Problem Sheet for Week 2 | ***MT1121*** | **Graphs and Functions**

***Reminder*:**

- attempt all 'easy' questions without delay and discuss any difficulties in your supervision class
- attempt the starred 'standard' questions and hand in your answers (complete or incomplete) to your supervisor before your supervision class
- the 'harder' questions are optional, offering an additional challenge as well as introducing problems of further interest

***Suggested reading*:** 'Stewart' pages 11–85 (chapter 1)

Easy Questions

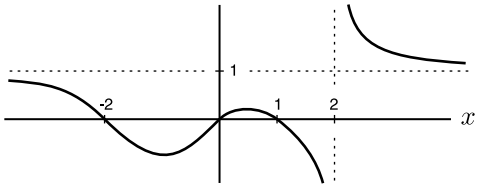
1. What are the domains of each of the following functions?
   (a) $\sqrt{x+4}$ (b) $\dfrac{t}{t-1}$ (c) $\sqrt{v}+\sqrt[3]{1-v}$ (d) $\dfrac{1}{\sqrt[4]{p(2-p)}}$
2. Sketch the graphs of the following functions. Explain very briefly how you got the graphs.
   (a) $\sqrt{x^2}$ (b) $\sqrt{t-4}$ (c) $|3s+1|$ (d) $|x^3+1|$
3. Are the following true or false? Give a corrected version of those that are false.
   (a) tan has range $\mathbb{R}$ (b) the range of cos is $\mathbb{R}$ (c) sec has domain $\mathbb{R}$
4. Sketch the graphs of the following functions (do not use a calculator).
   (a) 4^x-3 (b) -2^{-x} (c) $3-e^x$ (d) $\ln(5-x)-3$
5. Find the exact value of each expression (do not use a calculator).
   (a) $\sin^{-1}\left(\sqrt{3}/2\right)$ (b) $\tan^{-1}\sqrt{3}$ (c) $\sin\left(\sin^{-1}0.54\overline{321}\right)$ (d) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right)$
6. Find a formula for the inverse of each of the functions (be sure to identify the domain of each inverse)
   (a) $\sqrt{9-3x}$ (b) $\exp(x^3)$ (c) $\ln(x+3)$
7. Add, subtract, multiply and divide the functions $\sqrt{x(2-x)}$ and $1-x^2$.
   In each case, what is the domain?

Standard Questions

8. Sketch the graph of the relation $|x|+|y|=1$. Does this relation represent a function?
9. Sketch the graphs of the following functions. Explain very briefly how you got each graph.
   (a) $\dfrac{1}{1+t^2}$ ⋆(b) $\dfrac{3x+|x|}{x}$ (c) $\dfrac{(t+1)(3-2t)}{t^3}$ (d) $\sqrt{\dfrac{x}{x-1}}$
10. Sketch the graphs of the following functions. Explain very briefly how you got each graph.
    ⋆(a) $f(x)=\begin{cases}x+2 & \text{if } x\le -1\\ x^2 & \text{if } x>-1\end{cases}$ (b) $g(t)=\begin{cases}\sqrt{t-1} & \text{if } t\ge 1\\ -\sqrt{1-t} & \text{if } t<1\end{cases}$

⋆11. For the function $f(x)=1-2/x^2$, with domain $x>0$, find an explicit formula for the inverse function f^{-1}. Sketch the curves of $y=f(x)$, $y=f^{-1}(x)$ and $y=x$, all on the same graph.

⋆12. Suggest a function whose graph would have similar overall features to those in the sketch:



13. A function f is defined so that $f(x) = \sqrt{3 - e^{2x}}$

    (a) what is the domain of f

    (b) find a formula for the inverse function f^{-1}

    (c) what is the domain of f^{-1}

⋆14. Show that $\cos\big(\sin^{-1} x\big) = \sqrt{1 - x^2}$ for all angles x in the domain of $\sin^{-1}$.

15. Fully simplify the expression $\sin\big(\sin^{-1} x\big)$, making sure that the domain is properly identified.

⋆16. Use the definitions of cosh and sinh to show that

    (a) $\cosh^2 x - \sinh^2 x = 1$ for any value of x
    deduce that $\tanh^2 x = 1 - \operatorname{sech}^2 x$ where $\operatorname{sech} x = \dfrac{1}{\cosh x}$

    (b) $\cosh(a + b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$

    (c) $\sinh(a + b) = \cosh(a)\sinh(b) + \sinh(a)\cosh(b)$

Hint. for parts (b) and (c) write the right sides in terms of exponentials, multiply out and simplify

⋆17. A function with domain $[0, 2)$ is defined as $f(x) = \begin{cases} x + 1 & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x < 2 \end{cases}$

    (a) Sketch a graph of the function

    (b) Is the function an increasing function, a decreasing function or neither?

    (c) Can you find the inverse of the function and, if so, give a formula for f^{-1}.

Harder Questions

18. Sketch the graphs of the relation $|x|^a + |y|^a = 1$ where $a > 0$ is a constant. Consider how the graphs change as a increases or decreases from unity.

19. Consider the claim that any function $f(x)$ with domain $\mathbb{R}$ can be written as the sum of an even function $g(x)$ and an odd function $h(x)$. Can you verify this claim and, if so, how do you construct the even part $g(x)$ and the odd part $h(x)$ of the function from $f(x)$ itself?

20. Suppose that an even function $f(x)$, with domain $\mathbb{R}$, is also periodic with period $p \neq 0$.

    (a) show that the function shifted by p, namely $g(x) = f(x - p)$, is also an even function

    (b) show that the function shifted by $\frac{1}{2}p$, namely $h(x) = f\big(x - \frac{1}{2}p\big)$, is even as well

    If the function $f(x)$ was an odd function, would $g(x)$ and $h(x)$ be odd functions?

***Suggested reading for week 2*:** 'Stewart' Chapters 2, 3 and 4