Calculus and Vectors (2005) Problem Sheet for Week 1

Note: This problem sheet (for week 1) introduces some basic concepts in calculus and also serves to demonstrate how problem sheets are to be used in the module. The problem sheet for any week is made available via the web on the Thursday of the week before (`www.ma.umist.ac.uk/jwd/calculus`). You should print it out and work through it without delay. Sample answers will be made available on the web a fortnight later, on each Friday afternoon.

You will have a small-group supervision class each week and you must work through the problem sheet for that week and hand in your answers (or attempted answers) to the starred questions (*) *before* your calculus supervision. In the first week there may not be time to hand in work based on the problem sheet but make sure that you have worked through the sheet before your supervision class. You must also arrange with your supervisor where, when and how to hand in work for the second and all subsequent weeks.

The objective is not to hand in '*the right answer*' to each question. It is to attempt each question, identify where you have difficulties and to make sure that your difficulties are discussed in the supervision class. You should make sure that the supervisor is fully aware of your difficulties by handing in your work and, if necessary, by telling the supervisor at or before the start of the class. You should not be shy in these classes; make the best use of them.

The problem sheets are structured as follows:

- The first part consists of easy questions and basic revision material which can be used as warm-up exercises; you should ensure that you can confidently answer these questions. It is *essential* that you raise any difficulties with these questions in your supervision class and discuss them with your supervisor.
- The second part consists of standard questions that test your understanding; these are for use in supervision classes and for revision at later times. There are also many suitable problems in 'Stewart' with answers to all odd-numbered exercises provided in Appendix H at the back.
- The third part consists of deeper and more difficult (usually interesting) questions that can, optionally, be used to advance your knowledge of calculus even further.

Finally, this is a new module in a new teaching programme in the newly formed Manchester School of Mathematics. There could be errors or omissions in the problem sheets, the sample answers or any other aspect of the course. If there are or if you have any suggestions for improvement please do not hesitate to let me know. — John Dold

Problem Sheet for Week 1 **MT1121** Real and Complex Numbers

Suggested reading: 'Stewart' pages A2–A38 (appendices A to E)
pages A49–A53 (in appendix G)

Easy Questions

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1. In each case, for $x \in \mathbb{R}$, find all of the intervals where x satisfies the inequality.
   (a) $|x+3| < 1$ (b) $|3x-2| \le 4$ (c) $|x+1| \ge 5$ (d) $1/x \ge 2$
2. Sketch the graphs of the following relations between real values of x and y (that is the set of all points satisfying the equation or inequality). Explain very briefly how you got the graphs.
   (a) $2x+y=1$ (b) $2x+y\le 1$ (c) $y^2+x=4$ (d) $x^2+4y^2<4$
3. Are the following true or false? (assume that $x \in \mathbb{R}$ and $y \in \mathbb{R}$)
   (a) $3 \ge 2$ (b) $2 \ge 2$ (c) If $x \ge y$ then $x > y$ (d) If $x < y$ then $x \le y$
4. Are the following true or false? (assume that $z \in \mathbb{C}$)
   (a) $i^2 + (-i)^4 = 0$ (b) If $z \neq 0$ then $\mathrm{Im}(z) \notin \mathbb{R}$ (c) $\mathrm{Im}(z) = \mathrm{Im}(\bar{z})$
5. Are the following formulae correct for any a and b in $\mathbb{R}$? If not, what is the correct formula?
   (a) $\cos(a+b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ (b) $\sin^2(a) = \sin(a)\cos(a)$
   (c) $\sin(a+b) = \cos(a)\sin(b) - \sin(a)\cos(b)$ (d) $\cos^2(a) = \frac{1}{2}\big(1+\sin(2a)\big)$
6. Give the values of the following exactly, without using a calculator. For example $\sin(\frac{\pi}{4}) = 1/\sqrt{2}$.
   (a) $\cos(\frac{\pi}{4})$ (b) $\sin(\frac{\pi}{3})$ (c) $\tan(\frac{\pi}{6})$ (d) $\cos(\frac{2\pi}{3})$ (e) $\tan(\frac{7\pi}{6})$

Standard Questions

7. (a) Find the set of real numbers x satisfying $|x^2 - 2| < 1$
   (b) Find the set of real numbers u satisfying $|u^2 - 2| < 3$
   $\star$ (c) Find the set of real numbers y satisfying $1/y \geq -2$

8. Sketch the graphs of the following relations between real values of x and y (that is the set of all points satisfying the equation or inequality). Explain very briefly how you got the graphs.
   (a) $xy \geq 1$ $\quad\star$ (b) $x^2 - 4y^2 < 4$

9. (a) Sketch, in the complex plane, where $|z - i| < 1$ for $z \in \mathbb{C}$
   $\star$ (b) Sketch, in the complex plane, where $1 < |w - 2| \leq 3$ for $w \in \mathbb{C}$

$\star$ 10. For complex numbers z and w given by $z = \cos(A) + i\sin(A)$ and $w = \cos(B) + i\sin(B)$
   (a) what are the values of $|z|$ and $|w|$, given that A and B are real angles?
   (b) work out the product zw in terms of $\cos(A)$, $\sin(A)$, $\cos(B)$ and $\sin(B)$
   (c) show that this is the same as $zw = \cos(A + B) + i\sin(A + B)$
   (d) deduce how the following can be written in terms of A and B?
   i. $z\bar{w}$ ii. z/w iii. w/z iv. $\bar{z}\bar{w}$ v. z^2 vi. w^3 vii. w^p/z^q

11. Convert each of the following complex numbers into standard form (‘`real part`’ + i ‘`imaginary part`’)
   (a) $(5 + i)(2 + 3i)$ $\quad\star$ (b) $(3 + 4i)(3 - 4i)$ $\quad$ (c) $(-2 + i)^3$ $\quad\star$ (d) $(2 - i)^4$
   (e) $\dfrac{13}{3 - 2i}$ $\quad$ (f) $\dfrac{i(7 + 3i)}{3 - 7i}$ $\quad\star$ (g) $\dfrac{5 - 5i}{i(1 + i)(2 + i)}$ $\quad\star$ (h) $\dfrac{4 + 5i}{2 - 3i}$

Harder Questions

12. Is there any non-zero difference between 4 and $3.99999\cdots$ (repeated indefinitely)?
   Hints:
   - Note that repeated parts of decimals can be marked by an overline, i.e. $3.\overline{9} = 3.99999\cdots$.
   - Also, $3.\overline{9}$ cannot be greater than 4.
   - It is convenient to define a sequence of numbers $N_1 = 3.9$, $N_2 = 3.99$, $N_3 = 3.999$, etc.
   - For each of these numbers, $4 - N_k = 10^{-k}$ (for any value of k)
   - Note that, for any value of k, $N_k < 3.\overline{9}$.
   - If $4 \neq 3.\overline{9}$ then the difference can be written as (say) $\delta = 4 - 3.\overline{9}$, with $\delta > 0$.
   - If this is so, is there any value of k for which $10^{-k} < \delta$?
   - Why does this prove that δ cannot be positive?
   - If the difference between two numbers is zero, the numbers must be equal!

13. Use a calculator or suitable computer package (if you have access to one) to work out values of the formula
   $$\left(1 + \tfrac{1}{n}\right)^n$$
   for $n = 1, 2, 4, 8, \cdots, 2^k, \cdots$ and any other suitably large values of n you wish.
   How large does n need to be before the absolute value of the difference $e - \left(1 + \frac{1}{n}\right)^n$ is less than 0.01?
   Note that e is the base of the natural logarithm given by
   $$e \approx 2.718281828459045235360287471352662497757247093699959574966967627724076630\cdots$$
   with no repetition however many decimal places are evaluated.

***Suggested reading for week 1*:** ‘Stewart’ pages 11–85 (chapter 1)