Problem Sheet for Week 3 | **MT1121** | **Limits and Derivatives**

***Reminder*:**

- attempt all 'easy' questions without delay and discuss any difficulties in your supervision class
- attempt the starred 'standard' questions and hand in your answers (complete or incomplete) to your supervisor before your supervision class
- the 'harder' questions are optional, offering an additional challenge as well as introducing problems of further interest

***Suggested reading*:** 'Stewart' pages 92–315 (chapters 2, 3 and 4)

Easy Questions

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1. Differentiate the following functions. You should be able to write down the answers at once without any intermediate working. If necessary, practice on examples from the textbook until you can do this with confidence.

   (a) $(1-x^2)^3$ (b) $\sin^2 t$ (c) $\dfrac{1}{1-x}$ (d) $\theta^2 \tan\theta$

   (e) $\sqrt{u^3+u}$ (f) $e^{-t}\cosh t \sinh t$ (g) $x^3\exp(-2x)$ (h) $z^{-2}\ln(1+z^2)$

2. Find the following limits

   (a) $\displaystyle\lim_{x\to\infty} \sin^{-1}\frac{1-x}{1+x}$ (b) $\displaystyle\lim_{x\to-\infty} \frac{\ln(1+e^x)}{x}$ (c) $\displaystyle\lim_{x\to\infty} \frac{4+4x^2-x^4}{3+x^4}$ (d) $\displaystyle\lim_{x\to 0} \frac{\sinh(x)}{x}$

3. Find a value A (if it exists) for which the following functions will be continuous for all $x \in \mathbb{R}$

   (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$ (b) $g(x) = \begin{cases} 1/\sqrt{x^4} & \text{if } x \neq 0 \\ A & \text{if } x = 0 \end{cases}$

4. Find a general formula for the n^{th} derivative of (a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1-x}$

Standard Questions

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5. Find the following limits

   ⋆(a) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$ (b) $\displaystyle\lim_{x\to\infty} x\big(\ln(x+1)-\ln x\big)$ ⋆(c) $\displaystyle\lim_{x\to\pi} \frac{\sin(mx)}{\sin(nx)}$ for $n, m \in \mathbb{Z}$

6. Find the following derivatives. You should work efficiently, writing down as few intermediate steps as possible.

   (a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin^2 x\cos^3 x$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}u}(au+b)^{3/2}$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}a}(au+b)^{3/2}$

   ⋆(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{\dfrac{a+x}{a-x}}$ ⋆(e) $\dfrac{\mathrm{d}}{\mathrm{d}p}\, p\sin(s/p)$ ⋆(f) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\,\theta^n\sin^m(a\theta+b)$

7. Find a value A (if it exists) for which the following functions will be continuous for $t \geq 0$

   (a) $g(t) = \begin{cases} t^t & \text{if } t > 0 \\ A & \text{if } t = 0 \end{cases}$ ⋆(b) $g(t) = \begin{cases} \sqrt{t}\cos\left(1/\sqrt{t}\right) & \text{if } t > 0 \\ A & \text{if } t = 0 \end{cases}$

   Hint. You can write t^t as $e^{t\ln t}$. Also, you can note that $\left|\sqrt{t}\cos\left(1/\sqrt{t}\right)\right| \leq \sqrt{t}$.

8. Find the following limits.

   (a) $\displaystyle\lim_{x\to\infty} \frac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}}$ (b) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ (c) $\displaystyle\lim_{x\to 0} x^{-100}e^{-1/x^2}$

9. Suppose that $R(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$ with $a_n \neq 0$ and $b_m \neq 0$. Show that

$$\lim_{x\to\infty} R(x) = \begin{cases} \infty & \text{if } n > m \\ a_n/b_m & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

$\star$10. Are the following true or false?

(a) $\displaystyle\lim_{x\to 0} \frac{1}{1+e^{1/x}} = 0$ (b) $\displaystyle\lim_{x\to 1} \tan^{-1}\frac{1}{1-x} = \frac{\pi}{2}$ (c) $\displaystyle\lim_{x\to 0} x\sin\frac{1}{x} = 0$

11. Where are the functions $f(g(x))$ and $g(f(x))$ continuous or discontinuous if

$$f(x) = 1 + x^2 \quad \text{and} \quad g(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases}$$

12. The function $[\cdot]$ represents the 'integer part' of any number, defined such that $[x]$ is the highest integer value that is less than or equal to x. Thus $[\pi] = 3$, $[e] = 2$, $[5] = 5$, $[-4.9] = -5$, etc.

Where are the following functions discontinuous, if at all

(a) $x - [x]$ (b) $[\sin x]$ (c) $[1/x]$ (d) $(-1)^{[x]}$ (e) $(-1)^{2[\ln x^2]}$

Harder Questions

13. We say that the curve $y = f(x)$ has an asymptote $y = ax + b$ as $x \to \infty$ if $\displaystyle\lim_{x\to\infty}\big(f(x) - (ax+b)\big) = 0$.

(a) Show that if $y = f(x)$ has the asymptote $y = ax + b$ as $x \to \infty$, then

$$a = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{and} \quad b = \lim_{x\to\infty}\big(f(x) - ax\big)$$

(b) Conversely, show that if a and b have finite values given by these limits then $y = f(x)$ has the asymptote $y = ax + b$ as $x \to \infty$.

(c) Find asymptotes for $\sqrt{4x^2 - 8x} - x$ both as $x \to \infty$ and as $x \to -\infty$.

14. Are the following true or false?

(a) If f is discontinuous, then f^2 is also discontinuous.

(b) If f is continuous, then $|f|$ is also continuous.

(c) If f is continuous and g is discontinuous at a point, then $(f+g)$ is discontinuous at that point.

(d) If f is continuous and g is discontinuous at a point, then (fg) is discontinuous at that point.

(e) If f and g are both discontinuous at a point, then $(f+g)$ is discontinuous at that point.

(f) If f and g are both discontinuous at a point, then (fg) is discontinuous at that point.

Suggested reading for week 4: 'Stewart' Chapters 3, 4, 11 and Appendix G