Some Applications of Integration

Integration can be used to obtain many properties connected with curves.

Area Between Two Curves

Integration and area are closely related

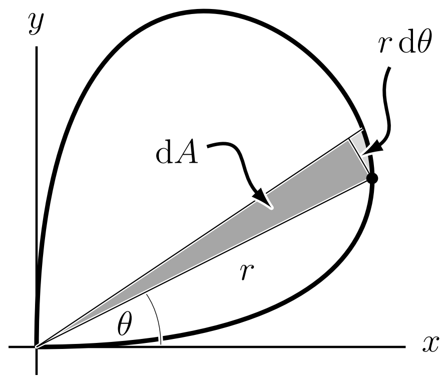
Example.

The curves $y = 2/x$ and $y = 2 - \frac{1}{6}(x-1)^2$ intersect at $x = 1$ and $x = 4$

The area between them is simply

Area in Polar Coordinates

Consider the curve: $r = \sin(2\theta)$ for $0 \le \theta \le \frac{1}{2}\pi$



If θ increases by $\mathrm{d}\theta$ the radial vector r sweeps through an area of $\mathrm{d}A = \frac{1}{2}r^2\mathrm{d}\theta$.

The whole area of the curve is

Length of a Curve

Suppose that $y = f(x)$ is plotted as a curve

and $s(x)$ is the 'arclength' of the curve from some point, then

if x changes by $\mathrm{d}x$ and

y changes by $\mathrm{d}y$

s changes by $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$

$$= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

so the length of the curve from $x = a$ to $x = b$ is

$$\int_{x=a}^{x=b} \mathrm{d}s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Circumference of a Circle

A circle of radius a can be parameterised as

$$x = \quad a\cos t \qquad\qquad y = a\sin t$$

with $\mathrm{d}x = -a\sin t\,\mathrm{d}t \qquad \mathrm{d}y = a\cos t\,\mathrm{d}t$

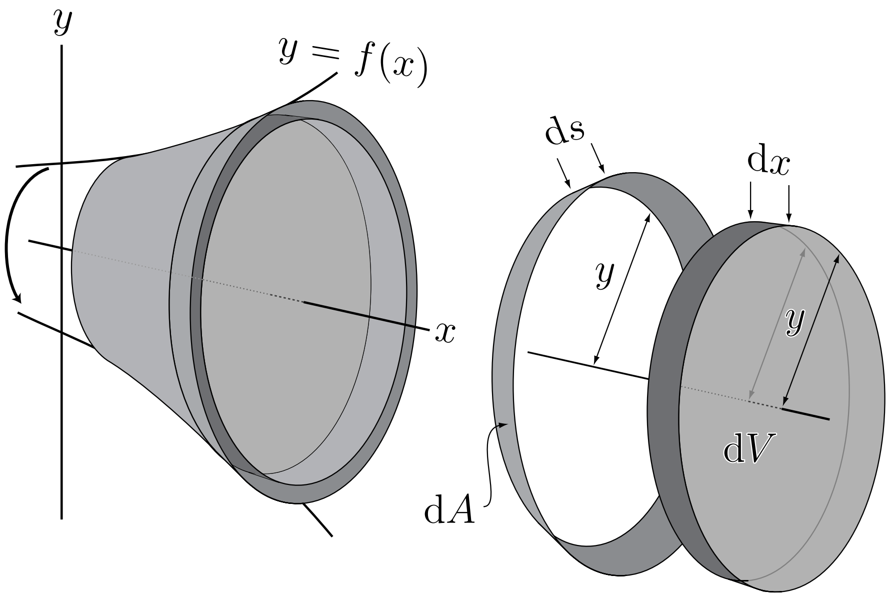
$$\begin{aligned}\mathrm{d}s &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,\mathrm{d}t \\ &= a\,\mathrm{d}t\end{aligned}$$

The arclength of the circumference is

Surfaces and Volumes of Revolution

Rotating a curve $y = f(x)$ about the x-axis creates

- a surface of revolution of area A, and
- a volume of revolution, V



As x increases by $\mathrm{d}x$, the area and volume increase by

$$\mathrm{d}A = 2\pi y\,\mathrm{d}s$$
$$\mathrm{d}V = \pi y^2\,\mathrm{d}x$$

so that the area and volume (from $x = a$ to $x = b$) are

$$A = \int_{x=a}^{x=b} 2\pi y\,\mathrm{d}s \quad \text{and} \quad V = \int_a^b \pi y^2\,\mathrm{d}x$$

Surface and Volume of a Bucket

Example. The line $y = \frac{1}{2}x$ rotated about the x-axis creates a bucket-shape for $2 \le x \le 4$.

Rotation about the y-axis

Rotating the curve

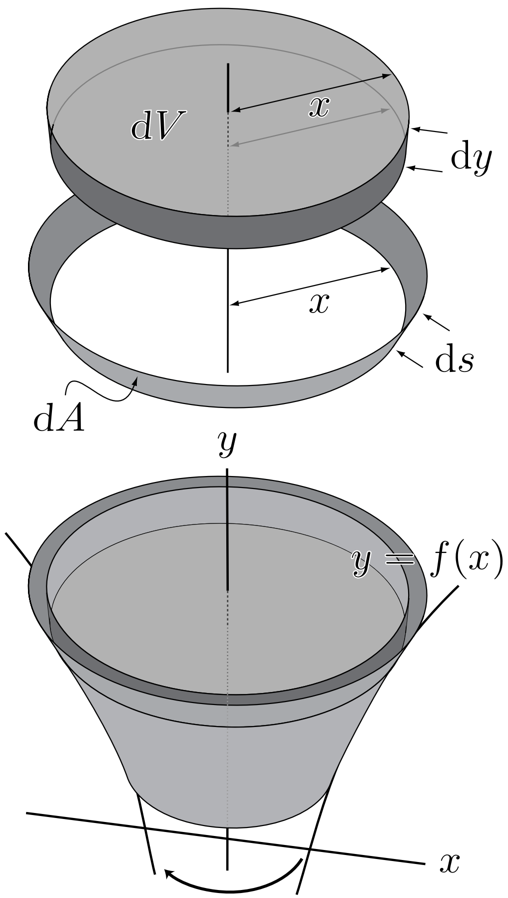
$$y = f(x)$$

about the y-axis, the surface area and volume increase by

$$\mathrm{d}A = 2\pi x \,\mathrm{d}s$$

$$\mathrm{d}V = \pi x^2 \,\mathrm{d}y$$

as x increases by $\mathrm{d}x$.



Hence the area and volume,

from $y = f(a)$

to $y = f(b)$

are

$$A = \int_{x=a}^{x=b} 2\pi x \,\mathrm{d}s \qquad \text{and} \qquad V = \int_a^b \pi x^2 \frac{\mathrm{d}y}{\mathrm{d}x} \,\mathrm{d}x$$

Surface and Volume of a Paraboloid

Example. The parabola $y = \frac{1}{2}x^2$ for $0 \le y \le 2$