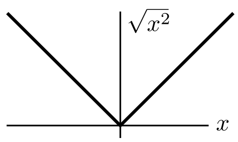
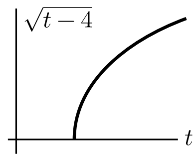
Easy Questions

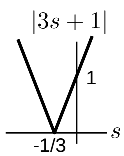
1. (a) $[-4,\infty)$ (b) $(-\infty,1)\cup(1,\infty)$ (c) $[0,\infty)$ (d) $(0,2)$

2. (a) 

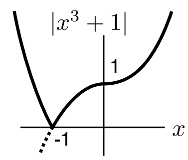
$\sqrt{x^2}=|x|$ which is simple to plot

(b) 

$\sqrt{t-4}=0$ at $t=4$, increasing with t. It is part of a rotated parabola

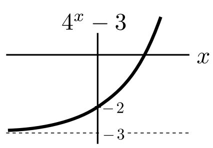
(c) 

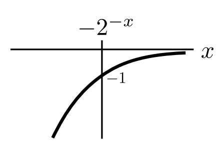
$|3s+1|=0$ at $s=-1/3$
$|3s+1|=1$ at $s=0$
sign changed for $3s+1<1$

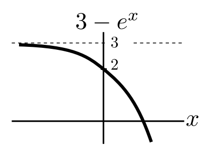
(d) 

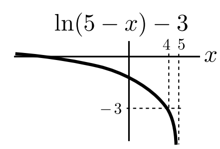
$|x^3+1|=0$ at $x=-1$
$|x^3+1|=1$ at $x=0$
function is a cubic with sign changed for $x^3+1<0$

3. (a) True (b) False. cos has range $[-1,1]$
(c) False. sec has domain $\mathbb{R}$ with points where $\cos=0$ removed (i.e. $\frac{1}{2}\pi+n\pi$ for $n\in\mathbb{Z}$).

4. (a) 

(b) 

(c) 

(d) 

5. (a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{3}$ (c) $0.54\overline{321}$ (d) $\dfrac{\pi}{3}$

6. (a) $3-\frac{1}{3}y^2$ for $y\ge 0$ (b) $\sqrt[3]{\ln y}$ for $y>0$ (c) e^y-3 for $y\in\mathbb{R}$

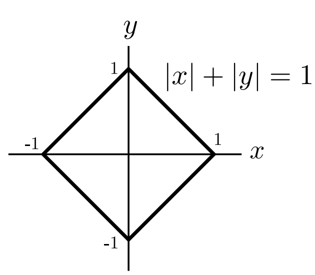
7. sum: $\sqrt{x(2-x)}+1-x^2$, domain $[0,2]$ product: $\sqrt{x(2-x)}(1-x^2)$, domain $[0,2]$

difference: $\sqrt{x(2-x)}-1+x^2$, domain $[0,2]$ or $1-x^2-\sqrt{x(2-x)}$, domain $[0,2]$

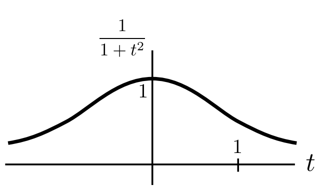
ratios: $\dfrac{1-x^2}{\sqrt{x(2-x)}}$, domain $(0,2)$ or $\dfrac{\sqrt{x(2-x)}}{1-x^2}$, domain $[0,1)\cup(1,2]$

Standard Questions

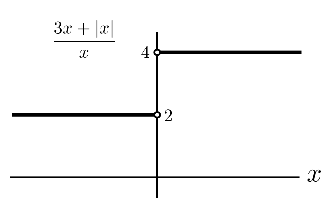
8. $|x|+|y|=1$ has the forms
$x+y=1$ for $x\ge 0$, $y\ge 0$
$x-y=1$ for $x\ge 0$, $y<0$
$-x-y=1$ for $x<0$, $y<0$
$-x+y=1$ for $x<0$, $y\ge 0$
Curve has 4 different forms (one in each quadrant)



The relation cannot represent a function because there are two values of y for each value of x between -1 and 1.

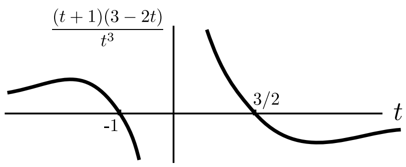
9. (a) 

$\frac{1}{1+t^2}>0$ for all $t\in\mathbb{R}$
$\frac{1}{1+t^2}=1$ at $t=0$
$\frac{1}{1+t^2}\to 0$ as $t\to\pm\infty$

$\star$ (b) 

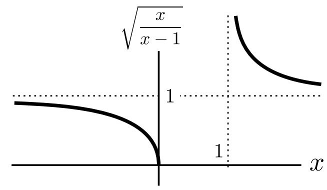
$\frac{3x+|x|}{x}=3+\frac{|x|}{x}$ and
$\frac{|x|}{x}=1$ for $x>0$, $\frac{|x|}{x}=-1$ for $x<0$
The function is not defined at $x=0$.

9. (c)



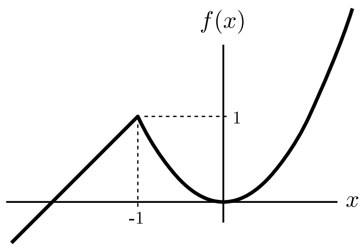
$\frac{(t+1)(3-2t)}{t^3} = 0$ at $t = -1$ and $t = \frac{3}{2}$
singularity like $1/t^3$ at $t = 0$
$\frac{(t+1)(3-2t)}{t^3} \to 0$ as $t \to \pm\infty$

$\star$ (d)



$\frac{x}{x-1} \geq 0$ for $x \leq 0$ and $x > 1$
with singularity at $x = 1$
and $\frac{x}{x-1} \to 1$ as $x \to \pm\infty$

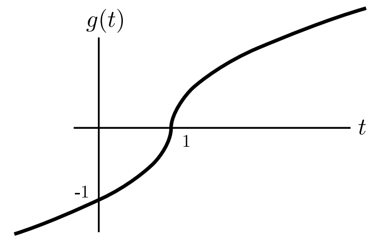
10. (a)



$f(x) = x^2$ is a parabola for $x > -1$ with base at $(0,0)$, passing through $(\pm 1, 1)$.

$f(x) = x+2$ is a straight line for $x \leq -1$ with slope 1, passing through $(-1, 1)$.

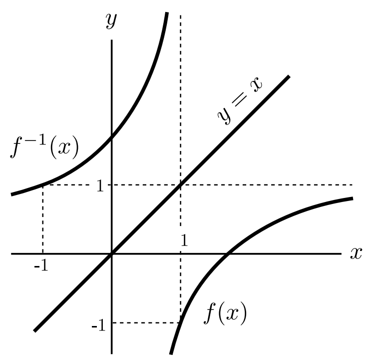
$\star$ (b)



$g(t) = \sqrt{t-1}$ is the upper half of a parabola on its side, for $t \geq 1$.

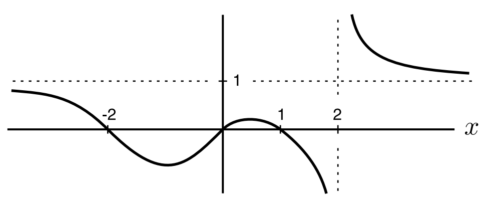
For $t < 1$, $g(t) = -\sqrt{1-t}$ is the same parabola rotated through 180°.

11. Let $y = f(x) = 1 - 2/x^2$ for $x > 0$,
having the range $y < 1$.
Solving for x gives $x = \sqrt{\frac{2}{1-y}}$
for $y < 1$, with range $x > 0$.
Thus the inverse function is
$f^{-1}(x) = \sqrt{\frac{2}{1-x}}$ with domain
$(-\infty, 1)$ and range $(0, \infty)$.



12. The function $\dfrac{x(x-1)(x+2)}{(x-2)(1+x^2)}$:

- has zeros at $x = 0, 1$ and -2
- approaches $\frac{x^3}{x^3} = 1$ as $x \to \pm\infty$
- has a singularity like $\frac{1}{x-2}$ at $x = 2$



13. $f(x) = \sqrt{3 - e^{2x}}$

(a) domain requires $3 - e^{2x} \geq 0$ or $3 \geq e^{2x}$ or $\ln 3 \geq 2x$ or $x \leq \frac{\ln 3}{2}$.
So the domain is $\left(-\infty, \frac{\ln 3}{2}\right]$.

We can note that the range is $\left[0, \sqrt{3}\right)$.

(b) Let $y = \sqrt{3 - e^{2x}}$ so that $y^2 = 3 - e^{2x}$, $e^{2x} = 3 - y^2$, $2x = \ln(3 - y^2)$
giving $x = \ln\sqrt{3 - y^2}$ for $y \in \left[0, \sqrt{3}\right)$.
So the inverse function is given by $f^{-1}(x) = \ln\sqrt{3 - x^2}$.

(c) The domain of f^{-1} is $\left[0, \sqrt{3}\right)$.

14. $\cos\left(\sin^{-1} x\right)$: The domain of $\sin^{-1}$ is $[-1,1]$ with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.
Suppose that $\theta = \sin^{-1} x$, having $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, then $x = \sin\theta$.
We know that $\cos^2\theta = 1 - \sin^2\theta$ and so, because $\cos\theta \geq 0$ for $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$, we can solve to obtain $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$.
But $\theta = \sin^{-1} x$ and so $\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$ for $x \in [-1,1]$.

15. $\sin\left(\sin^{-1} x\right)$: The domain of $\sin^{-1} x$ is $[-1,1]$ with range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.
Thus $\sin\left(\sin^{-1} x\right) = x$ provided $x \in [-1,1]$.

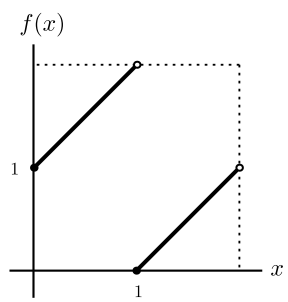
16. (a)
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \frac{(e^x+e^{-x})^2}{2^2} - \frac{(e^x-e^{-x})^2}{2^2} \\ &= \frac{(e^{2x}+2e^{x-x}+e^{-2x}) - (e^{2x}-2e^{x-x}+e^{-2x})}{4} \\ &= \frac{4e^0}{4} = 1\end{aligned}$$

Hence: $\dfrac{\cosh^2 x}{\cosh^2 x} - \dfrac{\sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$ so that $1 - \tanh^2 x = \text{sech}^2 x$ or $\tanh^2 x = 1 - \text{sech}^2 x$.

(b)
$$\cosh(a)\cosh(b) + \sinh(a)\sinh(b) = \frac{e^a+e^{-a}}{2}\frac{e^b+e^{-b}}{2} + \frac{e^a-e^{-a}}{2}\frac{e^b-e^{-b}}{2}$$
$$= \frac{(e^{a+b}+e^{a-b}+e^{b-a}+e^{-a-b}) + (e^{a+b}-e^{a-b}-e^{b-a}+e^{-a-b})}{4}$$
$$= \frac{2e^{a+b}+2e^{-(a+b)}}{4} = \frac{e^{a+b}+e^{-(a+b)}}{2} = \cosh(a+b)$$

(c)
$$\cosh(a)\sinh(b) + \sinh(a)\cosh(b) = \frac{e^a+e^{-a}}{2}\frac{e^b-e^{-b}}{2} + \frac{e^a-e^{-a}}{2}\frac{e^b+e^{-b}}{2}$$
$$= \frac{(e^{a+b}-e^{a-b}+e^{b-a}-e^{-a-b}) + (e^{a+b}+e^{a-b}-e^{b-a}-e^{-a-b})}{4}$$
$$= \frac{2e^{a+b}-2e^{-(a+b)}}{4} = \frac{e^{a+b}-e^{-(a+b)}}{2} = \sinh(a+b)$$

17. (a)



(b) The function is neither increasing nor decreasing over its domain $[0,2)$

(c) Let $y = f(x)$, then, for $0 \leq x < 1$ we have $y = x+1$, giving $x = y-1$ with $1 \leq y < 2$.

Also, for $1 \leq x < 2$ we have $y = x-1$, giving $x = y+1$ with $0 \leq y < 1$.

So the inverse is given by $f^{-1}(x) = \begin{cases} x+1 & \text{if } 0 \leq x < 1 \\ x-1 & \text{if } 1 \leq x < 2 \end{cases}$

Note. For this function we can observe that $f^{-1} = f$.

Harder Questions

18. $|x|^a + |y|^a = 1$

- The case $a = 1$ already appears in question 8.
- The case $a = 2$ amounts to having $x^2 + y^2 = 1$ which is a circle of unit radius.
- More generally, we can note that it is enough to consider only the 'first quadrant' where $x \geq 0$ and $y \geq 0$ because, as in question 8, the other quadrants just involve rotations of the first quadrant.
- In the first quadrant, we can note that for any $a > 0$, $y^a = 1 - x^a$ so that $y^a \leq 1$ requiring that $y \leq 1$. In exactly the same way, we must always have $x \leq 1$ for any value of $a > 0$.

- Thus the overall curve always lies within the square bounded by $x = \pm 1$ and $y = \pm 1$
- A further understanding of how the curve varies with a is found by looking at points along a diagonal where $y = x$ on the curve in the first quadrant. This gives $y^a + x^a = 2x^a = 1$, so that $x = y = \frac{1}{2}^{1/a}$.
- For $a = 2$ this is $1/\sqrt{2}$ as it must be for a circle.
- For $a = 1$, it is $\frac{1}{2}$ as it must be for the straight line of question 8
- If a decreases towards zero, $\frac{1}{2}^{1/a}$ involves increasing powers so that the point where the curve intersects $y = x$ approaches the origin.
- If a increases towards infinity, $\frac{1}{2}^{1/a}$ involves decreasing powers so that $\frac{1}{2}^{1/a}$ approaches unity and hence the point where the curve intersects $y = x$ approaches $(1, 1)$.

19. Supposing that $f(x) = g(x) + h(x)$ with g even and h odd, we can note that $f(-x) = g(-x) + h(-x) = g(x) - h(x)$ so that we can write

$$g(x) + h(x) = f(x) \qquad \text{and} \qquad g(x) - h(x) = f(-x)$$

Simply adding and subtracting leads to the formulae for g and h

$$g(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big) \qquad \text{and} \qquad h(x) = \tfrac{1}{2}\big(f(x) - f(-x)\big)$$

It is relatively simple to confirm that g is even and h is odd and that their sum is the function f.

20. If we have that $f(x) = f(-x)$ and that $f(x + p) = f(x)$ for any $x \in \mathbb{R}$ and $p \neq 0$, then:

(a) $g(x) = f(x - p) = f(p - x) = f(-x) = f(x)$. This shows that $f = g$ and so, because f is even, it follows that g must be even.

(b) $h(x) = f(x - \frac{1}{2}p) = f(\frac{1}{2}p - x) = f(p - \frac{1}{2}p - x) = f(-\frac{1}{2}p - x) = f(\frac{1}{2}p + x)$. Hence we have both that $h(x) = f(\frac{1}{2}p - x)$ and that $h(x) = f(\frac{1}{2}p + x)$. Changing the sign of x in the latter gives $h(-x) = f(\frac{1}{2}p - x)$ which the former shows gives $h(x) = h(-x)$. Hence h is also even.

Note. These are just examples of the ways in which (a) and (b) can be proven.
Can you find alternative arguments?

If $f(x)$ was an odd function then periodicity would ensure that $g(x)$ is odd, as above. Using very similar arguments to those above, you can also show that $h(x)$ must be odd.

An interesting corollary to this is that a function that has two points of symmetry or antisymmetry must also be periodic. An informative exercise is to sketch functions having points of symmetry or antisymmetry to see how the periodicity comes about.