Problem Sheet for Week 4 | *MT1121* | **Derivatives, Series, Complex Numbers**

Suggested reading: 'Stewart' Chapters 3, 4, 11 and Appendix G

Easy Questions

1. A straight line passing through the point (x_0, y_0) with slope m is $y - y_0 = m(x - x_0)$
   (a) What is the equation of the straight line parallel to $y = \frac{1}{4}x$ passing through $(-1, \pi)$?
   (b) What is the equation of the straight line at right angles to $y = \frac{1}{4}x$ passing through $(-1, \pi)$?

2. Find all critical points (if any) of the following functions and say whether each is a maximum, minimum or point of inflection
   (a) $\dfrac{1}{1+x}$ (b) $1 + x^4$ (c) $3x + x^3$ (d) $3x - x^3$
   (e) $\dfrac{x}{1-x}$ (f) $1 - x^{-4}$ (g) x^5 (h) $2 - x^6$

3. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ write the following complex numbers in the form $re^{i\theta}$. Sketch all of the points on a single diagram of the complex plane.
   (a) 1 (b) $-i$ (c) i (d) -1
   (e) $1 + i$ (f) $1 - i$ (g) $-1 + i$ (h) $-1 - i$
   (i) $1 + i\sqrt{3}$ (j) $\sqrt{3} - i$ (k) $-1 + i\sqrt{3}$ (l) $-\sqrt{3} - i$
   Are the answers you have given the only possible answers?

4. Write down the Taylor series expansions about $x = 0$ of the following functions
   (a) e^x (b) $\sin x$ (c) $\cos x$ (d) $\ln(1 + x)$
   You should know these Taylor expansions by heart; if not, then try to memorise them.

Standard Questions

5. Using the results you have given in question 3, what are the values of the following?
   (a) i^i ⋆(b) i^{-i} (c) $(-i)^i$ (d) $\ln i$ ⋆(e) $\ln(-1)$ (f) $\ln(-\sqrt{3} - i)$
   Are these the only possible values in each case.

⋆6. Sketch the function $\dfrac{8t}{4+t^2}$ for $t \in \mathbb{R}$, making sure to locate all critical points.

7. ⋆(a) Show that the curve A given by $2x = \dfrac{y^2}{2} - 2$ passes through $(x, y) = (0, 2)$ and $\left(-\frac{1}{2}, \sqrt{2}\right)$
   ⋆(b) For what values of b and c does the curve B, given by $2x = b - \dfrac{y^2}{c}$, intersect the curve A at right angles at $(0, 2)$?
   (c) For what values of p and q does the curve P, given by $2x = p - \dfrac{y^2}{q}$, intersect the curve A at right angles at $\left(-\frac{1}{2}, \sqrt{2}\right)$?

8. Find all critical points of the function $x^r e^{1-x}$ with $x \geq 0$ in the following cases:
   (a) the case $r = 1$. Sketch the function.
   (b) cases for which $0 < r < 1$. Sketch the function in the case $r = 1/2$.
   ⋆(c) cases for which $r > 1$. Sketch the function in the case $r = 2$.

   Note. Your sketches should take into account that each function has the value 1 at $x = 1$.

9. Find all of the roots of the following equations (in the form $re^{i\theta}$)
In each case, sketch the roots on a diagram of the complex plane.

(a) $z^2 = i$ (b) $z^3 = -1$ ⋆(c) $z^4 = -1 + i\sqrt{3}$ (d) $z^5 = -1 - i$ (e) $z^6 = -i$

⋆10. (a) Use Euler's formula to obtain the identity $\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

(b) Find a corresponding identity for $\sin(4\theta)$?

(c) From these identities, deduce similar identities for $\cosh(4A)$ and $\sinh(4A)$.

11. Use whatever means seems appropriate to find a formula for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in each of these cases:

(a) $y = e^{x^2}\sqrt{1+x}\sin^2 x$ (b) $x^2 - 4xy + y^2 = \dfrac{2}{xy}$ (c) $x = \cos\theta$ and $y = \sin\theta$

⋆(d) $y = t^3 - t$ and $x = t^2 - t$ ⋆(e) $y - e^{-xy} = x - e^{xy}$ (f) $y = \dfrac{x^2e^{1-x}\sqrt{x^2+2}}{(x+3)(x^2-1)}$

12. True or false. Briefly explain each answer.

(a) $e^{i\pi} + 1 = 0$ (b) $e^{i2\pi} = 1$ (c) $e^{i\pi/3} + e^{-i\pi/3} = 1$ (d) $e^{i2\pi/3} + e^{-i2\pi/3} + 1 = 0$

(e) A quadratic equation of the form $az^2 + bz + c = 0$ with real coefficients a, b and c always has exactly two roots which may be real or complex.

(f) It is impossible for the the quadratic equation in part (e) to have one real root and one non-real root.

⋆13. (a) Find the Taylor expansion of $\dfrac{1}{1+x}$ about $x = 0$.

(b) Use this to find the Taylor expansion for $\tan^{-1} z$ about $z = 0$ (see the hint below).

(c) What should be the radius of convergence of this series?

Hint. Recall that $\dfrac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z = \dfrac{1}{1+z^2}$ and substitute $x = z^2$ into the expansion of part (a).
The result is a Taylor expansion for $\dfrac{\mathrm{d}}{\mathrm{d}z}\tan^{-1} z$ which can be integrated.

Harder Questions

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14. (a) Polar coordinates (r, θ) are defined parametrically so that $x = r\cos\theta$ and $y = r\sin\theta$

i. treating θ as a constant, use parametric differentiation with respect to r to show that $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta = \tan\theta$

ii. on any curve for which r is held constant, find the formula for $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_r$

iii. show that any curve with constant r intersects any curve with constant θ at right angles.

(b) Parabolic coordinates (u, v) are defined parametrically so that $2x = u^2 - v^2$ and $y = uv$.

i. treating v as a constant, use parametric differentiation to show that $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_v = v/u$

ii. on any curve for which u is held constant, find the formula for $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_u$

iii. show that any curve with constant v intersects any curve with constant u at right angles.

iv. by eliminating u show that curves of constant v are given by $2x = \dfrac{y^2}{v^2} - v^2$. Similarly find the curves of constant u and use these to sketch both families of curves.

(c) Elliptic coordinates (s, t) are defined such that $x = \cosh s\cos t$ and $y = \sinh s\sin t$.
Show that any curve with constant s intersects any curve with constant t at right angles.
Sketch the families of curves with constant s and with constant t.

Suggested reading for week 4: 'Stewart' Chapters 5 and 7