

Use of basic rules and integrals

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$$\text{Exercise 1. } \int \frac{2 dx}{9x^2 - 12x + 8}$$

$$\text{Exercise 2. } \int \frac{-dz}{\sqrt{12z - 4z^2 - 4}}$$

$$\text{Exercise 3. } \int \frac{7 dx}{\sqrt{4x^2 - 4x - 2}}$$

Methods of Integration

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Not all formulae involving standard functions have an integral in terms of standard functions, such as

$$\int \sin \frac{1}{\theta} d\theta \quad \int \frac{e^t}{t} dt \quad \int e^{-x^2} dx$$

In such cases, the integrals lead to new functions, for example the 'error function' and the 'elliptic function'

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad K(z) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - z \sin^2 \theta}}$$

Note. In a *definite integral*, the variable used for integrating is a 'dummy variable'.

So θ or t can be replaced by s , u or anything that is not already used in the integral.

in this course

We will consider a variety of techniques for finding integrals that can be worked out using standard functions, such as

$$\int \frac{x-1}{x^2+3x-2} dx$$
$$\int \cos^7 \theta d\theta \quad \int \frac{t}{\sqrt{2-t^2}} dt$$

and many others.

a: Standard Integrals

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An integral may involve one or more standard anti-derivatives (indefinite integrals)

$$\text{Example. } \int (\operatorname{sech}^2 u + \sqrt[3]{u}) \, du = \tanh u + \frac{3}{4}u^{4/3} + C$$

because $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
 $\frac{d}{dx} \frac{3}{4}x^{4/3} = x^{1/3}$

$$\text{Exercise 1. } \int \sec^2\left(\frac{1}{2}\theta - \pi\right) \, d\theta$$

$$\text{Exercise 2. } \int \frac{4 \, dt}{2 - 3t}$$

$$\text{Exercise 3. } \int \frac{dz}{1 + (3z)^2}$$

$$\text{Exercise 4. } \int \frac{3 \, dx}{\sqrt{1 - (x/2)^2}}$$

b: Algebraic Manipulation

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Algebraic manipulation may turn the integral into one or more standard integrals

$$\text{Exercise 1. } \int \frac{(1+x)^2}{x} \, dx$$

$$\text{Exercise 2. } \int \sin^2 t \, dt$$

$$\text{Exercise 3. } \int \sin x \cos x \, dx$$

c: Integration by Parts

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$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

integration by parts

one part times integral of other, minus integral of derivative of the one times integral of other

Exercise 1. $\int t e^t dt$

(differentiating should simplify the underlined term)

Exercise 2. $\int \sin x \cos x dx$

Exercise 3. $\int \ln z dz$

c: Recurrence Formulae

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Sometimes many integrations by parts are needed to evaluate an integral.

In such cases a 'recurrence formula' can help

Example. Find $\int \sin^5 t dt$

$$\begin{aligned} \text{Let } I_n &= \int \sin^n t dt \\ &= \int \sin t \sin^{n-1} t dt \\ &= \\ &= -\cos t \sin^{n-1} t + (n-1) \int \cos^2 t \sin^{n-2} t dt \end{aligned}$$

$$\begin{aligned} \text{but } \int \cos^2 t \sin^{n-2} t dt &= \int (1 - \sin^2 t) \sin^{n-2} t dt \\ &= \int \sin^{n-2} t dt - \int \sin^n t dt \\ &= I_{n-2} - I_n \end{aligned}$$

$$\text{so } I_n = -\cos t \sin^{n-1} t + (n-1)I_{n-2} - (n-1)I_n$$

giving $nI_n = (n-1)I_{n-2} - \cos t \sin^{n-1} t$

which is an example of a 'recurrence formula'.

$$\begin{aligned} \text{So } I_5 &= \frac{1}{5}(4I_3 - \cos t \sin^4 t) \\ &= \frac{1}{5}\left(\frac{4}{3}(2I_1 - \cos t \sin^2 t) - \cos t \sin^4 t\right) \end{aligned}$$

$$\text{but } I_1 = \int \sin t dt = -\cos t + C \text{ and so}$$

$$I_5 = -\frac{8}{15} \cos t - \frac{4}{15} \cos t \sin^2 t - \frac{1}{5} \cos t \sin^4 t + D$$

Exercise. Find $\int \sin^6 t dt$

d: Substitution

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Making a *good* substitution can be a powerful technique in simplifying integrals

The key (central) point is

if we set $u = g(x)$ then

$$du = g'(x) dx \quad \text{or} \quad dx = \frac{du}{g'(x)}$$

Example. $\int \frac{t}{\sqrt{2-t^2}} dt$

Setting $u = 2 - t^2$ simplifies the integrand:

Let $u = 2 - t^2$ so $du = (-2t) dt$ or $dt = \frac{du}{-2t}$

$$\begin{aligned} \text{so } \int \frac{t}{\sqrt{2-t^2}} dt &= \int \frac{t}{\sqrt{u}} \frac{du}{-2t} \\ &= -\frac{1}{2} \int \frac{du}{\sqrt{u}} \quad (t \text{ eliminated}) \\ &= -\frac{1}{2} \int u^{-1/2} du = -\frac{1}{2} \frac{u^{1/2}}{1/2} \\ &= -\sqrt{u} \end{aligned}$$

Note. The answer must involve t , not u :

$$\begin{aligned} \text{So } \int \frac{t}{\sqrt{2-t^2}} dt &= -\sqrt{u} \quad (\text{eliminate } u) \\ &= -\sqrt{2-t^2} + C \end{aligned}$$

Note. Never forget the arbitrary constant.

d: Substitution

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For integrating

$$\int f(x) dx$$

the 'substitution' procedure is:

1. find a substitution to try $u = g(x)$
2. replace dx by $\frac{du}{g'(x)}$
3. use $u = g(x)$ to completely eliminate x
4. evaluate the integral (if possible)
5. use $u = g(x)$ to completely eliminate u
6. don't forget the arbitrary constant

If step 4 fails, either

- another substitution is needed, or
- the substitution method does not work

It helps to be able to identify situations when the method is most likely to work.

d: one factor is derivative of another¹⁷

Substitution is *very* useful if:

one factor is a constant times the derivative of another,
i.e.

$$\int f(g(x))g'(x) dx$$

for example

$$\int \frac{t}{\sqrt{2-t^2}} dt \quad \left(\text{where } t = \text{const.} \times \frac{d}{dt}(2-t^2)\right)$$

$$\int \frac{\cos \theta \sin \theta}{\cos^2 \theta - 1} d\theta \quad \left(\text{where } \sin \theta = -\frac{d}{d\theta} \cos \theta\right)$$

$$\int \frac{f'(x)}{f(x)} dx \quad \left(\text{where top is derivative of bottom}\right)$$

Exercise. Find $\int \cos t \sin t dt$

(we have integrated $\sin x \cos x$ in 3 different ways!)

d: a factor is derivative of another¹⁸

When one factor g' is the derivative of another g
it is usually best to substitute $u = g$, not $u = g'$.

Example. $\int \frac{2 \sin \theta}{\cos^2 \theta - 1} d\theta \quad \left(\sin \theta = -\frac{d}{d\theta} \cos \theta\right)$

- If we use $u = \sin \theta$, for which $d\theta = \frac{du}{\cos \theta}$

we obtain $\int \frac{2u}{\cos^2 \theta - 1} \frac{du}{\cos \theta}$.

It is tricky, but possible, to eliminate θ ,
using $\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - u^2}$

giving $\int \frac{2u}{-u^2} \frac{du}{\sqrt{1-u^2}} = \int \frac{-2}{u\sqrt{1-u^2}} du$

(which is still not simple)

- If we use $v = \cos \theta$, for which $d\theta = \frac{dv}{-\sin \theta}$

we obtain $\int \frac{2 \sin \theta}{v^2 - 1} \frac{dv}{-\sin \theta}$.

It is much simpler to eliminate θ in this case

giving $\int \frac{2 dv}{1-v^2} = \int \left(\frac{1}{1-v} + \frac{1}{1+v}\right) dv$
 $= -\ln |1-v| + \ln |1+v| + C$
 $= \ln \left| \frac{1+\cos \theta}{1-\cos \theta} \right| + C$

d: top is derivative of the bottom

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The special case $\int \frac{f'(x)}{f(x)} dx$ is easily integrated.

Let $u = f(x)$ so that $dx = \frac{du}{f'(x)}$, then

$$\begin{aligned}\int \frac{f'(x)}{f(x)} dx &= \int \frac{f'(x)}{u} \frac{du}{f'(x)} = \int \frac{du}{u} = \ln |u| \\ &= \ln |f(x)| + C\end{aligned}$$

This situation can be treated as a general rule, not needing its own substitution every time.

Exercise 1. $\int \tan x dx$

Exercise 2. $\int \frac{x^3}{4-x^4} dx$

Exercise 3. $\int \coth x dx$

d: simplifying substitutions

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In many cases, substitution helps just to simplify

Exercise. $\int \frac{3x}{\sqrt{4-x}} dx$

There are other (*often*) helpful substitutions:

| if $\int (\cdot) d\cdot$ contains | try this substitution |
|-----------------------------------|--------------------------------------|
| $\sqrt{ax+b}$ | $u^2 = ax+b$ |
| $\sqrt{a^2-x^2}$ or a^2-x^2 | $x = a \sin u$ or $x = a \cos u$ |
| $\sqrt{a^2+x^2}$ or a^2+x^2 | $x = a \sinh u$ or $x = a \tan u$ |
| $\sqrt{x^2-a^2}$ | $x = a \cosh u$ or $x = a \sec u$ |

d: more substitutions

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Exercise 1. $\int \sqrt{4 - t^2} dt$

Exercise 2. $\int \frac{dx}{\sqrt{9 + x^2}}$

d: more on simplifying substitutions²²

| if $\int (\cdot) d\cdot$ contains | try this substitution |
|--------------------------------------|--------------------------------------|
| $\sqrt{ax + b}$ | $u^2 = ax + b$ |
| $\sqrt{a^2 - x^2}$ or $a^2 - x^2$ | $x = a \sin u$ or $x = a \cos u$ |
| $\sqrt{a^2 + x^2}$ or $a^2 + x^2$ | $x = a \sinh u$ or $x = a \tan u$ |
| $\sqrt{x^2 - a^2}$ | $x = a \cosh u$ or $x = a \sec u$ |

These substitutions can be remembered because they depend on the identities:

$$\begin{array}{l} \sqrt{a^2 - x^2} \\ \text{or } a^2 - x^2 \end{array} \qquad \sin^2 + \cos^2 = 1$$

$$\begin{array}{l} \sqrt{a^2 + x^2} \\ \text{or } a^2 + x^2 \end{array} \qquad \begin{array}{l} \cosh^2 - \sinh^2 = 1 \\ \text{or } \sec^2 - \tan^2 = 1 \end{array}$$

$$\sqrt{x^2 - a^2} \qquad \begin{array}{l} \cosh^2 - \sinh^2 = 1 \\ \text{or } \sec^2 - \tan^2 = 1 \end{array}$$

e: Partial Fractions

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A rational function

$$\frac{\text{some polynomial}}{(a_1+b_1x)(a_2+b_2x)\cdots(p_1+q_1x+r_1x^2)(p_2+q_2x+r_2x^2)\cdots}$$

can be rewritten as

$$\begin{aligned} &\text{another polynomial} + \frac{A_1}{a_1 + b_1x} + \frac{A_2}{a_2 + b_2x} + \cdots \\ &+ \frac{P_1 + Q_1x}{p_1 + q_1x + r_1x^2} + \frac{P_2 + Q_2x}{p_2 + q_2x + r_2x^2} + \cdots \end{aligned}$$

Example 1. $\frac{x^2 + 15x - 1}{(2 - x)(x^2 + 2x + 3)}$

e: Partial Fractions

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Example 2. $\frac{3x^3 - 11x^2 + 15x - 6}{x^2 - 3x + 2}$