Easy Questions

1. (a) $\displaystyle\int \frac{14x^6 - 3x^2}{2x^7 - x^3 + 1}\,\mathrm{d}x = \ln|2x^7 - x^3 + 1| + C$

(b) $\displaystyle\int \frac{\sec^2\theta}{\tan\theta}\,\mathrm{d}\theta = \ln|\tan\theta| + C$

(c) $\displaystyle\int \frac{1/r}{\ln|r|}\,\mathrm{d}r = \ln\big|\ln|r|\big| + C$

(d) $\displaystyle\int \frac{1/\sqrt{1+t^2}}{\sinh^{-1} t}\,\mathrm{d}t = \ln|\sinh^{-1} t| + C$

(e) $\displaystyle\int \frac{1/\sqrt{1-s^2}}{\sin^{-1} s}\,\mathrm{d}s = \ln|\sin^{-1} s| + C$

(f) $\displaystyle\int \frac{1/(1+z^2)}{\tan^{-1} z}\,\mathrm{d}z = \ln|\tan^{-1} z| + C$

(g) $\displaystyle\int \frac{\sin\theta}{\cos\theta}\,\mathrm{d}\theta = -\ln|\cos\theta| + C$

(h) $\displaystyle\int \frac{\cos\theta}{\sin\theta}\,\mathrm{d}\theta = \ln|\sin\theta| + C$

(i) $\displaystyle\int \frac{\sinh y}{\cosh y}\,\mathrm{d}y = \ln|\cosh y| + C$

(j) $\displaystyle\int \frac{\cosh y}{\sinh y}\,\mathrm{d}y = \ln|\sinh y| + C$

(k) $\displaystyle\int \frac{\mathrm{cosec}^2 u}{\cot u}\,\mathrm{d}u = -\ln|\cot u| + C$

(l) $\displaystyle\int \frac{1/\sqrt{1-v^2}}{\cos^{-1} v}\,\mathrm{d}v = -\ln|\cos^{-1} v| + C$

(m) $\displaystyle\int \frac{\mathrm{sech}^2 w}{\tanh w}\,\mathrm{d}w = \ln|\tanh w| + C$

(n) $\displaystyle\int \frac{1/\sqrt{r^2-1}}{\cosh^{-1} r}\,\mathrm{d}r = \ln|\cosh^{-1} r| + C$

(o) $\displaystyle\int \frac{\mathrm{cosech}^2 x}{\coth x}\,\mathrm{d}x = -\ln|\coth x| + C$

(p) $\displaystyle\int \frac{1/(u^2-1)}{\tanh^{-1} u}\,\mathrm{d}u = -\ln|\tanh^{-1} u| + C$

(q) $\displaystyle\int \frac{1/(u^2-1)}{\coth^{-1} u}\,\mathrm{d}u = -\ln|\coth^{-1} u| + C$

(r) $\displaystyle\int \frac{x^n}{x^{n+1}+a}\,\mathrm{d}x = \tfrac{1}{n+1}\ln|x^{n+1}+a| + C$

Standard Questions

2. $\displaystyle\int \sin x\cos x\,\mathrm{d}x = \tfrac12\sin^2 x + C,\quad \int \sin x\cos x\,\mathrm{d}x = -\tfrac12\cos^2 x + C,\quad \int \sin x\cos x\,\mathrm{d}x = -\tfrac14\cos(2x) + C$

Differentiating: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\big(\tfrac12\sin^2 x + C\big) = \tfrac12 \times 2\sin x \times \cos x = \sin x\cos x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(-\tfrac12\cos^2 x + C\big) = -\tfrac12 \times 2\cos x \times (-\sin x) = \sin x\cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(-\tfrac14\cos(2x) + C\big) = -\tfrac14 \times 2\big(-\sin(2x)\big) = \tfrac12\sin(2x) = \tfrac12 \times 2\sin x\cos x = \sin x\cos x$$

which confirms that each integral is correct.

It is not true that $\tfrac12\sin^2 x = -\tfrac12\cos^2 x = -\tfrac14\cos(2x)$.

The differences are: $\tfrac12\sin^2 x - (-\tfrac12\cos^2 x) = \tfrac12(\sin^2 x + \cos^2 x) = \tfrac12$

$$\tfrac12\sin^2 x - \big(-\tfrac14\cos(2x)\big) = \tfrac12\sin^2 x + \tfrac14\big(\cos^2 x - \sin^2 x\big) = \tfrac14(\sin^2 x + \cos^2 x) = \tfrac14$$

$$-\tfrac12\cos^2 x - \big(-\tfrac14\cos(2x)\big) = -\tfrac12\cos^2 x + \tfrac14\big(\cos^2 x - \sin^2 x\big) = -\tfrac14(\sin^2 x + \cos^2 x) = -\tfrac14$$

so that the solutions each differ by a constant.
The 'arbitrary' constant of integration should simply be interpreted as a different constant for each indefinite integral.

3. (a) i. $\displaystyle\int 1 \times \underline{\sqrt{4+x^2}}\,\mathrm{d}x = x\sqrt{4+x^2} - \int x\frac{\frac12 \times 2x}{\sqrt{4+x^2}}\,\mathrm{d}x = x\sqrt{4+x^2} - \int \frac{4+x^2-4}{\sqrt{4+x^2}}\,\mathrm{d}x$

so $\displaystyle\int \sqrt{4+x^2}\,\mathrm{d}x = x\sqrt{4+x^2} - \int \sqrt{4+x^2}\,\mathrm{d}x + \int \frac{4\,\mathrm{d}x}{\sqrt{4+x^2}}$

so $\displaystyle 2\int \sqrt{4+x^2}\,\mathrm{d}x = x\sqrt{4+x^2} + \int \frac{2\,\mathrm{d}x}{\sqrt{1+(x/2)^2}} = x\sqrt{4+x^2} + 2\frac{\sinh^{-1}(x/2)}{1/2}$

so $\displaystyle\int \sqrt{4+x^2}\,\mathrm{d}x = \tfrac12 x\sqrt{4+x^2} + 2\sinh^{-1}(x/2) + C$

ii. $\displaystyle\int 1\times \underline{\sqrt{a^2-z^2}}\,\mathrm{d}z = z\sqrt{a^2-z^2} - \int z\frac{\frac{1}{2}\times(-2z)}{\sqrt{a^2-z^2}}\,\mathrm{d}z = z\sqrt{a^2-z^2} - \int \frac{a^2-z^2-a^2}{\sqrt{a^2-z^2}}\,\mathrm{d}z$

so $\displaystyle\int \sqrt{a^2-z^2}\,\mathrm{d}z = z\sqrt{a^2-z^2} - \int\sqrt{a^2-z^2}\,\mathrm{d}z + \int\frac{a^2\,\mathrm{d}z}{\sqrt{a^2-z^2}}$

so $\displaystyle 2\int \sqrt{a^2-z^2}\,\mathrm{d}z = z\sqrt{a^2-z^2} + \int\frac{a\,\mathrm{d}z}{\sqrt{1-(z/a)^2}} = z\sqrt{a^2-z^2} + a\frac{\sin^{-1}(z/a)}{1/a}$

so $\displaystyle\int \sqrt{a^2-z^2}\,\mathrm{d}z = \tfrac{1}{2}z\sqrt{a^2-z^2} + \tfrac{1}{2}a^2\sin^{-1}(z/a) + C$

iii. $\displaystyle\int 1\times\underline{\sqrt{9t^2-3}}\,\mathrm{d}t = t\sqrt{9t^2-3} - \int t\frac{\frac{1}{2}\times(18t)}{\sqrt{9t^2-3}}\,\mathrm{d}t = t\sqrt{9t^2-3} - \int\frac{9t^2-3+3}{\sqrt{9t^2-3}}\,\mathrm{d}t$

so $\displaystyle\int\sqrt{9t^2-3}\,\mathrm{d}t = t\sqrt{9t^2-3} - \int\sqrt{9t^2-3}\,\mathrm{d}t - \int\frac{3\,\mathrm{d}t}{\sqrt{9t^2-3}}$

so $\displaystyle 2\int\sqrt{9t^2-3}\,\mathrm{d}t = t\sqrt{9t^2-3} - \int\frac{\sqrt{3}\,\mathrm{d}t}{\sqrt{(\sqrt{3}t)^2-1}} = t\sqrt{9t^2-3} - \sqrt{3}\,\frac{\cosh^{-1}|\sqrt{3}t|}{\sqrt{3}}$

so $\displaystyle\int\sqrt{9t^2-3}\,\mathrm{d}t = \tfrac{1}{2}t\sqrt{9t^2-3} - \tfrac{1}{2}\cosh^{-1}|\sqrt{3}t| + C$

(b) i. $\displaystyle\int\sqrt{4+x^2}\,\mathrm{d}x$. Let $x = 2\sinh u$, $\mathrm{d}x = 2\cosh u\,\mathrm{d}u$, so that

$$\int\sqrt{4+x^2}\,\mathrm{d}x = \int\sqrt{4+4\sinh^2 u}\,2\cosh u\,\mathrm{d}u = 4\int\sqrt{1+\sinh^2 u}\,\cosh u\,\mathrm{d}u$$
$$= 4\int\cosh^2 u\,\mathrm{d}u = 4\int\tfrac{1}{2}\big(1+\cosh(2u)\big)\,\mathrm{d}u = 2\Big(u+\frac{\sinh(2u)}{2}\Big)$$
$$= 2u+\sinh(2u) = 2u + 2\sinh u\cosh u = 2u + 2\sinh u\sqrt{1+\sinh^2 u}$$
$$= 2\sinh^{-1}(x/2) + 2(x/2)\sqrt{1+(x/2)^2} = 2\sinh^{-1}(x/2) + \tfrac{1}{2}x\sqrt{4+x^2} + C$$

ii. $\displaystyle\int\sqrt{a^2-z^2}\,\mathrm{d}z$. Let $z = a\sin u$, $\mathrm{d}z = a\cos u\,\mathrm{d}u$, so that

$$\int\sqrt{a^2-z^2}\,\mathrm{d}z = \int\sqrt{a^2-a^2\sin^2 u}\,a\cos u\,\mathrm{d}u = a^2\int\sqrt{1-\sin^2 u}\,\cos u\,\mathrm{d}u = a^2\int\cos^2 u\,\mathrm{d}u$$
$$= a^2\int\tfrac{1}{2}\big(1+\cos(2u)\big)\,\mathrm{d}u = \tfrac{1}{2}a^2\Big(u+\frac{\sin(2u)}{2}\Big) = \tfrac{1}{2}a^2u + \tfrac{1}{4}a^2\,2\sin u\cos u$$
$$= \tfrac{1}{2}a^2u + \tfrac{1}{4}a^2\,2\sin u\sqrt{1-\sin^2 u} = \tfrac{1}{2}a^2\sin^{-1}(z/a) + \tfrac{1}{2}a^2(z/a)\sqrt{1-(z/a)}$$
$$= \tfrac{1}{2}a^2\sin^{-1}(z/a) + \tfrac{1}{2}z\sqrt{a^2-z^2} + C$$

iii. $\displaystyle\int\sqrt{9t^2-3}\,\mathrm{d}t = 3\int\sqrt{t^2-1/3}\,\mathrm{d}t$. Let $t = \frac{1}{\sqrt{3}}\cosh u$, $\mathrm{d}t = \frac{1}{\sqrt{3}}\sinh u\,\mathrm{d}u$, so that

$$\int\sqrt{9t^2-3}\,\mathrm{d}t = \int\sqrt{9\times\tfrac{1}{3}\cosh^2 u - 3}\,\frac{1}{\sqrt{3}}\sinh u\,\mathrm{d}u = \int\sqrt{\cosh^2 u - 1}\,\sinh u\,\mathrm{d}u$$
$$= \int\sinh^2 u\,\mathrm{d}u = \int\tfrac{1}{2}\big(\cosh(2u)-1\big)\,\mathrm{d}u = \tfrac{1}{2}\Big(\frac{\sinh(2u)}{2} - u\Big)$$
$$= \tfrac{1}{4}\times 2\sinh u\cosh u - \tfrac{1}{2}u = \tfrac{1}{2}\sqrt{\cosh^2 u - 1}\,\cosh u - \tfrac{1}{2}u$$
$$= \tfrac{1}{2}\sqrt{3t^2-1}\sqrt{3}t - \tfrac{1}{2}\cosh^{-1}(\sqrt{3}t) = \tfrac{1}{2}t\sqrt{9t^2-3} - \tfrac{1}{2}\cosh^{-1}(\sqrt{3}t) + C$$

4. (a) $\displaystyle\int x^2\ln(1+x^3)\,\mathrm{d}x$. Let $1+x^3 = u$, $3x^2\,\mathrm{d}x = \mathrm{d}u$ so that

$$\int x^2\ln(1+x^3)\,\mathrm{d}x = \int x^2\ln u\,\frac{\mathrm{d}u}{3x^2} = \tfrac{1}{3}\int\ln u\,\mathrm{d}u = \tfrac{1}{3}(u\ln u - u)$$
$$= \tfrac{1}{3}\big((1+x^3)\ln(1+x^3) - (1+x^3)\big) + C$$

(b) $\displaystyle\int t\sin(t^2)\,e^{1+2\cos t^2}\,\mathrm{d}t$. Let $u = \cos t^2$, $\mathrm{d}u = -2t\sin t^2\,\mathrm{d}t$ so that

$$\int t\sin(t^2)\,e^{1+2\cos t^2}\,\mathrm{d}t = \int t\sin(t^2)\,e^{1+2u}\,\frac{\mathrm{d}u}{-2t\sin t^2} = -\tfrac{1}{2}\int e^{1+2u}\,\mathrm{d}u = -\tfrac{1}{2}\,\frac{e^{1+2u}}{2}$$
$$= -\tfrac{1}{4}e^{1+2\cos t^2} + C$$

(c) $\displaystyle\int \frac{\cos\theta\sin\theta}{2\cos\theta-5}\,\mathrm{d}\theta$. Let $u = 2\cos\theta - 5$, $\mathrm{d}u = -2\sin\theta\mathrm{d}\theta$, $\cos\theta = \frac{1}{2}(u+5)$ so that

$$\int \frac{\cos\theta\sin\theta}{2\cos\theta-5}\,\mathrm{d}\theta = \int \frac{\frac{1}{2}(u+5)\sin\theta}{u}\,\frac{\mathrm{d}u}{-2\sin\theta} = -\tfrac{1}{4}\int \frac{u+5}{u}\,\mathrm{d}u = -\tfrac{1}{4}\int \left(1+\frac{5}{u}\right)\mathrm{d}u$$
$$= -\tfrac{1}{4}(u + 5\ln|u|) = -\tfrac{1}{4}\big(2\cos\theta - 5 + 5\ln|2\cos\theta-5|\big) + C$$

(d) $\displaystyle\int \frac{5-x}{\sqrt{2x-3}}\,\mathrm{d}x$. Let $u^2 = 2x-3$, $2u\,\mathrm{d}u = 2\,\mathrm{d}x$, $x = \frac{1}{2}(u^2+3)$ so that

$$\int \frac{5-x}{\sqrt{2x-3}}\,\mathrm{d}x = \int \frac{5-\frac{1}{2}(u^2+3)}{u}\,u\,\mathrm{d}u = \int \left(\tfrac{7}{2} - \tfrac{1}{2}u^2\right)\mathrm{d}u = \tfrac{7}{2}u - \tfrac{1}{6}u^3$$
$$= \tfrac{7}{2}\sqrt{2x-3} - \tfrac{1}{6}(2x-3)^{3/2} + C$$

(e) $\displaystyle\int \cosh(r^3-3r)\,(r^2-1)\,\mathrm{d}r$. Let $u = r^3 - 3r$, $\mathrm{d}u = (3r^2-3)\,\mathrm{d}r$ so that

$$\int \cosh(r^3-3r)\,(r^2-1)\,\mathrm{d}r = \int \cosh u\,(r^2-1)\,\frac{\mathrm{d}u}{3r^2-3} = \int \tfrac{1}{3}\cosh u\,\mathrm{d}u = \tfrac{1}{3}\sinh u$$
$$= \tfrac{1}{3}\sinh(r^3-3r) + C$$

(f) $\displaystyle\int z\sqrt{4-z^2}\,\mathrm{d}z$. Let $u = 4 - z^2$, $\mathrm{d}u = -2z\,\mathrm{d}z$ so that

$$\int z\sqrt{4-z^2}\,\mathrm{d}z = \int z\sqrt{u}\,\frac{\mathrm{d}u}{-2z} = -\tfrac{1}{2}\int \sqrt{u}\,\mathrm{d}u = -\tfrac{1}{2}\,\frac{u^{3/2}}{3/2}$$
$$= -\tfrac{1}{3}(4-z^2)^{3/2} + C$$

5. (a) $\displaystyle\int_0^{\frac{1}{2}\sqrt{\pi}} \left(x - \tfrac{1}{2}\sqrt{\pi}\right)\sin\!\left(\tfrac{1}{2}\pi + x\sqrt{\pi} - x^2\right)\mathrm{d}x$. Let $\frac{1}{2}\pi + x\sqrt{\pi} - x^2 = u$, $(\sqrt{\pi} - 2x)\,\mathrm{d}x = \mathrm{d}u$

$\frac{1}{2}\pi + 0\sqrt{\pi} - 0^2 = \frac{1}{2}\pi$, $\frac{1}{2}\pi + \frac{1}{2}\sqrt{\pi}\sqrt{\pi} - (\frac{1}{2}\sqrt{\pi})^2 = \frac{3}{4}\pi$

leading to

$$\int_{\frac{1}{2}\pi}^{\frac{3}{4}\pi} \left(x - \tfrac{1}{2}\sqrt{\pi}\right)\sin u\,\frac{\mathrm{d}u}{\sqrt{\pi}-2x} = -\tfrac{1}{2}\int_{\frac{1}{2}\pi}^{\frac{3}{4}\pi} \sin u\,\mathrm{d}u = \tfrac{1}{2}\big[\cos u\big]_{\frac{1}{2}\pi}^{\frac{3}{4}\pi} = \tfrac{1}{2}\big(\cos(\tfrac{3}{4}\pi) - \cos(\tfrac{1}{2}\pi)\big)$$
$$= \tfrac{1}{2}\big(-1/\sqrt{2} - 0\big) = -\frac{1}{2\sqrt{2}}$$

(b) $\displaystyle\int_{-2}^{5/2} \frac{t+1}{\sqrt{5-2t}}\,\mathrm{d}t$. Let $5 - 2t = u^2$, $-2\,\mathrm{d}t = 2u\,\mathrm{d}u$, $t = \dfrac{5-u^2}{2}$

$5 - 2(-2) = 9 = 3^2$, $5 - 2\times\frac{5}{2} = 0$

leading to

$$\int_3^0 \frac{(5-u^2)/2+1}{u}\,\frac{u\,\mathrm{d}u}{-1} = -\int_3^0 \frac{7-u^2}{2}\,\mathrm{d}u = \left[-\tfrac{7}{2}u + \tfrac{1}{6}u^3\right]_3^0 = 0 - \left(-\tfrac{21}{2} + \tfrac{27}{6}\right) = 6$$

(c) $\displaystyle\int_0^{\pi/2} \frac{\sin^2\theta\cos\theta}{2-\sin\theta}\,\mathrm{d}\theta$. Let $u = \sin\theta$, $\mathrm{d}u = \cos\theta\,\mathrm{d}\theta$

$\sin 0 = 0$, $\sin\frac{\pi}{2} = 1$

leading to

$$\int_0^1 \frac{u^2\cos\theta}{2-u}\,\frac{\mathrm{d}u}{\cos\theta} = \int_0^1 \frac{u^2}{2-u}\,\mathrm{d}u$$

Let $2 - u = v$, $-\mathrm{d}u = \mathrm{d}v$, $u^2 = (2-v)^2$

$2 - 0 = 2$, $2 - 1 = 1$

$$= -\int_2^1 \frac{(2-v)^2}{v}\,\mathrm{d}v = \int_1^2 \frac{4-4v+v^2}{v}\,\mathrm{d}v = \int_1^2 \left(\frac{4}{v} - 4 + v\right)\mathrm{d}v$$
$$= \left[4\ln|v| - 4v + \tfrac{1}{2}v^2\right]_1^2 = \big(4\ln 2 - 8 + 2\big) - \big(4\ln 1 - 4 + \tfrac{1}{2}\big) = 4\ln 2 - \tfrac{5}{2}$$

(d) $\displaystyle\int_0^3 \frac{4s}{\sqrt[3]{9-s^2}}\,\mathrm{d}s$. Let $9 - s^2 = u$, $-2s\,\mathrm{d}s = \mathrm{d}u$

$9 - 0 = 9$, $9 - 3^2 = 0$

leading to

$$\int_9^0 \frac{4s}{\sqrt[3]{u}}\,\frac{\mathrm{d}u}{-2s} = 2\int_0^9 u^{-1/3}\,\mathrm{d}u = 2\left[\tfrac{3}{2}u^{2/3}\right]_0^9 = 2\times\tfrac{3}{2}9^{2/3} - 2\lim_{u\to 0^+}\tfrac{3}{2}u^{2/3} = 3\times 9^{2/3} - \{0\} = 3\times 9^{2/3}$$

6. (a) $$\frac{5x^2-7x+26}{x^2-2x+5} = \frac{5(x^2-2x+5)+10x-25-7x+26}{x^2-2x+5} = 5 + \frac{3x+1}{x^2-2x+5} = 5 + \frac{\frac{3}{2}(2x-2)+3+1}{x^2-2x+5}$$
$$= 5 + \tfrac{3}{2}\,\frac{2x-2}{x^2-2x+5} + \frac{4}{(x-1)^2-1+5} = 5 + \tfrac{3}{2}\,\frac{2x-2}{x^2-2x+5} + \frac{4}{(x-1)^2+4}$$

so $\displaystyle\int \frac{5x^2-7x+26}{x^2-2x+5}\,\mathrm{d}x = \int \Big(5 + \tfrac{3}{2}\frac{2x-2}{x^2-2x+5} + \frac{1}{(\frac{x-1}{2})^2+1}\Big)\,\mathrm{d}x$

$= 5x + \tfrac{3}{2}\ln|x^2-2x+5| + 2\tan^{-1}\dfrac{x-1}{2} + C$

(b) $\dfrac{12x^2-16x}{4x^2-4x+2} = \dfrac{3(4x^2-4x+2)+12x-6-16x}{4x^2-4x+2} = 3 + \dfrac{-4x-6}{4x^2-4x+2}$

$= 3 + \dfrac{-\frac{1}{2}(8x-4)-2-6}{4x^2-4x+2} = 3 - \tfrac{1}{2}\dfrac{8x-4}{4x^2-4x+2} - \dfrac{8}{(2x-1)^2+1}$

so $\displaystyle\int \frac{12x^2-16x}{4x^2-4x+2}\,\mathrm{d}x = \int \Big(3 - \tfrac{1}{2}\frac{8x-4}{4x^2-4x+2} - \frac{8}{(2x-1)^2+1}\Big)\,\mathrm{d}x$

$= 3x - \tfrac{1}{2}\ln|4x^2-4x+2| - 4\tan^{-1}(2x-1) + C$

(c) $\dfrac{18x^3-3x^2+7x-62}{9x^2+12x+20} = \dfrac{2x(9x^2+12x+20)-24x^2-40x-3x^2+7x-62}{9x^2+12x+20} = 2x + \dfrac{-27x^2-33x-62}{9x^2+12x+20}$

$= 2x + \dfrac{-3(9x^2+12x+20)+36x+60-33x-62}{9x^2+12x+20} = 2x - 3 + \dfrac{3x-2}{9x^2+12x+20}$

$= 2x - 3 + \dfrac{\frac{1}{6}(18x+12)-2-2}{9x^2+12x+20} = 2x - 3 + \tfrac{1}{6}\dfrac{18x+12}{9x^2+12x+20} + \dfrac{-4}{(3x+2)^2+16}$

so $\displaystyle\int \frac{18x^3-3x^2+7x-62}{9x^2+12x+20}\,\mathrm{d}x = \int \Big(2x - 3 + \tfrac{1}{6}\frac{18x+12}{9x^2+12x+20} - \frac{1/4}{(\frac{3x+2}{4})^2+1}\Big)\,\mathrm{d}x$

$= x^2 - 3x + \tfrac{1}{6}\ln|9x^2+12x+20| - \dfrac{1/\cancel{4}}{3/\cancel{4}}\tan^{-1}\dfrac{3x+2}{4} + C$

(d) $\dfrac{3x^2-6x+6}{(1-2x)(x-1)(x+2)} = \dfrac{A}{1-2x} + \dfrac{B}{x-1} + \dfrac{C}{x+2} = \dfrac{A(x-1)(x+2)+B(1-2x)(x+2)+C(1-2x)(x-1)}{(1-2x)(x-1)(x+2)}$

So: $3x^2-6x+6 = A(x-1)(x+2) + B(1-2x)(x+2) + C(1-2x)(x-1)$

For $x=\frac{1}{2}$: $\frac{3}{4}-3+6 = A(-\frac{1}{2})(\frac{5}{2})$ so $A = -\frac{15}{5}\frac{4}{5} = -3$
For $x=1$: $3 = B\times(-3)$ so $B=-1$
For $x=-2$: $30 = C\times(-15)$ so $C=-2$.

Hence $\displaystyle\int \frac{3x^2-6x+6}{(1-2x)(x-1)(x+2)}\,\mathrm{d}x = \int \Big(\frac{-3}{1-2x} + \frac{-1}{x-1} + \frac{-2}{x+2}\Big)\,\mathrm{d}x$

$= \tfrac{3}{2}\ln|1-2x| - \ln|x-1| - 2\ln|x+2| + C$

(e) $\dfrac{2x^2+21x+35}{(x+3)^2(x+1)} = \dfrac{A}{(x+3)^2} + \dfrac{B}{x+3} + \dfrac{C}{x+1} = \dfrac{A(x+1)+B(x+3)(x+1)+C(x+3)^2}{(x+3)^2(x+1)}$

So: $2x^2+21x+35 = A(x+1) + B(x+3)(x+1) + C(x+3)^2$

For $x=-3$: $18-63+35 = A\times(-2)$ or $-10=-2A$ so $A=5$
For $x=-1$: $2-21+35 = C\times 4$ or $16=4C$ so $C=4$
Coeffs of x^2: $2 = B+4$ so $B=-2$.

Hence $\displaystyle\int \frac{2x^2+21x+35}{(x+3)^2(x+1)}\,\mathrm{d}x = \int \Big(\frac{5}{(x+3)^2} - \frac{2}{x+3} + \frac{4}{x+1}\Big)\,\mathrm{d}x$

$= \dfrac{-5}{x+3} - 2\ln|x+3| + 4\ln|x+1| + C$

(f) $\dfrac{16x^2-12x-3}{(2x+3)(4x^2-4x+2)} = \dfrac{A}{2x+3} + \dfrac{Bx+C}{4x^2-4x+2} = \dfrac{A(4x^2-4x+2)+(Bx+C)(2x+3)}{(2x+3)(4x^2-4x+2)}$

So: $16x^2-12x-3 = A(4x^2-4x+2) + (Bx+C)(2x+3)$

For $x=-\frac{3}{2}$: $36+18-3 = A(9+6+2)$ or $51=17A$ so $A=3$
Coeffs of x^2: $16 = 12+2B$ so $B=2$
Coeffs of x^0: $-3 = 6+3C$ so $C=-3$

Hence $\displaystyle\int \frac{16x^2-12x-3}{(2x+3)(4x^2-4x+2)}\,\mathrm{d}x = \int \Big(\frac{3}{2x+3} + \frac{2x-3}{(2x-1)^2+1}\Big)\,\mathrm{d}x = \int \Big(\frac{3}{2x+3} + \frac{\frac{1}{4}(8x-4)-2}{(2x-1)^2+1}\Big)\,\mathrm{d}x$

$= \tfrac{3}{2}\ln|2x+3| + \tfrac{1}{4}\ln|(2x-1)^2+1| - \dfrac{\cancel{2}}{\cancel{2}}\tan^{-1}(2x-1) + C$