Sample Answer Sheet for Week 3 — *MT1121* — Limits and Derivatives

Easy Questions

1. (a) $-6(1-x^2)^2$ (b) $2\sin t\cos t$ (c) $\dfrac{1}{(1-x)^2}$ (d) $2\theta\tan\theta+\theta^2\sec^2\theta$

(e) $\dfrac{3u^2+1}{2\sqrt{u^3+u}}$ (f) $-\,e^{-t}\cosh t\sinh t + e^{-t}\sinh^2 t + e^{-t}\cosh^2 t$ (g) $3x^2\exp(-2x) - 2x^3\exp(-2x)$ (h) $-\,2z^{-3}\ln(1+z^2) + z^{-2}\dfrac{2z}{1+z^2}$

2. (a) $\lim_{x\to\infty}\sin^{-1}\dfrac{1-x}{1+x}$. For large x: $\dfrac{1-x}{1+x}=\dfrac{2-1-x}{1+x}=-1+\dfrac{2}{1+x}\in[-1,1]$ (domain of $\sin^{-1}$).

$\lim_{x\to\infty}\sin^{-1}\dfrac{1-x}{1+x}=\sin^{-1}\lim_{x\to\infty}\dfrac{1/x-1}{1/x+1}=\sin^{-1}\left\{\dfrac{-1}{1}\right\}=\sin^{-1}(-1)=-\dfrac{\pi}{2}$

(b) $\lim_{x\to-\infty}\dfrac{\ln(1+e^x)}{x}=\left\{\dfrac{\ln 1}{-\infty}\right\}=\left\{\dfrac{0}{-\infty}\right\}=0$

(c) $\lim_{x\to\infty}\dfrac{4+4x^2-x^4}{3+x^4}=\lim_{x\to\infty}\dfrac{4/x^4+4/x^2-1}{3/x^4+1}=\left\{\dfrac{-1}{1}\right\}=-1$

(d) $\lim_{x\to 0}\dfrac{\sinh(x)}{x}=\left\{\dfrac{0}{0}\right\}=\lim_{x\to 0}\dfrac{\cosh(x)}{1}=\left\{\dfrac{1}{1}\right\}=1$

3. (a) $f(x)=\begin{cases}e^{-1/x^2} & \text{if } x\neq 0\\ A & \text{if } x=0.\end{cases}$ $\lim_{x\to 0}e^{-1/x^2}=\left\{e^{-\infty}\right\}=0.$ So choose $A=0$.

(b) $g(x)=\begin{cases}1/\sqrt{x^4} & \text{if } x\neq 0\\ A & \text{if } x=0.\end{cases}$ $\lim_{x\to 0}\dfrac{1}{\sqrt{x^4}}=\lim_{x\to 0}\dfrac{1}{|x|^2}=\left\{\dfrac{1}{0}\right\}=\infty.$

So no choice of value of A can make $g(x)$ continuous.

4. (a) Consecutive derivatives of $\dfrac{1}{1+x}$ are: $\dfrac{-1}{(1+x)^2},\ \dfrac{2}{(1+x)^3},\ \dfrac{-2\times 3}{(1+x)^4},\ \dfrac{2\times 3\times 4}{(1+x)^5},\ \ldots$

So we can write: $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\dfrac{1}{1+x}=\dfrac{(-1)^n n!}{(1+x)^{n+1}}$

(b) Consecutive derivatives of $\dfrac{1}{1-x}$ are: $\dfrac{1}{(1+x)^2},\ \dfrac{2}{(1+x)^3},\ \dfrac{2\times 3}{(1+x)^4},\ \dfrac{2\times 3\times 4}{(1+x)^5},\ \ldots$

So we can write: $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}\dfrac{1}{1+x}=\dfrac{n!}{(1+x)^{n+1}}$

Standard Questions

5. ⋆(a) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(x\sqrt{1+1/x}-x\right)=\left\{\infty-\infty\right\}=\lim_{x\to\infty}\dfrac{\sqrt{1+1/x}-1}{1/x}=\left\{\dfrac{0}{0}\right\}$

$=\lim_{x\to\infty}\dfrac{\frac{1}{2}/\sqrt{1+1/x}(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}\dfrac{1/2}{\sqrt{1+1/x}}=\dfrac{1}{2}$

***Alternatively*:**

$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\dfrac{\cancel{x}^2+x-\cancel{x}^2}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\dfrac{1}{\sqrt{1+1/x}+1}=\dfrac{1}{2}$

(b) $\lim_{x\to\infty}x\big(\ln(x+1)-\ln x\big)=\lim_{x\to\infty}x\ln\dfrac{x+1}{x}=\lim_{x\to\infty}\ln(1+1/x)^x=\ln e=1$

⋆(c) For $n,\,m\in\mathbb{Z}$: $\lim_{x\to\pi}\dfrac{\sin(mx)}{\sin(nx)}=\left\{\dfrac{0}{0}\right\}=\lim_{x\to\pi}\dfrac{m\cos(mx)}{n\cos(nx)}=\dfrac{m}{n}\dfrac{\cos(m\pi)}{\cos(n\pi)}=\dfrac{m}{n}(-1)^{m-n}$

Note. $\cos(m\pi)=(-1)^m$, $\cos(n\pi)=(-1)^n$, so $\dfrac{\cos(m\pi)}{\cos(n\pi)}=\dfrac{(-1)^m}{(-1)^n}=(-1)^{m-n}$.

6. (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^2 x\cos^3 x=2\sin x\cos^4 x-3\sin^3 x\cos^2 x$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}u}\,(au+b)^{3/2} = \tfrac{3}{2}a(au+b)^{1/2}$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}a}\,(au+b)^{3/2} = \tfrac{3}{2}u(au+b)^{1/2}$

⋆(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{\dfrac{a+x}{a-x}} = \dfrac{1}{2}\sqrt{\dfrac{a-x}{a+x}} \times \dfrac{a-\cancel{x}+(a+\cancel{x})}{(a-x)^2} = \dfrac{a}{(a-x)^2}\sqrt{\dfrac{a-x}{a+x}}$

⋆(e) $\dfrac{\mathrm{d}}{\mathrm{d}p}\,p\sin(s/p) = \sin(s/p) - \cancel{p}\dfrac{s}{p^{\cancel{2}}}\cos(s/p) = \sin(s/p) - \dfrac{s}{p}\cos(s/p)$

⋆(f) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\,\theta^n\sin^m(a\theta+b) = n\,\theta^{n-1}\sin^m(a\theta+b) + ma\,\theta^n\sin^{m-1}(a\theta+b)\cos(a\theta+b)$

7. (a) $g(t) = \begin{cases} t^t & \text{if } t>0 \\ A & \text{if } t=0. \end{cases}$ $\quad \lim\limits_{t\to0^+} t^t = \lim\limits_{t\to0^+} e^{t\ln t} = \left\{e^{0\times(-\infty)}\right\}.$

$\lim\limits_{t\to0^+} t\ln t = \lim\limits_{t\to0^+}\dfrac{\ln t}{1/t} = \left\{\dfrac{-\infty}{\infty}\right\} = \lim\limits_{t\to0^+}\dfrac{1/t}{-1/t^2} = \lim\limits_{t\to0^+}\dfrac{t}{-1} = 0.$

Hence $\lim\limits_{t\to0^+} e^{t\ln t} = e^0 = 1$. So setting $A=1$ makes $g(t)$ continuous on $[0,\infty)$.

⋆(b) $g(t) = \begin{cases} \sqrt{t}\,\cos\left(1/\sqrt{t}\right) & \text{if } t>0 \\ A & \text{if } t=0. \end{cases}$ $\quad$ Since $\left|\sqrt{t}\,\cos\left(1/\sqrt{t}\right)\right| \le \sqrt{t}$ for $t\ge0$ we can note that $\lim\limits_{t\to0^+}\left|\sqrt{t}\,\cos\left(1/\sqrt{t}\right)\right| \le \lim\limits_{t\to0^+}\sqrt{t} = 0$. This requires that $\lim\limits_{t\to0^+}\sqrt{t}\,\cos\left(1/\sqrt{t}\right) = 0$.

Hence, setting $A=0$ makes $g(t)$ continuous on $[0,\infty)$.

8. (a) $\lim\limits_{x\to\infty}\dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}} = \lim\limits_{x\to\infty}\dfrac{(2-3/x)^{20}(3+2/x)^{30}}{(2+1/x)^{50}} = \dfrac{2^{20}3^{30}}{2^{50}} = \left(\tfrac{3}{2}\right)^{30}.$

(b) $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right) = \{\infty-\infty\} = \lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{1+x^{-1}\sqrt{x+\sqrt{x}}}-1\right) = \{\infty\times0\}$

$= \lim\limits_{x\to\infty}\dfrac{\left(1+x^{-1/2}(1+x^{-1/2})^{1/2}\right)^{1/2}-1}{1/\sqrt{x}} = \left\{\dfrac{0}{0}\right\}$

$= \lim\limits_{x\to\infty}\dfrac{\frac{1}{2}\left(1+x^{-1/2}(1+x^{-1/2})^{1/2}\right)^{-1/2}\left(-\frac{1}{2}x^{-3/2}(1+x^{-1/2})^{1/2}+\frac{1}{2}x^{-1/2}(1+x^{-1/2})^{-1/2}(-\frac{1}{2}x^{-3/2})\right)}{-\frac{1}{2}x^{-3/2}}$

$= \tfrac{1}{2}$ (after dividing above and below by $-\tfrac{1}{2}x^{-3/2}$)

***Alternatively*:**

$\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right) = \lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

$= \lim\limits_{x\to\infty}\dfrac{\cancel{x}+\sqrt{x+\sqrt{x}}-\cancel{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim\limits_{x\to\infty}\dfrac{\sqrt{1+x^{-1}\sqrt{x}}}{\sqrt{1+x^{-1}\sqrt{x+\sqrt{x}}}+1} = \tfrac{1}{2}.$

(c) $\lim\limits_{x\to0} x^{-100}e^{-1/x^2} = \{\infty\times0\} = \lim\limits_{x\to0}\dfrac{x^{-100}}{e^{1/x^2}} = \left\{\dfrac{\infty}{\infty}\right\} = \lim\limits_{x\to0}\dfrac{-100\,x^{-101}}{e^{1/x^2}(-2/x^3)}$

$= \lim\limits_{x\to0} 50\,\dfrac{x^{-98}}{e^{1/x^2}} = \lim\limits_{x\to0} 50\times49\,\dfrac{x^{-96}}{e^{1/x^2}} = \cdots = \lim\limits_{x\to0}\dfrac{50!}{e^{1/x^2}} = 0.$

9. $R(x) = \dfrac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0}$

For $n=m$: $\lim\limits_{x\to\infty} R(x) = \lim\limits_{x\to\infty}\dfrac{x^n(a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n)}{x^n(b_n + b_{n-1}/x + \cdots + b_1/x^{n-1} + b_0/x^n)} = \dfrac{a_n}{b_m}$

For $n<m$: $\lim\limits_{x\to\infty} R(x) = \lim\limits_{x\to\infty} x^{n-m}\dfrac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_m + b_{m-1}/x + \cdots + b_1/x^{m-1} + b_0/x^m} = 0$

For $n>m$: $\lim\limits_{x\to\infty} R(x) = \lim\limits_{x\to\infty} x^{n-m}\dfrac{a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n}{b_m + b_{m-1}/x + \cdots + b_1/x^{m-1} + b_0/x^m} = \infty$

⋆10. Are the following true or false?

(a) $\lim_{x\to 0}\dfrac{1}{1+e^{1/x}}=0$. This is false because $\lim_{x\to 0^+}\dfrac{1}{1+e^{1/x}}=0$ and $\lim_{x\to 0^-}\dfrac{1}{1+e^{1/x}}=\frac{1}{2}$.
The two-sided limit does not exist.

(b) $\lim_{x\to 1}\tan^{-1}\dfrac{1}{1-x}=\dfrac{\pi}{2}$. This is false because $\lim_{x\to 1^-}\tan^{-1}\dfrac{1}{1-x}=\left\{\tan^{-1}\infty\right\}=\frac{\pi}{2}$ and

$\lim_{x\to 1^+}\tan^{-1}\dfrac{1}{1-x}=\left\{\tan^{-1}(-\infty)\right\}=-\frac{\pi}{2}$. The two-sided limit does not exist.

(c) $\lim_{x\to 0}x\sin\dfrac{1}{x}=0$ is true because $\lim_{x\to 0}\left|x\sin\dfrac{1}{x}\right|\le\lim_{x\to 0}|x|=0$, requiring $\lim_{x\to 0}x\sin\dfrac{1}{x}=0$.

11. $f(x)=1+x^2$ and $g(x)=\begin{cases}1 & \text{for } x\ge 0\\ -1 & \text{for } x<0.\end{cases}$

$f\big(g(x)\big)=\left\{\begin{matrix}1+1^2 & \text{for } x\ge 0\\ 1+(-1)^2 & \text{for } x<0\end{matrix}\right\}=2$ for all $x\in\mathbb{R}$, so that $f\big(g(x)\big)$ is continuous

$f(x)=1+x^2>0$ for all $x\in\mathbb{R}$ so $g\big(f(x)\big)=1$ for all x, so that $g\big(f(x)\big)$ is continuous.

12. (a) $x-[x]$ is discontinuous at every integer $x\in\mathbb{Z}$

(b) $[\sin x]$ is discontinuous where $\sin x=0$ (i.e. $x=k\pi$ for $k\in\mathbb{Z}$)
and where $\sin x=1$ (i.e. $x=\frac{\pi}{2}+2k\pi$ for $k\in\mathbb{Z}$).
It is not discontinuous where $\sin x=-1$ (why?).

(c) $[1/x]$ is discontinuous where $1/x\in\mathbb{Z}$ and at $x=0$.

(d) $(-1)^{[x]}$ is discontinuous at every integer $x\in\mathbb{Z}$, where the value swops between -1 and 1.

(e) $(-1)^{2[\ln x^2]}=\big((-1)^2\big)^{[\ln x^2]}=1^{[\ln x^2]}$ is only discontinuous at $x=0$ where it is not defined.

Harder Questions

13. (a) Given that $\lim_{x\to\infty}\big(f(x)-(ax+b)\big)=0$
we must have $\lim_{x\to\infty}\big(f(x)-ax\big)-b=0$ and $\lim_{x\to\infty}\left(\dfrac{f(x)}{x}-\dfrac{ax+b}{x}\right)=0$
so that $\lim_{x\to\infty}\big(f(x)-ax\big)=b$ and $\lim_{x\to\infty}\dfrac{f(x)}{x}=\lim_{x\to\infty}\dfrac{ax+b}{x}=a$.

(b) Given that $\lim_{x\to\infty}\big(f(x)-ax\big)=b$ it follows that $\lim_{x\to\infty}\big(f(x)-ax\big)-b=\lim_{x\to\infty}\big(f(x)-(ax+b)\big)=0$
which means that $y=f(x)$ has the asymptote $y=ax+b$ as $x\to\infty$.

(c) Since $\lim_{x\to\infty}\dfrac{\sqrt{4x^2-8x}-x}{x}=\lim_{x\to\infty}\big(\sqrt{4-8/x}-1\big)=1$ and

$$\lim_{x\to\infty}\big((\sqrt{4x^2-8x}-x)-x\big)=\lim_{x\to\infty}\frac{(\sqrt{4x^2-8x}-2x)(\sqrt{4x^2-8x}+2x)}{\sqrt{4x^2-8x}+2x}=\lim_{x\to\infty}\frac{4x^2-8x-4x^2}{\sqrt{4x^2-8x}+2x}$$
$$=\lim_{x\to\infty}\frac{-8x}{\sqrt{4x^2-8x}+2x}=\lim_{x\to\infty}\frac{-8}{\sqrt{4-8/x}+2}=\frac{-8}{4}=-2,$$

the asymptote as $x\to\infty$ is $x-2$.

Also, since $\lim_{x\to-\infty}\dfrac{\sqrt{4x^2-8x}-x}{x}=\lim_{x\to-\infty}\big(-\sqrt{4-8/x}-1\big)=-3$ and

$$\lim_{x\to-\infty}\big((\sqrt{4x^2-8x}-x)+3x\big)=\lim_{x\to-\infty}\big(\sqrt{4x^2-8x}+2x\big)=\lim_{x\to-\infty}\big(2|x|\sqrt{1-2/x}-2|x|\big)$$
$$=\lim_{x\to-\infty}2\frac{\sqrt{1-2/x}-1}{-1/x}=2\lim_{x\to-\infty}\frac{\frac{1}{2}/\sqrt{1-2/x}\times 2/x^2}{1/x^2}=2,$$

the asymptote as $x\to\infty$ is $-3x+2$.

14. (a) False. A function such as $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is discontinuous at $x = 1$, but f^2 is continuous.

(b) True. If f is continuous, then $|f|$ is also continuous.

(c) True. If f is continuous and g is discontinuous at a point, then $(f + g)$ is discontinuous at that point.

(d) False. If f is continuous and equal to zero and g is discontinuous at a point, then (fg) is always zero and so is continuous wherever it is defined.

(e) False. If f and g are both discontinuous at a point, then $(f + g)$ is not necessarily discontinuous at that point. We could, for example have $f = -g$ so that $(f + g)$ is always zero.

(f) False. If f and g are both discontinuous at a point, then (fg) is not necessarily discontinuous at that point. We could, for example have $f = 1/g$ so that (fg) is always equal to 1.