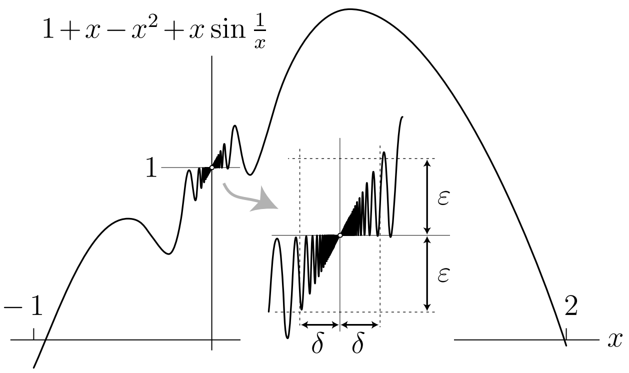
Limit of a Function

We often need to address the question

what is $\lim_{x \to a} f(x)$?

for a function $f(x)$ that is defined around a, although not necessarily at a itself.

Example. The function $f(x) = 1 + x - x^2 + x \sin \frac{1}{x}$ is not defined at $x = 0$



A limit arises, as $x \to 0$, because we can ensure that $|f(x) - 1|$ is smaller than *any* chosen number ($\varepsilon > 0$), simply by restricting $|x - 0|$ to small enough values ($|x - 0| < \delta$)

We can say that $\lim_{x \to 0} \left(1 + x - x^2 + x \sin \frac{1}{x}\right) = 1$

Left and Right Limits

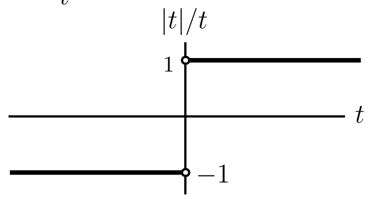
only values $x < a$ are considered in finding a ***left limit***, written as $\lim_{x \to a^-} f(x)$

only values $x > a$ are considered in finding a ***right limit***, written as $\lim_{x \to a^+} f(x)$

the ***two-sided limit*** $\lim_{x \to a} f(x)$ exists, if and only if left and right limits are the same, i.e.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example. Limits of $\dfrac{|t|}{t}$ as $t \to 0^-$, $t \to 0^+$, $t \to 0$



note that: $\lim_{t \to 0^-} \dfrac{|t|}{t} = \lim_{t \to 0^-} \dfrac{-t}{t} = \lim_{t \to 0^-} -1 = -1$

$\lim_{t \to 0^+} \dfrac{|t|}{t} = \lim_{t \to 0^+} \dfrac{t}{t} = \lim_{t \to 0^-} 1 = 1$

The limit $\lim_{x \to 0} \frac{|t|}{t}$ *does not exist* because the left and right limits are not equal.

Limits and Infinity

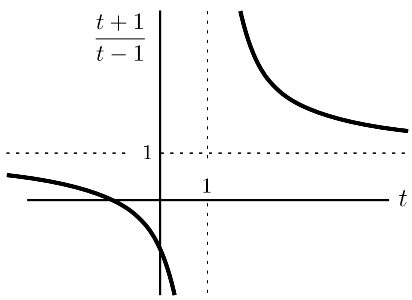
We may also ask

what is $\lim_{x \to \infty} f(x)$ or $\lim_{x \to -\infty} f(x)$?

for a function $f(x)$ whose domain extends towards ∞ or $-\infty$

Situations may also arise in which $|f(x)|$ can be made arbitrarily large by choosing x close to some value a

Example. The function $f(t) = \dfrac{t+1}{t-1}$



Four limits involving infinity arise in this case

$\lim_{t \to -\infty} f(t) = 1 \qquad \lim_{t \to 1^-} f(t) = -\infty$

$\lim_{t \to \infty} f(t) = 1 \qquad \lim_{t \to 1^+} f(t) = \infty$

The double-sided limit $\lim_{t \to 1} f(t)$ does not exist.

Combining Limits

Suppose that the functions f and g have the finite or infinite limits $\lim_{t \to \ell} f(t) = A$ and $\lim_{t \to \ell} g(t) = B$ (in which ℓ could represent a, a^+, a^-, ∞ or $-\infty$) then

$$\lim_{t \to \ell} (f+g)(t) = A + B$$
$$\lim_{t \to \ell} (f-g)(t) = A - B$$
$$\lim_{t \to \ell} kf(t) = kA \text{ for a constant } k$$
$$\lim_{t \to \ell} (fg)(t) = AB$$
$$\lim_{t \to \ell} (f/g)(t) = A/B \text{ provided } B \neq 0$$

and, if $\lim_{t \to B} f(t) = C$ then

$$\lim_{t \to \ell} (f \circ g)(t) = \lim_{t \to \ell} f\big(g(t)\big) = C$$

except for 'indefinite' results $\left(\frac{0}{0}, \frac{\infty}{\infty}, 0^0, \infty - \infty, \text{etc.}\right)$

l'Hôpital's rule

If $\lim_{t \to \ell} f(t) = 0$ and $\lim_{t \to \ell} g(t) = 0$

or $\lim_{t \to \ell} f(t) = \pm\infty$ and $\lim_{t \to \ell} g(t) = \pm\infty$

a useful formula is

$$\lim_{t \to \ell} \frac{f(t)}{g(t)} = \lim_{t \to \ell} \frac{f'(t)}{g'(t)}$$

where f' and g' are the derivatives of f and g

Examples of Some Limits

In the following $r > 0$:

$$\lim_{x\to 0} x^r = 0 \qquad \lim_{x\to\infty} x^{-r} = 0$$
$$\lim_{x\to 0^+} x^{-r} = \infty \qquad \lim_{x\to\infty} x^r = \infty$$
$$\lim_{x\to -\infty} e^x = 0 \qquad \lim_{x\to\infty} e^x = \infty$$
$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0} \ln x = -\infty$$
$$\lim_{x\to\infty} x^r e^{-x} = 0 \qquad \lim_{x\to 0} x^r \ln x = 0$$
$$\lim_{x\to\infty} x^{-r} e^x = \infty \qquad \lim_{x\to\infty} x^{-r} \ln x = 0$$

as a rule: e^x dominates x^r which dominates $\ln x$
(when one factor $\to \pm\infty$ and the other $\to 0$)

Sketches.

Order Notation

It is sometimes useful to compare the way in which two functions approach a limit.

If $\lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| \le C$ for some constant $C > 0$

we say that $f(x)$ is of the order of $g(x)$ as $x \to a$.

This is written as $f(x) = \mathbf{O}\big(g(x)\big)$ as $x \to a$

Example 1. $\sin(t) = \mathbf{O}(t)$ as $t \to 0$
because $\lim_{t\to 0} \frac{\sin(t)}{t} = 1.$

The notation extends to limits where $x \to \infty$

Example 2. $\frac{2 - s^4}{1 + 2s^2} = \mathbf{O}(s^2)$ as $s \to \infty$
because $\lim_{s\to\infty} \frac{2 - s^4}{1 + 2s^2}\Big/ s^2 = -\frac{1}{2}.$

Example 3. $\sin(t) = t - \frac{1}{6}t^3 + \mathbf{O}(t^5)$ as $t \to 0$
because $\lim_{t\to 0} \frac{\sin(t) - (t - \frac{1}{6}t^3)}{t^5} = \frac{1}{5!}.$

The final example shows that order notation can be used to describe quite small errors in approximations of functions (if t is small, t^5 is very small)

Continuity

A function $f(x)$ is ***continuous at a point*** a in its domain if
$$\lim_{x\to a} f(x) = f(a)$$

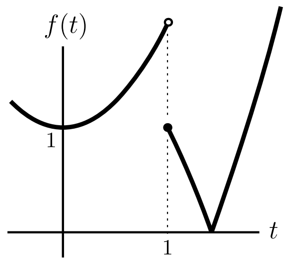
Broadly speaking: the value of f at a is consistent with the values of the function around the point a

A function $f(x)$ is ***continuous on an interval*** if it is continuous at all points in the interval

A function $f(x)$ is ***discontinuous*** at any point where it is not continuous

Example 1. The function $1 + x - x^2 + x\sin(1/x)$ is discontinuous at $x = 0$ because it is not defined there

Example 2. Where is $f(t) = \begin{cases} |2 - t^2| & \text{for } t \ge 1 \\ 1 + t^2 & \text{for } t < 1 \end{cases}$ continuous or discontinuous?



Mean value theorem

The mean value theorem states that:

> If $f(x)$ is continuous on an interval $[a, b]$ with $f(a) \ne f(b)$
>
> and if N is a number between $f(a)$ and $f(b)$
>
> then there is some number $c \in [a, b]$ such that $f(c) = N$

The mean value theorem highlights the fact that the range of a continuous function over any interval cannot contain any gaps.

Example. If we define

$$f(x) = \begin{cases} 1 + x - x^2 + x\sin\frac{1}{x} & \text{for } x \ne 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Is $f(x)$ continuous on the interval $[-1, 1]$?

Is there a value $A \in [f(-1), f(1)]$ for which there are infinitely many possible values of c such that $f(c) = A$?