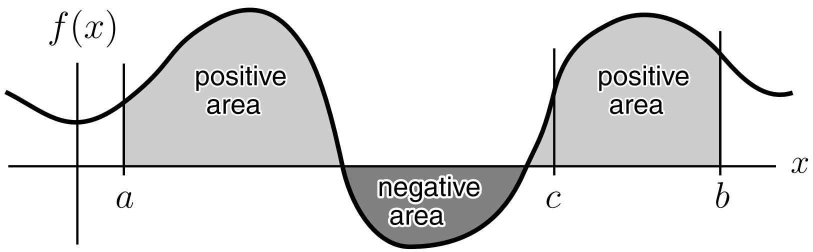
Definite Integrals

The '***definite integral***' symbol $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ in spoken words is:

'the integral from a to b of the function $f(x)$'

Graphically, it represents an area



__between__: the function $f(x)$ and the x-axis, from the vertical line $x = a$ on the left to the vertical line $x = b$ on the right

(where $f(x) < 0$ the 'area' is taken to be negative)

__also__ if the direction of integration is reversed the sign of the area is changed

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

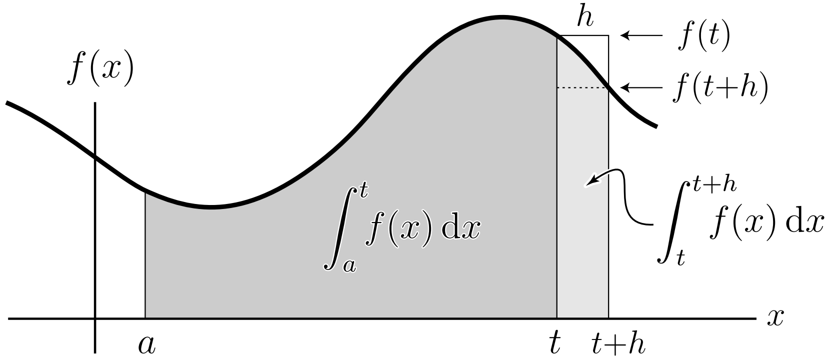
The integral can be split up into separate parts

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

covering the same overall domain, for any $c \in (a, b)$

Fundamental Theorem of Calculus

Suppose that the right end-point is t and that it can change from t to $t + h$



For h small, the light area is approximately $h \times f(t)$

$$\int_t^{t+h} f(x)\,\mathrm{d}x = \int_a^{t+h} f(x)\,\mathrm{d}x - \int_a^t f(x)\,\mathrm{d}x$$
$$= hf(t) + \mathbf{O}(h^2) \quad \text{as} \quad h \to 0$$

so that differentiating the integral gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(x)\,\mathrm{d}x = \lim_{h\to 0} \frac{1}{h}\left(\int_a^{t+h} f(x)\,\mathrm{d}x - \int_a^t f(x)\,\mathrm{d}x\right)$$
$$= \lim_{h\to 0} \big(f(t) + \mathbf{O}(h)\big)$$
$$= f(t)$$

(differentiation and integration are inverse operations)

This fact is 'the fundamental theorem of calculus'

Evaluating Definite Integrals

As a result of the fundamental theorem of calculus, the integral from a to b of $f'(x)$ is simply

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

provided: $f'(x)$ is continuous for all $x \in [a, b]$ and $f(x)$ has derivative $f'(x)$

but, if $f'(x)$ is continuous only for $x \in (a, b)$, then

$$\int_a^b f'(x)\,\mathrm{d}x = \lim_{x \to b^-} f(x) - \lim_{x \to a^+} f(x) = \Big[f(x)\Big]_a^b$$

provided: both limits exist and are finite.
Otherwise, the integral is said to '*diverge*'

Exercise. Integrate $\dfrac{1}{s^2}$ from $s = 1$ to ∞.

Indefinite Integrals

Any function whose derivative is $f'(x)$ is called an 'indefinite integral' of $f'(x)$, written as

$$\int f'(x)\,\mathrm{d}x = f(x) + C$$

for any constant C

Example. $\displaystyle\int \frac{1}{s}\,\mathrm{d}s = \ln|s| + C$ $\quad (s \neq 0)$

Exercise 1. $\displaystyle\int e^t\;\mathrm{d}t =$

Exercise 2. $\displaystyle\int x^n\;\mathrm{d}x =$ $\quad (n \neq -1)$

Exercise 3. $\displaystyle\int \frac{1}{\sqrt[4]{v}}\,\mathrm{d}v =$ $\quad (v > 0)$

Exercise 4. $\displaystyle\int \sec^2\theta\,\mathrm{d}\theta =$

Exercise 5. $\displaystyle\int \sinh w\,\mathrm{d}w =$

Exercise 6. $\displaystyle\int f'(|z|)\,\mathrm{d}z =$

Singularities in Definite Integrals

If the 'integrand' $f(t)$ is singular at $t = c \in (a, b)$, the domain *must be* divided so that $t = c$ is an end-point

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$$

Examples:

$$\int_0^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x \qquad \left\{ \begin{array}{r} \frac{\mathrm{d}}{\mathrm{d}x} 2\sqrt{x} \\ = 1/\sqrt{x} \end{array} \right.$$

$$\int_{-8}^{27} |x|^{-2/3}\,\mathrm{d}x \qquad \left\{ \begin{array}{r} \frac{\mathrm{d}}{\mathrm{d}x} 3x^{1/3} \\ = x^{-2/3} \end{array} \right.$$

singularities like $|x|^{-r}$ are 'integrable' if $r \in (0, 1)$

Non-Integrable Singularities

More Examples:

$$\int_0^2 \frac{1}{t}\,\mathrm{d}t \qquad \left\{ \begin{array}{r} \frac{\mathrm{d}}{\mathrm{d}t} \ln|t| \\ = 1/t \end{array} \right.$$

$$\int_{-2}^{1} \frac{2}{t^3}\,\mathrm{d}t \qquad \left\{ \begin{array}{r} \frac{\mathrm{d}}{\mathrm{d}t}(-1/t^2) \\ = 2/t^3 \end{array} \right.$$

singularities like $|x|^{-r}$ are 'non-integrable' if $r \geq 1$

Some Key Basic Integrals

$f(x)$	$\int f(x)\,\mathrm{d}x$
e^x	$e^x + C$
x^n for $n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$
$1/x$ for $x \neq 0$	$\ln\lvert x\rvert + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\dfrac{1}{x^2+1}$	$\tan^{-1} x + C$
$\dfrac{1}{1-x^2}$ for $\lvert x\rvert < 1$	$\tanh^{-1} x + C$
$\dfrac{1}{x^2-1}$ for $\lvert x\rvert > 1$	$-\coth^{-1} x + C$
$\dfrac{1}{x^2-1}$ for $\lvert x\rvert \neq 1$	$\dfrac{1}{2}\ln\left\lvert\dfrac{x-1}{x+1}\right\rvert + C$
$\dfrac{1}{\sqrt{1-x^2}}$ for $\lvert x\rvert < 1$	$\sin^{-1} x + C$
$\dfrac{1}{\sqrt{x^2+1}}$	$\sinh^{-1} x + C$
$\dfrac{1}{\sqrt{x^2-1}}$ for $\lvert x\rvert > 1$	$\cosh^{-1}\lvert x\rvert + C$
$\dfrac{1}{\sqrt{x^2\pm 1}}$	$\ln\left\lvert x+\sqrt{x^2\pm 1}\right\rvert + C$

Rules for Integration

$$\boxed{\int c\,f(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x}$$

integral with a constant factor

Exercise. integrate $\pi\cos\theta$ w.r.t. θ

$$\boxed{\int f'(ax+b)\,\mathrm{d}x = \frac{1}{a}f(ax+b) + C}$$

integral of a function of a linear function

Exercise. find $\displaystyle\int \sec^2(\pi t - 2)\,\mathrm{d}t$

$$\boxed{\int \big(f(x)+g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x}$$

integral of a sum

Exercise. find $\displaystyle\int \frac{2}{1-x^2}\,\mathrm{d}x$