Use of basic rules and integrals

Exercise 1. $\displaystyle\int \frac{2\,\mathrm{d}x}{9x^2 - 12x + 8}$

Exercise 2. $\displaystyle\int \frac{-\mathrm{d}z}{\sqrt{12z - 4z^2 - 4}}$

Exercise 3. $\displaystyle\int \frac{7\,\mathrm{d}x}{\sqrt{4x^2 - 4x - 2}}$

Methods of Integration

Not all formulae involving standard functions have an integral in terms of standard functions, such as

$$\int \sin\frac{1}{\theta}\,\mathrm{d}\theta \qquad \int \frac{e^t}{t}\,\mathrm{d}t \qquad \int e^{-x^2}\,\mathrm{d}x$$

In such cases, the integrals lead to new functions, for example the 'error function' and the 'elliptic function'

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t \quad \mathrm{K}(z) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - z\sin^2\theta}}$$

Note. In a *definite integral*, the variable used for integrating is a 'dummy variable'.
So θ or t can be replaced by s, u or anything that is not already used in the integral.

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in this course

We will consider a variety of techniques for finding integrals that can be worked out using standard functions, such as

$$\int \frac{x-1}{x^2 + 3x - 2}\,\mathrm{d}x$$

$$\int \cos^7\theta\,\mathrm{d}\theta \qquad \int \frac{t}{\sqrt{2 - t^2}}\,\mathrm{d}t$$

and many others.

a: Standard Integrals

An integral may involve one or more standard anti-derivatives (indefinite integrals)

Example. $\displaystyle\int \left(\operatorname{sech}^2 u + \sqrt[3]{u}\right)\mathrm{d}u = \tanh u + \tfrac{3}{4}u^{4/3} + C$

because $\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \operatorname{sech}^2 x$

$\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{3}{4}x^{4/3} = x^{1/3}$

Exercise 1. $\displaystyle\int \sec^2(\tfrac{1}{2}\theta - \pi)\,\mathrm{d}\theta$

Exercise 2. $\displaystyle\int \frac{4\,\mathrm{d}t}{2 - 3t}$

Exercise 3. $\displaystyle\int \frac{\mathrm{d}z}{1 + (3z)^2}$

Exercise 4. $\displaystyle\int \frac{3\,\mathrm{d}x}{\sqrt{1 - (x/2)^2}}$

b: Algebraic Manipulation

Algebraic manipulation may turn the integral into one or more standard integrals

Exercise 1. $\displaystyle\int \frac{(1+x)^2}{x}\,\mathrm{d}x$

Exercise 2. $\displaystyle\int \sin^2 t\,\mathrm{d}t$

Exercise 3. $\displaystyle\int \sin x\cos x\,\mathrm{d}x$

c: Integration by Parts

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

integration by parts

one part times integral of other, minus integral of derivative of the one times integral of other

Exercise 1. $\int t\,e^t\,\mathrm{d}t$

(differentiating should simplify the underlined term)

Exercise 2. $\int \sin x \cos x\,\mathrm{d}x$

Exercise 3. $\int \ln z\,\mathrm{d}z$

c: Recurrence Formulae

Sometimes many integrations by parts are needed to evaluate an integral.

In such cases a ‘recurrence formula’ can help

Example. Find $\int \sin^5 t\,\mathrm{d}t$

Let $I_n = \int \sin^n t\,\mathrm{d}t$
$= \int \sin t\,\sin^{n-1} t\,\mathrm{d}t$
$=$
$= -\cos t \sin^{n-1} t + (n-1)\int \cos^2 t \sin^{n-2} t\,\mathrm{d}t$

but $\int \cos^2 t \sin^{n-2} t\,\mathrm{d}t = \int (1-\sin^2 t)\sin^{n-2} t\,\mathrm{d}t$
$= \int \sin^{n-2} t\,\mathrm{d}t - \int \sin^n t\,\mathrm{d}t$
$= I_{n-2} - I_n$

so $I_n = -\cos t \sin^{n-1} t + (n-1)I_{n-2} - (n-1)I_n$

giving $\boxed{nI_n = (n-1)I_{n-2} - \cos t \sin^{n-1} t}$

which is an example of a ‘recurrence formula’.

So $I_5 = \frac{1}{5}(4I_3 - \cos t \sin^4 t)$
$= \frac{1}{5}\left(\frac{4}{3}(2I_1 - \cos t \sin^2 t) - \cos t \sin^4 t\right)$

but $I_1 = \int \sin t\,\mathrm{d}t = -\cos t + C$ and so

$I_5 = -\frac{8}{15}\cos t - \frac{4}{15}\cos t \sin^2 t - \frac{1}{5}\cos t \sin^4 t + D$

Exercise. Find $\int \sin^6 t\,\mathrm{d}t$

d: Substitution

Making a *good* substitution can be a powerful technique in simplifying integrals

The key (central) point is

if we set $u = g(x)$ then

$$\mathrm{d}u = g'(x)\,\mathrm{d}x \quad \text{or} \quad \mathrm{d}x = \frac{\mathrm{d}u}{g'(x)}$$

Example. $\int \frac{t}{\sqrt{2-t^2}}\,\mathrm{d}t$

Setting $u = 2 - t^2$ simplifies the integrand:

Let $u = 2 - t^2$ so $\mathrm{d}u = (-2t)\,\mathrm{d}t$ or $\mathrm{d}t = \frac{\mathrm{d}u}{-2t}$

so $\int \frac{t}{\sqrt{2-t^2}}\,\mathrm{d}t = \int \frac{t}{\sqrt{u}}\,\frac{\mathrm{d}u}{-2t}$

$= -\frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}}$ (t eliminated)

$= -\frac{1}{2}\int u^{-1/2}\mathrm{d}u = -\frac{1}{2}\frac{u^{1/2}}{1/2}$

$= -\sqrt{u}$

Note. The answer must involve t, not u:

So $\int \frac{t}{\sqrt{2-t^2}}\,\mathrm{d}t = -\sqrt{u}$ (eliminate u)

$= -\sqrt{2-t^2} + C$

Note. Never forget the arbitrary constant.

d: Substitution

For integrating

$$\int f(x)\,\mathrm{d}x$$

the ‘substitution’ procedure is:

1. find a substitution to try $u = g(x)$
2. replace $\mathrm{d}x$ by $\frac{\mathrm{d}u}{g'(x)}$
3. use $u = g(x)$ to completely eliminate x
4. evaluate the integral (if possible)
5. use $u = g(x)$ to completely eliminate u
6. don’t forget the arbitrary constant

If step 4 fails, either

- another substitution is needed, or
- the substitution method does not work

It helps to be able to identify situations when the method is most likely to work.

d: one factor is derivative of another

Substitution is ***very*** useful if:

one factor is a constant times the derivative of another, i.e.

$$\int f\big(g(x)\big)g'(x)\,\mathrm{d}x$$

for example

$$\int \frac{t}{\sqrt{2-t^2}}\,\mathrm{d}t \qquad \Big(\text{where } t = \text{const.} \times \frac{\mathrm{d}}{\mathrm{d}t}(2-t^2)\Big)$$

$$\int \frac{\cos\theta\sin\theta}{\cos^2\theta - 1}\,\mathrm{d}\theta \qquad \Big(\text{where } \sin\theta = -\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta\Big)$$

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x \qquad \Big(\text{where top is derivative of bottom}\Big)$$

Exercise. Find $\int \cos t \sin t\,\mathrm{d}t$

(we have integrated $\sin x\cos x$ in 3 different ways!)

d: a factor is derivative of another

When one factor g' is the derivative of another g it is usually best to substitute $u = g$, not $u = g'$.

Example. $\int \frac{2\sin\theta}{\cos^2\theta - 1}\,\mathrm{d}\theta \qquad \Big(\sin\theta = -\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta\Big)$

- If we use $u = \sin\theta$, for which $\mathrm{d}\theta = \frac{\mathrm{d}u}{\cos\theta}$

  we obtain $\int \frac{2u}{\cos^2\theta - 1}\,\frac{\mathrm{d}u}{\cos\theta}$.

  It is tricky, but possible, to eliminate θ, using $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-u^2}$

  giving $\int \frac{2u}{-u^2}\,\frac{\mathrm{d}u}{\sqrt{1-u^2}} = \int \frac{-2}{u\sqrt{1-u^2}}\,\mathrm{d}u$

  (which is still not simple)

- If we use $v = \cos\theta$, for which $\mathrm{d}\theta = \frac{\mathrm{d}v}{-\sin\theta}$

  we obtain $\int \frac{2\sin\theta}{v^2-1}\,\frac{\mathrm{d}v}{-\sin\theta}$.

  It is much simpler to eliminate θ in this case

  giving
  $$\int \frac{2\,\mathrm{d}v}{1-v^2} = \int \Big(\frac{1}{1-v} + \frac{1}{1+v}\Big)\mathrm{d}v$$
  $$= -\ln|1-v| + \ln|1+v| + C$$
  $$= \ln\left|\frac{1+\cos\theta}{1-\cos\theta}\right| + C$$

d: top is derivative of the bottom

The special case $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$ is easily integrated.

Let $u = f(x)$ so that $\mathrm{d}x = \frac{\mathrm{d}u}{f'(x)}$, then

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \int \frac{f'(x)}{u}\,\frac{\mathrm{d}u}{f'(x)} = \int \frac{\mathrm{d}u}{u} = \ln|u|$$
$$= \ln|f(x)| + C$$

This situation can be treated as a general rule, not needing its own substitution every time.

Exercise 1. $\int \tan x\,\mathrm{d}x$

Exercise 2. $\int \frac{x^3}{4-x^4}\,\mathrm{d}x$

Exercise 3. $\int \coth x\,\mathrm{d}x$

d: simplifying substitutions

In many cases, substitution helps just to simplify

Exercise. $\int \frac{3x}{\sqrt{4-x}}\,\mathrm{d}x$

There are other (***often***) helpful substitutions:

if $\int (\,\cdot\,)\,\mathrm{d}\cdot$ contains	try this substitution
$\sqrt{ax+b}$	$u^2 = ax+b$
$\sqrt{a^2-x^2}$ or a^2-x^2	$x = a\sin u$ or $x = a\cos u$
$\sqrt{a^2+x^2}$ or a^2+x^2	$x = a\sinh u$ or $x = a\tan u$
$\sqrt{x^2-a^2}$	$x = a\cosh u$ or $x = a\sec u$

d: more substitutions

Exercise 1. $\int \sqrt{4-t^2}\,\mathrm{d}t$

Exercise 2. $\int \frac{\mathrm{d}x}{\sqrt{9+x^2}}$

d: more on simplfying substitutions

if $\int (\,\cdot\,)\,\mathrm{d}\cdot$ contains	try this substitution
$\sqrt{ax+b}$	$u^2 = ax+b$
$\sqrt{a^2-x^2}$ or a^2-x^2	$x = a\sin u$ or $x = a\cos u$
$\sqrt{a^2+x^2}$ or a^2+x^2	$x = a\sinh u$ or $x = a\tan u$
$\sqrt{x^2-a^2}$	$x = a\cosh u$ or $x = a\sec u$

These substitutions can be remembered because they depend on the identities:

$\sqrt{a^2-x^2}$ or a^2-x^2	$\sin^2+\cos^2=1$
$\sqrt{a^2+x^2}$ or a^2+x^2	$\cosh^2-\sinh^2=1$ or $\sec^2-\tan^2=1$
$\sqrt{x^2-a^2}$	$\cosh^2-\sinh^2=1$ or $\sec^2-\tan^2=1$

e: Partial Fractions

A rational function

$$\frac{\text{some polynomial}}{(a_1+b_1x)(a_2+b_2x)\cdots(p_1+q_1x+r_1x^2)(p_2+q_2x+r_2x^2)\cdots}$$

can be rewritten as

$$\text{another polynomial} + \frac{A_1}{a_1+b_1x} + \frac{A_2}{a_2+b_2x} + \cdots + \frac{P_1+Q_1x}{p_1+q_1x+r_1x^2} + \frac{P_2+Q_2x}{p_2+q_2x+r_2x^2} + \cdots$$

Example 1. $\frac{x^2+15x-1}{(2-x)(x^2+2x+3)}$

e: Partial Fractions

Example 2. $\frac{3x^3-11x^2+15x-6}{x^2-3x+2}$