Infinite Power Series

An infinite ***power series*** is a polynomial of 'infinite' order

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots$$

The notation

$$\sum_{k=0}^{\infty} a_k x^k$$

is used to represent the summation of all terms $a_k x^k$ starting from $k = 0$.

The infinite summation is only meaningful if the series '***converges***', meaning that

$$\sum_{k=0}^{\infty} a_k x^k = \lim_{N \to \infty} \sum_{k=0}^{N} a_k x^k$$

provided the limit of N-term sums exists, as $N \to \infty$.

The series is said to '***diverge***' if it does not converge.

Example. A string of length 1 is cut into two halves. One of those pieces is cut into two halves, of which one of the pieces is again cut in half, and so on 'ad infinitum'. The total length of all of the pieces must be

$$\begin{aligned} 1 &= \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + \cdots + (\tfrac{1}{2})^k + \cdots \\ &= \sum_{k=1}^{\infty} (\tfrac{1}{2})^k \end{aligned}$$

Derivatives of Power Series

It is most useful to consider $z \in \mathbb{C}$ in the power series 'about the point $z = z_0$'

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Written in this way, the series defines a function $f(z)$ for every value of z at which the series converges.

All terms in the power series can be differentiated to give

$$\begin{aligned} f'(z) &= \sum_{k=1}^{\infty} k\, a_k\, (z - z_0)^{k-1} \\ f''(z) &= \sum_{k=2}^{\infty} k(k-1)\, a_k\, (z - z_0)^{k-2} \\ &\;\vdots \\ f^{(n)}(z) &= \sum_{k=n}^{\infty} \big(k(k-1)\cdots(k-n+1)\big)\, a_k\, (z - z_0)^{k-n} \end{aligned}$$

etc.

Note. The index (k in the series above) is a dummy variable that can always be changed. So that

$$\begin{aligned} f''(z) &= \sum_{j=2}^{\infty} j(j-1)\, a_j\, (z - z_0)^{j-2} \\ f''(z) &= \sum_{\ell=0}^{\infty} (\ell+2)(\ell+1)\, a_{\ell+2}\, (z - z_0)^{\ell} \end{aligned}$$

are exactly the same series for $f''(z)$, as above, obtained simply by rewriting $k = j = 2 + \ell$.

Radius of Convergence

Any power series about a point $z = z_0$

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

has a '***radius of convergence***' R such that

- the series converges for all $|z - z_0| < R$
- the series diverges for all $|z - z_0| > R$
- it might converge or diverge if $|z - z_0| = R$
- the differentiated series has the same radius of convergence

A function $f(z)$ is said to be '***analytic***' at a point $z = a$ if there is an infinite power series that converges to $f(z)$, with $z = a$ within its radius of convergence.

Example 1. $\sum_{n=0}^{\infty} (z - i)^n$ converges for $|z - i| < 1$, diverges for $|z - i| \geq 1$, and so it has radius of convergence $R = 1$

Example 2. $\sum_{n=0}^{\infty} z^n/n!$ converges for all $z \in \mathbb{C}$ and so it has radius of convergence $R = \infty$

Example 3. $\sum_{n=0}^{\infty} n!(z - \pi)^n$ diverges for all $z \neq \pi$ and so it has radius of convergence $R = 0$

More on Radius of Convergence

The radius of convergence of a function given by

$$f(z) = \sum_{k=0}^{\infty} a_k \, (z - z_0)^k$$

extends as far as the nearest singularity of $f(z)$, from z_0, in the complex plane.

Example 1. What is the radius of convergence of the series $\dfrac{1+t}{9+t^2} = \displaystyle\sum_{n=0}^{\infty} a_n (t-4)^n$?

What is the radius of convergence of $\dfrac{1+t}{9+t^2} = \displaystyle\sum_{n=0}^{\infty} b_n (t-1-4i)^n$?

Taylor Series

Supposing that a function $f(z)$ has the series

$$f(z) = \sum_{k=0}^{\infty} a_k \, (z - z_0)^k$$

then, it's n^{th} derivative has the series

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \big(k(k-1) \cdots (k-n+1)\big) \, a_k \, (z - z_0)^{k-n}$$

If we now set $z = z_0$ all terms become zero except for the first, which involves $(z - z_0)^0$, showing that

$$f(z_0) = a_0$$
$$f^{(n)}(z_0) = n! \, a_n$$

The coefficient a_n in the series is therefore given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \text{for} \quad n = 0,\ 1,\ 2,\ \text{etc.}$$

in which $f^{(0)}$ represents f and $0! = 1$.

This leads to ***Taylor's series*** for the function $f(z)$, 'expanded' about the point z_0

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \, (z - z_0)^k$$

(also called a '***Taylor expansion***' about z_0)

Series Expansions of sin, cos and exp

Infinite power series can now be found for functions that can be differentiated an unlimited number of times.

$\boxed{\exp z}$

about
$z = 0$

$\boxed{\sin t}$

about
$t = 0$

$\boxed{\cos x}$

about
$x = 0$

Series Approximations

It is never practically possible to add together an infinite number of terms. Instead, functions are often approximated using truncated series expansions.

Example 1. Approximations for $\sin x$ and $\cos x$ are

$$\sin x = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 + \mathbf{O}(x^7)$$
$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \mathbf{O}(x^6)$$

giving an error of less than $\frac{1}{3} \times 10^{-3}$ for $|x| \le \frac{1}{4}\pi$, using only 3 terms.

Example 2. Find the Taylor series expansion for $\ln x$ about $x = 1$. Write down a 3-term approximation for $\ln x$.

Euler's Formula

We already know that

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots \\ \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \end{aligned}$$

A remarkable property of complex numbers arises from the power series for e^{ix}, taking x to be real

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

The result is ***Euler's formula***

$$e^{ix} = \cos x + i \sin x$$

making a direct link between trigonometric functions and the exponential function for complex numbers

Applications of Euler's Formula

The formula $e^{ix} = \cos x + i \sin x$ can be used to obtain trigonometric identities

Example. $e^{i(A+B)} = \cos(A+B) + i\sin(A+B)$

and

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B - \sin A\sin B \\ &\quad + i(\cos A\sin B + \sin A\cos B) \end{aligned}$$

Equating real and imaginary parts yields the well-known trigonometric identities

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \cos A\sin B + \sin A\cos B \end{aligned}$$

Exercise 1. Find formulae for $\cos(3A)$ and $\sin(3A)$

Complex forms of $\sin$ and $\cos$

Euler's formula and its complex conjugate are

$e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$

Solving for $\cos x$ and $\sin x$, gives

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} = \cosh(ix) \\ i\sin x &= \frac{e^{ix} - e^{-ix}}{2} = \sinh(ix) \end{aligned}$$

which link trigonometric and hyperbolic functions.

Any trigonometric identity mirrors a hyperbolic identity (and vice versa)

Exercise 1. Find the hyperbolic equivalent of

$$\cos(3A) = \cos A\,(1 - 4\sin^2 A)$$

Exercise 2. Find the trigonometric equivalent of

$$\sinh(3A) = \sinh A\,(4\cosh^2 A - 1)$$

de Moivre's Theorem

Because of Euler's formula, the polar form of a complex number can be written, equivalently, as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

with 'modulus' r and 'argument' θ.

Sketch:

In polar form, multiplying or dividing complex numbers

$$z = r\,e^{i\theta} \quad \text{and} \quad w = s\,e^{i\phi}$$

simply involves adding or subtracting angles

$$zw = rs\,e^{i(\theta+\phi)} \quad \text{and} \quad \frac{z}{w} = \frac{r}{s}\,e^{i(\theta-\phi)}$$

As a special case:

de Moivre's Theorem gives powers of $z = r\,e^{i\theta}$, namely:

$$\begin{aligned} z^n &= \big(r\,(\cos\theta + i\sin\theta)\big)^n \\ &= r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \end{aligned}$$

Complex Roots

de Moivre's theorem helps to find the roots of any complex number.

If $w \in \mathbb{C}$ and n is a natural number, then the values of z that satisfy

$$z^n = w$$

are the n^{th} roots of w. There are n roots if $w \neq 0$.

Example. The roots of unity: $z^n = 1 = e^{i0}$

If $z = re^{i\theta}$ then, by de Moivre's theorem

$$z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) = 1.$$

Since $r^n = |1| = 1$, we must have $r = 1$.

Equating real and imaginary parts gives

$$\sin(n\theta) = 0 \;\text{ and }\; \cos(n\theta) = 1$$

so that $n\theta = 0$, 2π, 4π, etc., and so

$$1^{1/n} = e^{i0},\ e^{i2\pi/n},\ e^{i4\pi/n},\ \ldots\ e^{i2\pi(n-1)/n}$$

(roots are successively rotated by $2\pi/n$)

Exercise. Find all roots of $z^5 = i - \sqrt{3}$