10, and 10 conjugate to 20,
thus meaning 1 is conjugate to 20 and we need only keep 1. However, the code
does not recognise this directly, so the user must see this themselves by
looking through rels.

rels := {@@};
keep := {@@};
no := {@@};
for i in [1..#H] do
 I := H[i];
 cnt2 := 0;
 if i in no then continue; end if;
 check := 0;
 for j in [1..50000] do
  el := Random(test);
  if j mod 100 eq 0 then j,i; end if;
  if I^el in H and el notin I then
   Include(~keep,i);
   for k in [1..#H] do
    if I^el eq H[k] then Include(~rels, Sort({@i,k@})); Include(~no,k);
    rels; "----------------"; "KEEP = ",keep;
    "----------------"; "NO = ",no; "---------------"; end if;
   end for;
  check := 1; end if;
 end for;
 if check eq 0 then Include(~keep,i); end if;
end for;
```

9.2.11 Brauer Character Code

The following function is used to calculate Brauer characters inside $E_8(3)_{ad}$.

```
function BrauerCharacter1(g)
//g is a 248 dimensional matrix from E_8(3)
```

```
p:=CharacteristicPolynomial(g);
R, S<w>:=RootsInSplittingField(p);

// Here we find the roots of the polynomial p
// in its minimal splitting field S.

k:=#R;
o:=Order(w);
// We now need to map elements of the finite field K to
// suitable roots of unity in the complex field.
// We deal with the case where S has prime order separately to
// avoid the situation where w equals 1 and so does
// not generate the multiplicative group of S.

if IsPrime(#S) then
 Q<x>:=CyclotomicField(#S-1);
 f:=pmap< S -> Q | [5^i -> x^i : i in [1..#S-1]]>;
else
 Q<x>:=CyclotomicField(o);
 f:=pmap< S -> Q | [w^i -> x^i : i in [1..o]]>;
 //When S is not of prime order w will generate S*.
end if;
//Now we sum the relevant powers of roots of unity.
c:=0;
for j:=1 to k do
 v:=R[j,2]*f(R[j,1]);
 c:=c+v;
end for;

return [c,#S];
end function;
```

9.2.12 Feasible Character Code

The following function outputs the feasible decompositions for a given subgroup H.

```
//This function takes as argument:
-G, this is the group you wish to calculate the feasible decompositions of
-I, the irreducible modules of G over GF(3), this can be obtained by
I:=IrreducibleModules(G,GF(3))
-S, a set or sequence consisting of sequences of size 2 that are of the form
[a,b] where a is an element order from G and b is a corresponding Brauer character
value inside E_8(3). We do this for all element orders in G, providing that the
corresponding Brauer character in E_8(3) is an integer. If it isnt, the method of solving
simultaneous equations is used instead. For example, consider G:=PSL(2,8). This
has elements of order 1,2,7. Hence we set S:=[ [1,248],[2,24],[2,-8],[7,52],[7,3] ].
```

The output of the function is a sequence containing each feasible decomposition which are given in the form [a,b,c,d,...] where a,b,c,d,... are integers corresponding to the coefficients of the feasible decomposition.

This code is taken from Peter Neuhaus' thesis - found on pages 130-134.

The code works by finding every feasible decomposition sequence of H and checking if the corresponding feasible decomposition is a feasible decomposition.
It does this by building the feasible decomposition sequences in reverse decomposition order starting at [248,0,...,0].

```
Feasible:= function(G,I,S);
B:= [BrauerCharacter(i): i in I]; //calculate Brauer character table for G
S1:= {@@};
for s in S do
Include(~S1,Integers()!s[1]);
end for;
C:= ConjugacyClasses(G);
C1:= {@@};
for i in [1..#C] do
Include(~C1,{@-i,C[i][1]@});
end for;
C2:= {@@};
for i in [1..#C] do
if C[i][1] in S1 then
Include(~C2,{@-i,C[i][1]@});
end if;
end for;

// C1 is the set of all conjugacy class orders of G as {@-i, order@}
// and C2 is the set of conjugacy class orders of G which we know
// Brauer character values of the E8(2) conjugacy classes of that order.

IS:= {@@};
for i in [1..#I] do
if Dimension(I[i]) lt 249 then
Include(~IS,{@-i,Dimension(I[i])@});
end if;
end for;

//IS is the set of dimensions of modules in I as {@-i,dimension@}.

Combs:= {@@}; //output set
t:= [1..#IS];
```

```
for i in [1..#IS] do
t[i]:= 0;
end for;
t[1]:= 248; //This sets up the base test case of 248 trivial modules.
vis:= 0;
repeat t1:= t;
vis:= vis+1;
for i in [2..#IS] do
check:= 0;
for j in [1..i-1] do
check:= check + (t[j]*IS[j][2]);
end for;
// check is now the total dimension of modules smaller than I[i] in the
// previous possible decomposition
if check gt IS[i][2]-1 then t[i]:= t[i]+1;
// the first time this condition is satisfied a new feasible
// decomposition is found and replaces t, it will then be
// tested as to whether it is a feasible decomposition before the whole process repeats
for j in [1..i-1] do
t[j]:= 0;
end for;
count:= 0;
for j in [1..#IS] do
count := count + (t[j]*IS[j][2]);
end for;
t[1]:= 248-count; // t is now the new possible decomposition to be tested.
In:= 0;
// In is a indicator of whether t is a feasible decomposition, if any
// test shows it can't be the In will be set to 1.
for c in C2 do
b:=0;
j:= -c[1];
for i in [1..#IS] do
b:= b + (t[i]*B[i][j]);
end for;
x:= [c[2],b];
Alt:= 0;
for s in S do
if x eq s then
Alt:= 1;
end if;
end for;
if Alt eq 0 then
In:=1;
end if;
end for;
```

```
// for each conjugacy class in C2 this checks that the feasible
// decomposition's character values match with some conjugacy class of
// E8(3) using data from S.
if In eq 0 then
Include(~Combs,t);
end if;
break i;
// this ensures that once a new feasible decomposition has been
// found and tested the entire loop is started from the beginning
// rather than moving on to the next module in I.
end if;
end for;
if vis mod 500 eq 0 then
vis,t;
end if;
until t eq t1;
return Combs;
end function;

//This procedure takes as argument:
-H, the group you have obtained the feasible decompositions of
-I, the irreducible modules of H over GF(3)
-bct, the Brauer character table of H corresponding to I
-coeffs, a feasible decomposition of H given in the form [a,b,c,...]
for integers a,b,c... being the coefficients in the decomposition

The procedure outputs strings of information of the form
o,d,b where o is a semisimple element order, d is the dimension
of fixed space and b is the Brauer character of the elements from
that H-class should H embed into E_8(3) in the way described by the given
feasible decomposition.

    dims := procedure(H,I,bct,coeffs);
//A function which details the fusion pattern of a given feasible decomposition,
by displaying Brauer character values and eigenspace dimensions
res := [];
c := Classes(H);
inds := [i : i in [1..#c] | c[i][1] mod 3 ne 0];
for ind in inds do
sumdims := 0;
bc := 0;
for i in [2..#coeffs] do
m := MatrixGroup(I[i]);
cm:=Classes(m);
sumdims := sumdims + coeffs[i]*Dimension(Eigenspace(cm[ind][3],1));
bc := bc + bct[i][ind]*coeffs[i];
end for;
```

```
cm[ind][1],sumdims+coeffs[1],bc+coeffs[1]*bct[1][ind];
end for;
end procedure;
```

9.2.13 Constructing $L_2(11)$

Showing involutions are conjugate

Given here is the code implementing the algorithm shown in Procedure 5.3. Given a group G and two involutions x, y, the function `FindConjInv` returns a conjugating element from x to y. The function works as follows. To begin, random elements of G, denoted k, are generated until the element $x^k y$ has odd order; the dihedral group $\langle x^k, y\rangle$ will consequently have order $2n$ where n is odd. For computational purposes, it is advantageous to have n as small as possible, therefore an iteration is used (iterating `LoopLimit` times) to try to reduce n. The smallest value of n (and the element k from which it arose) from the iterations is saved and used for the final part of the algorithm. Having found a suitable dihedral group $D = \langle x^k, y\rangle$, the function `IsConjugate` is employed to find a conjugating element $\tau \in D$ from x^k to y. Finally, the element $k\tau \in G$ is a conjugating element from x to y.

This function assumes the inputted involutions x, y are G-conjugate. If this is not the case, (or if no suitable element $k \in G$ is found within the time limit `TimeLimit`), an error message is printed and the identity of G is returned.

```
FindConjInv := function(G,x,y : LoopLimit:=10, TimeLimit:=10);
start := Realtime(); //timing purposes
Q := GL(Ncols(x),BaseRing(G)); //define general linear group
repeat
k := Random(G); ord := Order(x^k*y); finish := Realtime();
if finish-start gt TimeLimit then
"Couldnt find in time, returning identity";
return Id(G);
end if;
until ord mod 2 eq 1;
for i in [1..LoopLimit] do
start := Realtime();
repeat
el := Random(G); finish := Realtime();
if finish-start gt TimeLimit then "Couldnt find in time, returning identity";
return Id(G);
end if;
until Order(x^el*y) mod 2 eq 1;
if Order(x^el*y) lt ord then ord := Order(x^el*y); k := el; end if;
end for;
D := sub<Q|x^k,y>; "Dihedral found, has order: ",Order(x^k*y)*2; //dihedral group
bool,conj := IsConjugate(D,x^k,y); //find conjugating element
return k*conj; //return conjugating element from x -> y
```

```
end function;
```

Constructing the stabiliser S

The code given here illustrates how the stabiliser S, described in Section 5, was constructed. This type of procedure was used thousands of times before a suitable stabiliser was found.

```
//Define stabv to be the stabiliser subgroup we have gathered so far
n := Order(stabv);
repeat
C := CentraliserOfInvolution(CGfext,Element(CGfext,2)); //find centraliser of a random
involution in the extended centraliser
el := Element(C,2); //take a random involution from within this centraliser
C2 := CentraliserOfInvolution(C,el); //construct its centraliser inside the other centraliser
#CompositionFactors(GModule(C2)); //number of composition factors of the second centraliser
on the 248-module
if #CompositionFactors(GModule(C2)) gt 40 then //if there are suitable many composition factors,
this is a good way of gauging how large the centraliser is without wasting time
calculating the order
teststab := Stabiliser(C2,v); //stabiliser of our 1-dimensional subspace
"and the size of teststab is: ",Order(teststab);
stabv := sub<Q|stabv,teststab>; Order(stabv); //combine the stabilisers that we have
end if;
until Order(stabv) gt n; //repeat until the stabiliser subgroup grows in size
```

The 10-dimensional model

Here is the code used to construct the 10-dimensional model described in Section 5.1.1.

```
Q := GL(10,81); load Element;
H := SU(5,9); V := GModule(H);
G := MatrixGroup(DirectSum(V,Dual(V)));
G := sub<Q|G>; //10dim rep of SU(5,9);

B:=ZeroMatrix(GF(81),5,5);
I := Identity(GL(5,81));
top := HorizontalJoin(B,I);
bottom := HorizontalJoin(I,B);
t:=Q!VerticalJoin(top,bottom); //inverse transpose map constructed using blocks
Gext := sub<Q|G,t>; //10dim rep of our extended centraliser
f := Element(LMGCenter(G),5); //element of order 5 in the center of SU(5,9)
```

Finding the last Cz orbit

```
//Assuming subs is the set of conjugacy class representatives for subgroups
of order 14400 inside Cz

load Element;
```

```
cnt := 0;
keep := {@@};
for k in subs do
cnt := cnt+1;
cnter:=0;
repeat
cnter:=cnter+1;
repeat t:=Element(k,2); until Dimension(Eigenspace(t,1)) gt 2;
if cnter lt 50 then gens := {@t, Random(k)@}; end if;
if cnter ge 50 then gens := {@t,Random(k),Random(k)@}; end if;
if cnter ge 100 then gens := {@t,Random(k),Random(k),Random(k)@}; end if;
until k eq sub<k|gens>;
"found gens";

Ct := CentraliserOfInvolution(Gext,t);
Ct := sub<Q|Ct,CentraliserOfInvolution(Gext,t)>;
Ct := sub<Q|Ct,CentraliserOfInvolution(Gext,t)>;
Ct := sub<Q|Ct,CentraliserOfInvolution(Gext,t)>;
"found ct, has order: ",LMGFactoredOrder(Ct);

C2 := Centraliser(Ct,gens[2]);
if #gens gt 2 then
for el in gens[2..#gens] do
C2 := Centraliser(C2,el);
end for;
end if;
"found c2";

c := Classes(C2);
invs:={@i[3] : i in c | i[1] eq 2 and f^i[3] eq f^-1@};
keep := keep join invs;
cnt,#keep;
end for;
```

Final module check

The code given here implements the final check described at the end of Section 5.1.2.

```
//Let g,f,v be as described and let 'master' be the set of all 4012
orbit representatives.
V := VectorSpace(GF(3),248); //standard 248-dimensional vector space
keep := {@@}; cnt := 0; //we shall store orbit representatives that could give a
\PSL{2}{11} in the set keep
for t in master do //loop through all the orbit representatives
F := sub<Q|f,t>; //Frobenius group of order 55
cnt :=cnt+1;
```

```
U := sub<V| {v} join {(v^t)^i : i in sub<Q|g>} >; //the subspace generated by
{v,v^t,(v^t)^g,(v^t)^{g^2},...,(v^t)^{g^{10}}}
if sub<V|U^F> eq U then Include(~keep,t); end if;
cnt,#keep;
end for;
#keep;
//3
```

9.2.14 Projective Covers

Here is the code used to determine the structures of the projective indecomposable modules for various irreducible GF(3) H-modules presented in Section 8. The code identifies the irreducible factors in each socle layer.

```
//This function takes as input a sequence of irreducible modules I
and returns the socle layers for each module inside this sequence.

ProjectiveSocleLayers := function(I);
for i in [2..#I] do
        "We look at the projective cover P(varphi_",i,").";
        P := ProjectiveCover(I[i]);
        SFs := SocleFactors(P);
        for block in [1..#SFs] do
                "Block " , block, " is made up of ";
                c:=CompositionFactors(SFs[block]);
                for factor in c do
                        for m in [1..#I] do
                                if IsIsomorphic(factor,I[m]) then
                                        m;
                                end if;
                        end for;
                end for;
        end for;
        "-------------------------------------------------------------";
end for;
return I;
end function;
```

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