Problem Sheet for Week 5 | **_MT1121_** | **Integration**

Suggested reading: 'Stewart' Chapters 5 and 7

Easy Questions

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1. Provide formulae for the following indefinite integrals

(a) $\displaystyle\int \sin\theta\,\mathrm{d}\theta$ (b) $\displaystyle\int \cos x\,\mathrm{d}x$ (c) $\displaystyle\int \sec^2 r\,\mathrm{d}r$

(d) $\displaystyle\int s^{13}\,\mathrm{d}s$ (e) $\displaystyle\int \sqrt[7]{t}\,\mathrm{d}t$ (f) $\displaystyle\int \frac{1}{u}\,\mathrm{d}u$

(g) $\displaystyle\int \frac{1}{|v|}\,\mathrm{d}v$ (h) $\displaystyle\int \sinh w\,\mathrm{d}w$ (i) $\displaystyle\int \cosh x\,\mathrm{d}x$

(j) $\displaystyle\int \mathrm{sech}^2 y\,\mathrm{d}y$ (k) $\displaystyle\int e^z\,\mathrm{d}z$ (l) $\displaystyle\int \frac{1}{\sqrt{r^2-1}}\,\mathrm{d}r$

(m) $\displaystyle\int \frac{1}{\sqrt{s^2+1}}\,\mathrm{d}s$ (n) $\displaystyle\int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$ (0) $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$

(p) $\displaystyle\int \frac{1}{1-u^2}\,\mathrm{d}u$ (for $|u|<1$) (q) $\displaystyle\int \frac{1}{1-v^2}\,\mathrm{d}v$ (for $|v|>1$) (r) $\displaystyle\int \frac{1}{1-w^2}\,\mathrm{d}w$ (for $|w|\neq 1$)

Try to memorise all of these basic integrals.

2. Calculate the following definite integrals.

(a) $\displaystyle\int_{-\pi/2}^{\pi} \sin\theta\,\mathrm{d}\theta$ (b) $\displaystyle\int_{-\pi/2}^{\pi} \cos x\,\mathrm{d}x$ (c) $\displaystyle\int_0^1 s^{13}\,\mathrm{d}s$

(d) $\displaystyle\int_0^{128} \sqrt[7]{t}\,\mathrm{d}t$ (e) $\displaystyle\int_1^e \frac{1}{u}\,\mathrm{d}u$ (f) $\displaystyle\int_0^{\ln 2} \sinh w\,\mathrm{d}w$

(g) $\displaystyle\int_0^{\ln 3} \cosh x\,\mathrm{d}x$ (h) $\displaystyle\int_0^{\ln 5} \mathrm{sech}^2 y\,\mathrm{d}y$ (i) $\displaystyle\int_{-1}^{1} e^z\,\mathrm{d}z$

(j) $\displaystyle\int_1^{\cosh 2} \frac{1}{\sqrt{r^2-1}}\,\mathrm{d}r$ (k) $\displaystyle\int_0^{\sinh 3} \frac{1}{\sqrt{s^2+1}}\,\mathrm{d}s$ (l) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$

(m) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$

Standard Questions

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3. (a) By direct differentiation show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\left|\dfrac{x-1}{x+1}\right|+C\right)=\dfrac{1}{x^2-1}$

   i. for any $x>1$
   ii. for any $|x|<1$
   iii. for any $x<-1$

   (b) By direct differentiation show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left|x+\sqrt{x^2+1}\right|+C\right)=\dfrac{1}{\sqrt{x^2+1}}$ for any $x\in\mathbb{R}$

   ★(c) By direct differentiation show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left|x+\sqrt{x^2-1}\right|+C\right)=\dfrac{1}{\sqrt{x^2-1}}$

   i. for any $x\geq 1$
   ii. for any $x\leq -1$

This confirms that $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2\pm 1}}=\ln\left|x+\sqrt{x^2\pm 1}\right|+C$ and $\displaystyle\int \frac{\mathrm{d}x}{x^2-1}=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C.$

4. Provide formulae for the following indefinite integrals

(a) $\displaystyle\int \tfrac{3}{2}\sin(4\theta-\pi)\,\mathrm{d}\theta$ (b) $\displaystyle\int \left(\sinh\frac{3-w}{2}+\cosh\frac{w-2}{3}+\operatorname{sech}^2\frac{w-1}{4}+e^{5w-1/5}\right)\mathrm{d}w$

★(c) $\displaystyle\int \left(3+7(4s-\tfrac{1}{3})^4\right)\mathrm{d}s$ ★(d) $\displaystyle\int \left(\frac{\pi}{\sqrt{(2-r)^2-1}}-\frac{2}{\sqrt{(r-3)^2+1}}+\frac{3}{\sqrt{1-(1-2r)^2}}\right)\mathrm{d}r$

(e) $\displaystyle\int \left(\sqrt[5]{2t-3}-\frac{4}{5-t/2}\right)\mathrm{d}t$ ★(f) $\displaystyle\int \left(\frac{3}{1-(2+x/2)^2}-\frac{2}{1+(3-x/3)^2}\right)\mathrm{d}x$

5. Find the following integrals

(a) $\displaystyle\int \frac{3}{8-4x+x^2}\,\mathrm{d}x$ (b) $\displaystyle\int \frac{3}{\sqrt{12x-6-4x^2}}\,\mathrm{d}x$ (c) $\displaystyle\int \frac{2}{9x^2-6x-3}\,\mathrm{d}x$

★(d) $\displaystyle\int \frac{7}{\sqrt{25x^2-40x+16}}\,\mathrm{d}x$ (e) $\displaystyle\int \frac{4}{\sqrt{x^2-6x+7}}\,\mathrm{d}x$ ★(f) $\displaystyle\int \frac{5}{\sqrt{4x^2+4x+10}}\,\mathrm{d}x$

6. Show that the following recursion relations hold for the integrals given.

★(a) $K_n = nK_{n-1}$ for $n>0$, where $K_n = \displaystyle\int_0^\infty x^n e^{-x}\,\mathrm{d}x$.

i. by direct integration calculate K_0 and hence find the value of K_n for any $n\in\mathbb{N}$.

ii. given that $K_{1/2}=\frac{1}{2}\sqrt{\pi}$ what is the value of $K_{7/2}$?

(In fact, the Gamma function, defined by $\Gamma(r+1)=\displaystyle\int_0^\infty x^r e^{-x}\,\mathrm{d}x$ extends the factorial $r!$ to $r\notin\mathbb{N}$)

(b) $I_n = \dfrac{n-1}{n}I_{n-2}-\dfrac{1}{n}\cos x\sin^{n-1}x$ where $I_n=\displaystyle\int \sin^n x\,\mathrm{d}x$. Why is this true only for $n\geq 2$?

★i. by direct integration calculate I_0 and use the formula to find $\displaystyle\int \sin^6 x\,\mathrm{d}x$

ii. by direct integration calculate I_1 and use the formula to find $\displaystyle\int \sin^7 x\,\mathrm{d}x$

(c) $J_n = \dfrac{n-1}{n}J_{n-2}+\dfrac{1}{n}\sin x\cos^{n-1}x$ where $I_n=\displaystyle\int \cos^n x\,\mathrm{d}x$. Why is this true only for $n\geq 2$?

i. by direct integration calculate J_1 and use the formula to find $\displaystyle\int \cos^5 x\,\mathrm{d}x$

ii. by direct integration calculate J_0 and use the formula to find $\displaystyle\int \cos^4 x\,\mathrm{d}x$

7. Determine whether or not the following integrals converge and, if they do, evaluate the integrals.

(a) $\displaystyle\int_{-e}^{e^2}\ln|t|\,\mathrm{d}t$ (b) $\displaystyle\int_0^\pi \sec^2\theta\,\mathrm{d}\theta$ (c) $\displaystyle\int_0^9 \frac{\mathrm{d}x}{\sqrt[3]{(x-1)^2}}$ ★(d) $\displaystyle\int_{-1}^1 \frac{\mathrm{d}u}{1-u^2}$ ★(e) $\displaystyle\int_{-1}^1 \frac{\mathrm{d}v}{\sqrt{1-v^2}}$

Harder Questions

8. (a) Confirm, by differentiating, that $\displaystyle\int f'(|x|)\,\mathrm{d}x = \frac{x}{|x|}f(|x|)+C$ for $x\neq 0$.

i. Without splitting the domain of integration, use this formula to calculate $\int_{-\pi/2}^{\pi/2}\sin|x|\,\mathrm{d}x$.

ii. Now calculate the same integral by splitting the domain, $\int_{-\pi/2}^{0}\sin|x|\,\mathrm{d}x+\int_0^{\pi/2}\sin|x|\,\mathrm{d}x$.

Which of these results is the correct one? (*Hint.* It may help to draw a graph of $\sin|x|$.)

(b) Also confirm, by differentiating, that $\displaystyle\int f'(|x|)\,\mathrm{d}x = \frac{x}{|x|}\big(f(|x|)-f(0)\big)+C$ for $x\neq 0$.

i. Without splitting the domain of integration, use this formula to calculate $\int_{-\pi/2}^{\pi/2}\sin|x|\,\mathrm{d}x$.

ii. Now calculate the same integral by splitting the domain, $\int_{-\pi/2}^{0}\sin|x|\,\mathrm{d}x+\int_0^{\pi/2}\sin|x|\,\mathrm{d}x$.

(c) Can you explain why the two formulae for $\displaystyle\int f'(|x|)\,\mathrm{d}x$ differ when integrating across $x=0$?

Suggested reading for week 4: 'Stewart' Chapters 5, 6 and 7