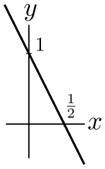
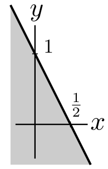
Easy Questions

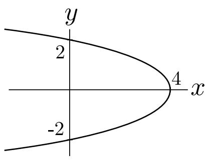
1. (a) $x \in (-4, -2)$ (b) $x \in [-\frac{2}{3}, 2]$ (c) $x \in (\infty, -6]$ or $x \in [4, \infty)$ (d) $x \in (0, \frac{1}{2}]$

2. (a) 

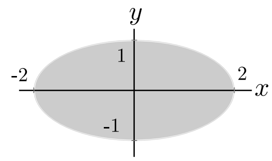
$2x + y = 1$
straight line with
$y = 1$ at $x = 0$
$x = \frac{1}{2}$ at $y = 0$

(b) 

$2x + y \leq 1$
as in (a) but including
more negative values
of x and y

(c) 

$y^2 + x = 4$
parabola with
$x = 4$ at $y = 0$
$y = \pm 2$ at $x = 0$

(d) 

$x^2 + 4y^2 < 4$
inside of ellipse with
$x = \pm 2$ at $y = 0$
$y = \pm 1$ at $x = 0$

3. (a) True (b) True (c) False. It is possible that $x = y$ (d) True

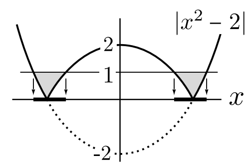
4. (a) True (b) False. The imaginary part is always real (c) False. $\text{Im}(z) = -\text{Im}(\bar{z})$

5. All formulae are incorrect. The correct formulae are:
   (a) $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ (b) $\sin^2(a) = \frac{1}{2}\big(1 - \cos(2a)\big)$
   (c) $\sin(a + b) = \cos(a)\sin(b) + \sin(a)\cos(b)$ (d) $\cos^2(a) = \frac{1}{2}\big(1 + \cos(2a)\big)$

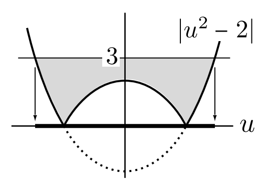
6. (a) $\cos(\frac{\pi}{4}) = 1/\sqrt{2}$ (b) $\sin(\frac{\pi}{3}) = \sqrt{3}/2$ (c) $\tan(\frac{\pi}{6}) = 1/\sqrt{3}$
   (d) $\cos(\frac{2\pi}{3}) = -\frac{1}{2}$ (e) $\tan(\frac{7\pi}{6}) = 1/\sqrt{3}$

Standard Questions

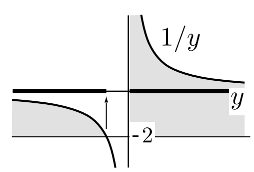
7. (a) $|x^2 - 2| < 1$ is satisfied for $-1 < x^2 - 2 < 1$
   or $1 < x^2 < 3$, for which $\pm x \in (1, \sqrt{3})$.
   Hence $x \in (-\sqrt{3}, -1)$ or $x \in (1, \sqrt{3})$.

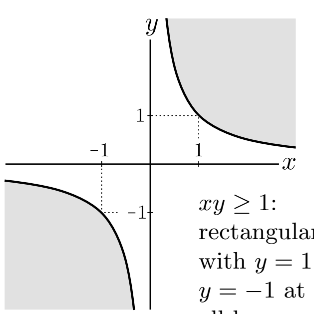
   sketch: 

   (b) $|u^2 - 2| < 3$ is satisfied for $-3 < u^2 - 2 < 3$
   or $-1 < u^2 < 5$.
   For $u \in \mathbb{R}$ we always have $u^2 > -1$
   so this only requires that $\pm u \in [0, \sqrt{5}]$.
   Hence $u \in (-\sqrt{5}, \sqrt{5})$.

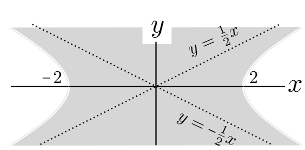
   sketch: 

   ⋆ (c) $1/y \geq -2$ is satisfied for both
   $y > 0$, for which $1/y > 0 \geq -2$, and
   $0 < -\frac{1}{y} \leq 2$, for which $-y \geq \frac{1}{2}$, or $y \leq -\frac{1}{2}$.
   Hence $y \in (-\infty, -\frac{1}{2})$ or $y \in (0, \infty)$.

   sketch: 

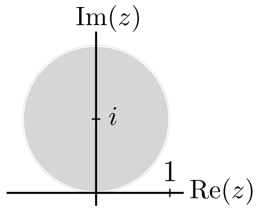
8. (a) 

$xy \geq 1$:
rectangular hyperbolae
with $y = 1$ at $x = 1$
$y = -1$ at $x = -1$
all larger values of xy

⋆ (b) 

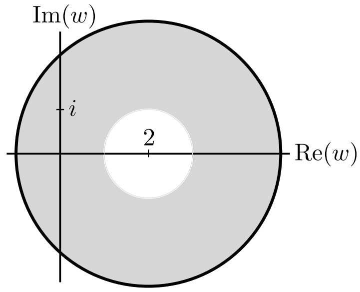
$x^2 - 4y^2 < 4$:
bounded by hyperbolae
with $x = \pm 2$ at $y = 0$
asymptotes $y = \pm\frac{1}{2}x$
all larger values of y
and smaller values of x

9. (a)



inside of circle with radius 1 and centre at $z = i$

$\star$ (b)



space between circles with centre at $w = 2$ and radii 1 and 3, including the circle with radius 3 but not the circle with radius 1

10. $z = \cos(A) + i\sin(A)$ and $w = \cos(B) + i\sin(B)$

(a) $|z| = \sqrt{\cos^2(A) + \sin^2(A)} = \sqrt{1} = 1.$ Similarly $|w| = 1.$

(b) $$\begin{aligned} zw &= \big(\cos(A) + i\sin(A)\big)\big(\cos(B) + i\sin(B)\big) \\ &= \cos(A)\cos(B) + i\sin(A)\cos(B) + \cos(A)i\sin(B) + i^2\sin(A)\sin(B) \\ &= \cos(A)\cos(B) - \sin(A)\sin(B) + i\big(\sin(A)\cos(B) + \cos(A)\sin(B)\big) \end{aligned}$$

(c) $\cos(A+B) + i\sin(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) + i\big(\sin(A)\cos(B) + \cos(A)\sin(B)\big)$
which is exactly the same as in part (b).
Hence $zw = \cos(A+B) + i\sin(A+B)$.

That is: $\big(\cos(A) + i\sin(A)\big)\big(\cos(B) + i\sin(B)\big) = \cos(A+B) + i\sin(A+B)$

(d) i. $$\begin{aligned} z\bar{w} &= \big(\cos(A) + i\sin(A)\big)\big(\cos(B) - i\sin(B)\big) \\ &= \big(\cos(A) + i\sin(A)\big)\big(\cos(-B) + i\sin(-B)\big) \\ &= \cos(A-B) + i\sin(A-B) \end{aligned}$$

ii. $$\begin{aligned} \frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} &= \frac{(\cos(A) + i\sin(A))(\cos(B) - i\sin(B))}{|w|^2} \\ &= \big(\cos(A) + i\sin(A)\big)\big(\cos(-B) + i\sin(-B)\big) \\ &= \cos(A-B) + i\sin(A-B) \end{aligned}$$

iii. $w/z = \cos(B-A) + i\sin(B-A)$ — similar to part ii.

iv. $$\begin{aligned} \bar{z}\bar{w} &= \big(\cos(A) - i\sin(A)\big)\big(\cos(B) - i\sin(B)\big) \\ &= \big(\cos(-A) + i\sin(-A)\big)\big(\cos(-B) + i\sin(-B)\big) \\ &= \cos(-A-B) + i\sin(-A-B) \end{aligned}$$

v. $$\begin{aligned} z^2 &= \big(\cos(A) + i\sin(A)\big)\big(\cos(A) + i\sin(A)\big) \\ &= \cos(2A) + i\sin(2A) \end{aligned}$$

vi. $$\begin{aligned} w^3 &= \big(\cos(2B) + i\sin(2B)\big)\big(\cos(B) + i\sin(B)\big) \\ &= \cos(3B) + i\sin(3B) \end{aligned}$$

vii. $w^p/z^q = \cos(pB - qA) + i\sin(pB - qA)$

11. (a) $(5+i)(2+3i) = 10 + 2i + 15i + 3i^2 = 10 + (2+15)i - 3 = 7 + 17i$

(b) $(3+4i)(3-4i) = 9 - 16i^2 = 9 + 16 = 25$ ← note the difference of two squares

(c) $(-2+i)^3 = (-2)^3 + \frac{3}{1}(-2)^2 i + \frac{3\times \cancel{2}}{1\times \cancel{2}}(-2)i^2 + i^3$ ← note use of the binomial theorem

$= -8 + 12i + 6 - i = -2 + 11i$

(d) $(2-i)^4 = 2^4 + \frac{4}{1}2^3(-i) + \frac{4\times 3}{1\times 2}2^2(-i)^2 + \frac{4\times \cancel{3}\times \cancel{2}}{1\times \cancel{2}\times \cancel{3}}2(-i)^3 + (-i)^4$ ← binomial theorem

$= 16 - 32i - 24 + 8i + 1 = -7 - 24i$

(e) $\dfrac{13}{3-2i} = \dfrac{13}{3-2i}\dfrac{3+2i}{3+2i} = \dfrac{13(3+2i)}{9+4} = 3 + 2i$

(f) $\dfrac{i(7+3i)}{3-7i} = \dfrac{7i-3}{3-7i} = -\dfrac{3-7i}{3-7i} = -1$ ← always try to simplify first

(g) $\dfrac{5-5i}{i(1+i)(2+i)} = \dfrac{5(1-i)}{(i-1)(2+i)} = -\dfrac{5}{2+i}\dfrac{2-i}{2-i} = -\dfrac{5(2-i)}{4+1} = -2+i$

(h) $\dfrac{4+5i}{2-3i} = \dfrac{4+5i}{2-3i}\dfrac{2+3i}{2+3i} = \dfrac{(4+5i)(2+3i)}{4+9} = \dfrac{8-15+(10+12)i}{13} = -\frac{7}{13}+\frac{22}{13}i$

Harder Questions

12.
- Note that $3.\overline{9}$ cannot be bigger than 4, so that we must have $4-3.\overline{9} \geq 0$.
- Assume (for the moment) that $4-3.\overline{9} > 0$.
- Let $\delta = 4-3.\overline{9} > 0$, then $\log_{10}\delta$ exists. ← For any base a, $\log_a x$ exists only if $x > 0$.
- We can then choose any number k such that $k > -\log_{10}\delta$.
- For such a value of k we must have $10^{-k} < \delta$.
- As a result, we have $4 - N_k = 10^{-k} < 4-3.\overline{9}$ (where $N_1 = 3.9$, $N_2 = 3.99$, $N_3 = 3.999$, etc.)
- It follows that $N_k > 3.\overline{9}$.
- This cannot be true, because we must always have $N_k < 3.\overline{9}$ for any value of k.
- The assumption, that $4-3.\overline{9} > 0$, must therefore be false.
- It follows that $4-3.\overline{9} = 0$, so that $4 = 3.\overline{9}$.

13. Using the computer package Maple, $e - \left(1+\frac{1}{n}\right)^n$ falls below 0.01 for $n = 135$.
It decreases further as n increases further.
Can you think of the reason why $e \approx \left(1+\frac{1}{n}\right)^n$ when n is very large?
