Real and Complex Numbers

Number Sets. there are 5 basic number sets[1]:

$\mathbb{N} = \{1, 2, 3, \cdots\}$ the '***natural numbers***'
(positive whole numbers as used for counting)

$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ '***integers***'
(include zero and negative whole numbers)

$\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z},\ n \in \mathbb{N}\}$ '***rational numbers***'
(include all fractions)

$\mathbb{R}$ the set of '***real numbers***'
(equivalent to all finite or infinite decimal Nos.)

$\mathbb{C}$ the set of '***complex numbers***'
(include all 'imaginary' numbers)

Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

We will examine the last two in more detail now.

[1] a summary of set notation is available on the web page

Real Numbers, $\mathbb{R}$

a real number measures distance along a '***real line***'

(you will often hear a real number called '*a point*')

Real numbers satisfy all algebraic and inequality laws.
these can be summarised as sets of 'axioms':

<u>field axioms</u> for any x, y, z in $\mathbb{R}$

$$x + 0 = x \qquad x + y = y + x$$
$$x - x = 0 \qquad (x + y) + z = x + (y + z)$$
$$1\,x = x \qquad xy = yx$$
$$x/x = 1 \ \ (\text{for } x \neq 0) \qquad (xy)z = x(yz)$$
$$x(y + z) = xy + xz$$

<u>ordering axioms</u> for any x, y, z in $\mathbb{R}$

only one of '$x > y$', '$x = y$' or '$x < y$' can hold
if $x < y$ and $y < z$ then $x < z$
if $x < y$ then $x + z < y + z$
if $x < y$ and $z > 0$ then $xz < yz$

(You will learn more about 'fields' in other courses)

a few points to note

All you know about the algebra and inequalities can be worked out from the axioms.

Example 1. Show that $0x = 0$ for any real number x.

Example 2. If $x > y > 0$ then $\frac{1}{y} > \frac{1}{x} > 0$.

Example 3. If $x > y$ and $z < 0$ then $zx < zy$.

Example 4. Is infinity (∞) a real number?

Note. We use x, y, z, etc. only as names or labels.

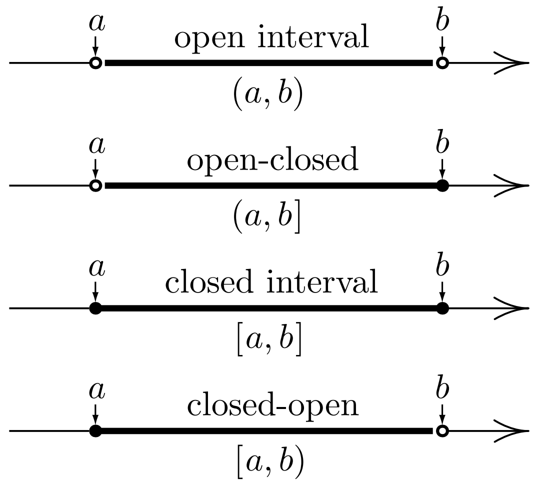
All of these definitions, formulae, etc., remain exactly the same if we use other names, such as s, t, u, etc.

Intervals

$a < x \leq b$ represents any value of x in an 'interval'

There is a special notation for intervals:

- if $a < x \leq b$ we write $x \in (a, b]$
- if an end-point is not in the interval it is called an 'open' end point, marked by a curved bracket
- if an end-point is in the interval it is called a 'closed' end point, marked by a square bracket



- if an interval extends to infinity use ($-\infty$ or ∞) e.g. $x \geq a$ means $x \in [a, \infty)$

  Note: infinity can only be an open end point.

Absolute Value or 'Modulus'

The absolute value $|x|$ of a real number x gives its 'size' but not its sign.

We can define: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

sketch:

Example 1. In what interval of x is $|x+3| < 1$?

Example 2. In what intervals of x is $|x-1| \geq 7$?

Squares and Square Roots

For any real number x:

$$\begin{array}{rlll} \text{either} & x > 0 & \text{so} & x^2 > 0 \\ \text{or} & x = 0 & \text{so} & x^2 = 0 \\ \text{or} & x < 0 & \text{so} & x^2 > 0 \end{array}$$

so we always have $x^2 \geq 0$

Hence we can also define

$$|x| = \sqrt{x^2}$$

Note: for any real number a:

- $\sqrt{a}$ only exists if $a \geq 0$
- $\sqrt{a}$ is the positive square root of a

Example 1. Solve $x^2 = 1$ for x.

Example 2. Can you solve $x^2 = -1$ for x?

Imaginary Numbers

There is no real number x for which $x^2 < 0$

but it is (remarkably) useful to invent new numbers (imaginary numbers) for which $x^2 < 0$

Let i be defined as

$$i = \sqrt{-1}$$

Example 1. Now can you solve $x^2 = -1$ for x?

Note:

- clearly $i \notin \mathbb{R}$ (i is not 'real')
- i is an 'imaginary' square root of -1
- if $x \in \mathbb{R}$ then xi is an imaginary number such that

$$(xi)^2 = x^2 i^2 = -x^2$$

Example 2. What are the values of i^2, i^3, i^4, i^5, etc.?

Complex Numbers, $\mathbb{C}$

Complex numbers arise if we add real and imaginary numbers.

If x and y are real, then

$$z = x + iy \quad \text{is a complex number}$$

we can write: $x + iy \in \mathbb{C}$

<u>Standard Form</u> of a complex number is

$$\text{`}\texttt{real part}\text{'} + i \, \text{`}\texttt{imaginary part}\text{'}$$

If x and y are real, then we define

$$\text{`real part'}: \quad \operatorname{Re}(x + iy) = x$$

$$\text{`imaginary part'}: \quad \operatorname{Im}(x + iy) = y$$

Note: the real part is a real number
the imaginary part is a real number

Example. Give the real and imaginary parts of $7 - 9i$

Complex Plane and Polar Form

We can extend the ‘real line’ to a ‘complex plane’

A complex number $z = x + iy$ is a point with

horizontal coordinate x
vertical coordinate y

It is also useful to use polar coordinates:

$$\begin{aligned} z &= r\big(\cos(\theta) + i\sin(\theta)\big) \\ &= r\cos(\theta) + ir\sin(\theta) \end{aligned}$$

having: $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Names: r is the ‘modulus’. θ is the ‘argument’.

Example. $-1 + i\sqrt{3} =$

Modulus and Complex Conjugate

The modulus is the distance from the origin in the complex plane

so for $z = x + iy$ we define

$$|z| = \sqrt{x^2 + y^2}$$

and for $z = r\big(\cos(\theta) + i\sin(\theta)\big)$ we have, simply

$$|z| = r$$

The complex conjugate or simply ***the conjugate*** of a complex number has the sign of the imaginary part changed

so if $z = x + iy$ we define the complex congugate of z as

$$\overline{z} = x - iy$$

Note

$$\begin{aligned} z\overline{z} &= (x + iy)(x - iy) \\ &= x^2 - i^2y^2 \\ &= x^2 + y^2 \end{aligned}$$

So: $z\overline{z}$ is real

$z\overline{z} = |z|^2$

or $|z| = \sqrt{z\overline{z}}$

Algebra of Complex Numbers

Complex numbers can be added, subtracted, multiplied, divided just like real numbers (simply use $i^2 = -1$ where it appears)

For $z = x + iy$ and $w = u + iv$ give the following in 'standard form'

Example 1. $z + w$

Example 2. $z - w$

Example 3. zw

Example 4. z/w (provided $w \neq 0$)

Two Examples

Convert to standard form:

$$\frac{3+4i}{1-2i} =$$

Where is $\left|z - (\pi + i\sqrt{2}\,)\right| \leq 1$ in the complex plane?