Functions

functions are a key concept in calculus

Note. In this part of the course we are concerned with functions having real values.

Example 1: $\sqrt{a}$ for real values of $a \geq 0$

- for any number $a \geq 0$
- the function $\sqrt{\cdot}$ 'operates' on a
- and provides a value $\sqrt{a} \geq 0$

Example 2: $\cos\theta$ for real values of θ

- for any number $\theta \in \mathbb{R}$
- the function $\cos$ 'operates' on θ
- and provides a value $\cos\theta \in [-1, 1]$

Example 3: $\sup(A)$ for sets of real numbers A

- for any set A of real numbers
- the function $\sup$ 'operates' on A
- and provides a value $\sup(A) \in \mathbb{R} \cup \{\infty\}$

e.g. $\sup\{1, 2\} = 2$, $\sup[0, \pi) = \pi$, $\sup(\mathbb{N}) = \infty$

What is a Function?

Any function is associated with two important sets:

Domain. the function is able to operate meaningfully on all elements in a set called its 'domain'

Range. when the function operates on an element in its 'domain', it returns an element in a set called its 'range'

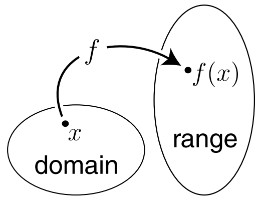
Hence: $\sqrt{\cdot}$ has domain $[0, \infty)$ and range $[0, \infty)$

$\cos$ has domain $\mathbb{R}$ and range $[-1, 1]$

$\sup$ has range $\mathbb{R} \cup \{\infty\}$. Its domain is the set of all non-empty subsets of $\mathbb{R}$

We can say that

a function 'maps' any element in its domain to an element in its range.



Note that:

- a function is ***defined for all elements*** in its domain.
- a function maps any one element in its domain to ***only one element*** in its range.

Notation

Any function can be written symbolically as f, meaning

if x is in the domain of the function
then $f(x)$ takes a value in the range of the function

The function can also be written as $f(\cdot)$ with the dot $(\cdot)$ indicating where the "argument" of the function is to appear.

Examples. sine function : $\sin(\cdot)$
exponential : $e^{\cdot}$
square root : $\sqrt{\cdot}$
modulus : $|\cdot|$

Note. a function is the rule that assigns a value to any argument in the domain.

Thus, $\sin(x)$ is not strictly a function
– it is the value of the function $\sin$ when applied to a number x in its domain

The function $\sin$ is the same in each of

$$\sin(x),\ \sin(t),\ \sin(\tfrac{\pi}{2}),\ \sin(10^4),\ \text{etc.}$$

but these only represent values of the function at different points, not (strictly) functions themselves.

Loose Notation (Defining Functions)

We will often use the loose terminology

'the function $\sin(x)$' or 'the function $\sqrt{1-t^2}$'.

Strictly, this is an abbreviation for the correct definition.

For example. $\sqrt{1-t^2}$ stands for:

a function f defined such that $f(t) = \sqrt{1-t^2}$ for values of t where this formula makes sense

it also suggests that the symbol t is expected to be used as the argument of the function.

The definition can also be written in the form

$$f \ : \ t \ \mapsto \ \sqrt{1-t^2}$$

meaning the function f maps a number t to $\sqrt{1-t^2}$

(for values of t where $\sqrt{1-t^2}$ is meaningful)

Note. This defines the same function $f(\cdot)$ whatever symbol is used in place of t

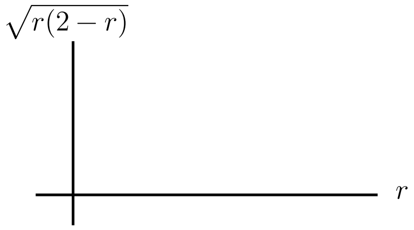
Exercise: State the domain and range of 'the function' $\sqrt{r(2-r)}$

Graphs of Functions

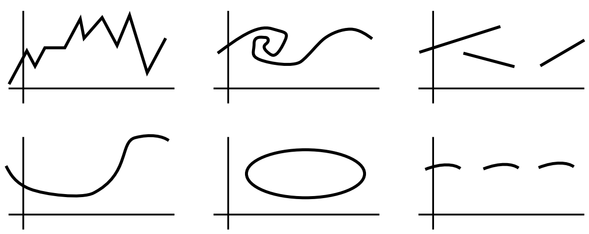
Real-valued functions of real-valued arguments can (usually) be drawn graphically.

Conventionally, the domain is on a horizonatal axis.

Example 1. sketch a graph of the function $\sqrt{r(2-r)}$



Example 2. Which of these curves can represent a function?



Note. Because a function can only be single-valued, the graph of a function cannot intersect a vertical line more than once

Combining Functions

Functions can be added, subtracted, multiplied and divided to create new functions

If f and g are functions with domains A and B, we can define $f+g$, $f-g$, fg and f/g such that

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \quad \text{for } x \in A \cap B \\ (f-g)(x) &= f(x)-g(x) \quad \text{for } x \in A \cap B \\ (fg)(x) &= f(x)g(x) \quad \text{for } x \in A \cap B \\ (f/g)(x) &= f(x)/g(x) \end{aligned}$$

In the latter case, the domain must exclude points where $g(x)=0$. Thus, the domain is

$$\{x \mid x \in A \cap B \text{ and } g(x) \neq 0\}$$

Exercise. Add, subtract, multiply and divide the functions $\sqrt{x(2-x)}$ and $1-x^2$. In each case, what is the domain?

Composition. A 'composite' function $f \circ g$ can be defined such that

$$(f \circ g)(x) = f\big(g(x)\big)$$

Exercise. If $f(x)=\sqrt{x}$ and $g(x)=1-x^2$, what are the functions $f \circ g$ and $g \circ f$ and what are their domains?

Inverse Functions

If a function f maps x to $f(x)$, the inverse process would map $f(x)$ back to x.

A function that does this reverse mapping is called the 'inverse' of the function f, written as f^{-1}.

The domain/range of f^{-1} is the range/domain of f.

The composition of f and f^{-1} simply maps x to itself:

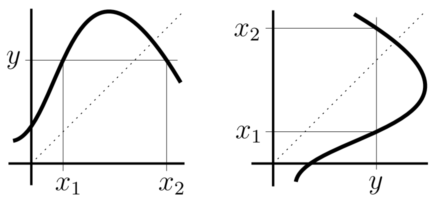
$$f^{-1}\big(f(x)\big) = x \qquad f\big(f^{-1}(x)\big) = x$$

or $$(f^{-1} \circ f)(x) = x \qquad (f \circ f^{-1})(x) = x$$

Example. What is the inverse function of $\sqrt{x}$?

Note. An inverse function cannot be constructed if different points in the domain map to the same point in the range.

Sketch: note reflection about a line at $45°$



If $y=f(x_1)=f(x_2)$ for $x_1 \neq x_2$ then we would have $f^{-1}(y)=x_1$ and $f^{-1}(y)=x_2$.

Inverse Functions

How to find the inverse of a function $f(x)$:

suppose $y=f(x)$ and solve for $x=f^{-1}(y)$

Example 1. Find the inverse of the function $\sqrt[3]{z^5+1}$. What are its domain and range?

Example 2. Limit the domain of $\sqrt[4]{t(8-t)}$ to an interval (including $t=0$) in which the inverse function can be found.

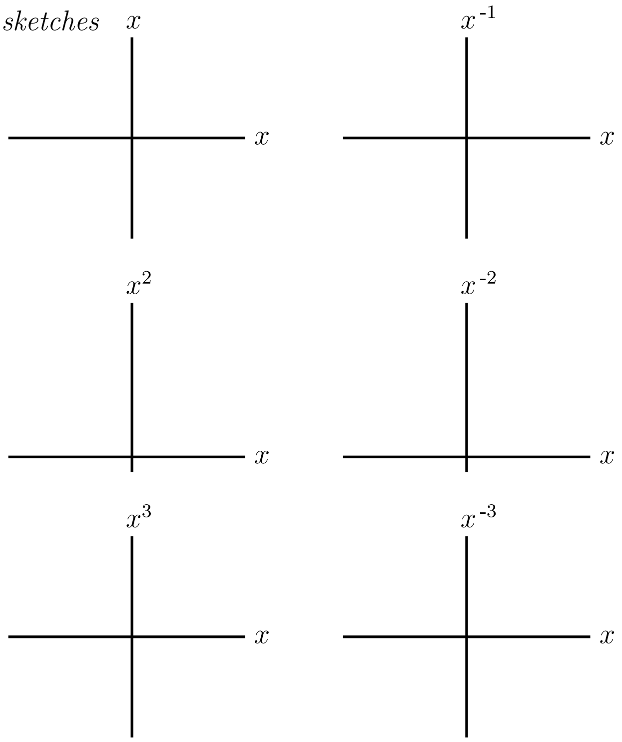
What is the inverse function and what are its domain and range?

Standard Functions

Many functions are well-known:

Powers, e.g. x, x^2, x^3, x^{-1} x^{-2}, x^{-3} etc.

sketches



Polynomials, e.g. $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

The 'order' of a polynomial is the highest power

we also say:

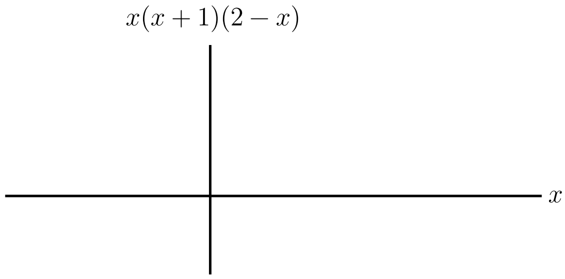
'constant'	for order 0	a
'linear'	for order 1	$ax + b$
'quadratic'	for order 2	$ax^2 + bx + c$
'cubic'	for order 3	$ax^3 + bx^2 + cx + d$

sketches for polynomials depend very much on the order and the values of the coefficients (a_0, a_1, etc.)

useful features include
- the term with the highest power
- where the polynomial is zero
- positions of maxima and minima

Note. the term with the highest power gives the behaviour when $|x|$ is large

Example. sketch the function $x(x+1)(2-x)$



Rational Functions, e.g. $\dfrac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_mx^m}$

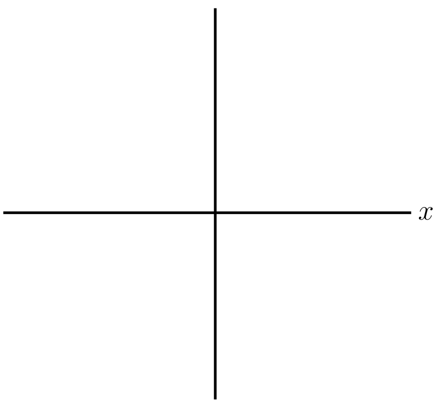
Rational functions are ratios of polynomials.

useful features include
- coefficients of the highest powers
- where the polynomials are zero
- positions of maxima and minima

Notes. the ratio $\dfrac{a_nx^n}{b_mx^m}$ dominates for large $|x|$

zeros arise at zeros of the numerator, and singularities arise at zeros of the denominator (except if numerator and denominator are both zero)

Example. sketch the function $\dfrac{4(x^2-1)(x^2-9)}{(1+x^2)(2x+3)(x-2)}$



Algebraic Functions. e.g. $\sqrt[3]{z^2 - 2}$

Algebraic functions can be written using the usual algebra of adding, subtracting, multiplying, dividing, powers and roots.

Examples: $\sqrt{1-t^4}$, $\dfrac{1-x}{1+x^2}$, $|z|$, $\sqrt[3]{1-\sqrt{y}+y^2}$

Transcendental Functions

Any function that cannot be written as an algebraic function is called a transcendental function.

transcendental functions include

$\sin$	$\cos$	$\tan$	$\cot$
$\sec$	cosec	$\sin^{-1}$	$\cos^{-1}$
$\exp$	$\ln$	$\log$	$\cosh$
$\sinh$	$\sinh^{-1}$	$\cosh^{-1}$	

and many more ...

Exponential and Logarithmic Functions

Exponential functions have the form a^x
for some number $a > 0$ (normally with $a \neq 1$)

The inverse function is written as $\log_a(\cdot)$
for the same number $a > 0$ (if $a \neq 1$)

Hence, if $y = a^x$ then $x = \log_a y$

Notes. a^x has domain $\mathbb{R}$ and range $(0, \infty)$
$\log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$

$$a^{-x} = (\tfrac{1}{a})^x \qquad a^x a^y = a^{x+y}$$

$$\log_a \tfrac{1}{x} = -\log_a x \qquad \log_a(xy) = \log_a x + \log_a y$$

Sketch

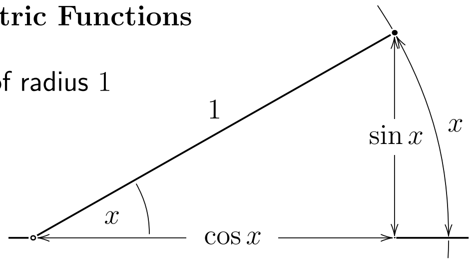
The base e of natural logarithms is the value of a
for which the graph of a^x has slope 1 at $x = 0$

This has a special notation:

$$\exp x = e^x \qquad \ln x = \log_e x$$

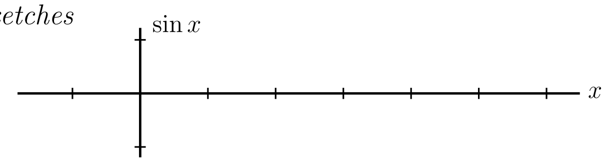
Trigonometric Functions

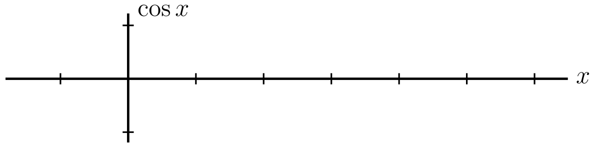
For a circle of radius 1



- angle x is length along the circumference
- $\sin x$ and $\cos x$ are the lengths shown
- x increases by 2π around one full circle
- $\sin x$ and $\cos x$ repeat as x increases by 2π
- $\sin$ and $\cos$ have domain $\mathbb{R}$ and range $[-1, 1]$

sketches



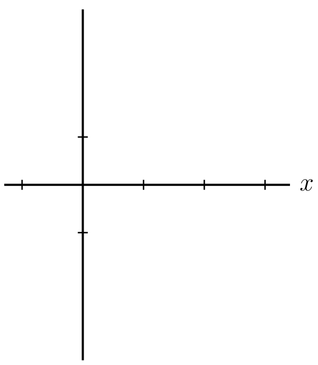


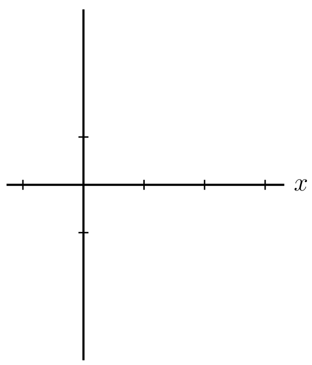
We also define

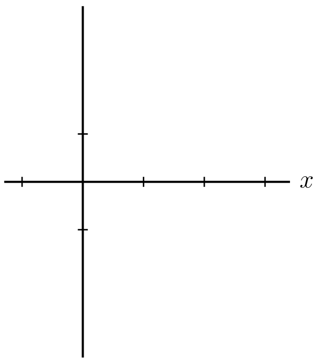
$$\tan = \frac{\sin}{\cos}, \quad \cot = \frac{\cos}{\sin}, \quad \sec = \frac{1}{\cos}, \quad \operatorname{cosec} = \frac{1}{\sin}$$

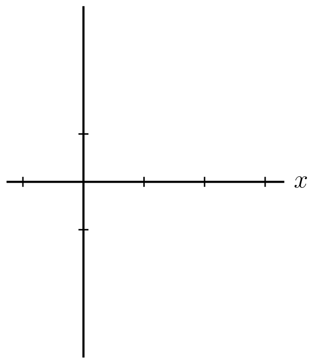
Trigonometric Functions

more sketches



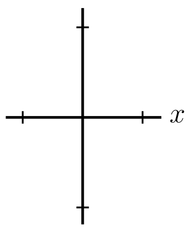


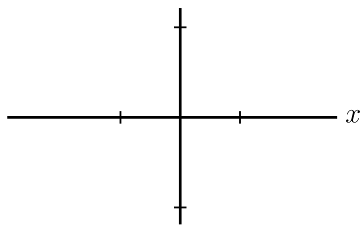


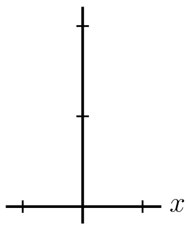


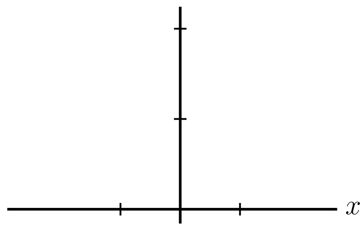
Inverse Trigonometric Functions

for inverses of $\sin$, $\cos$, $\tan$ and $\cot$, the domains are restricted so that the functions are single-valued









What are the domain and range of $\sin^{-1}(\sin x)$?

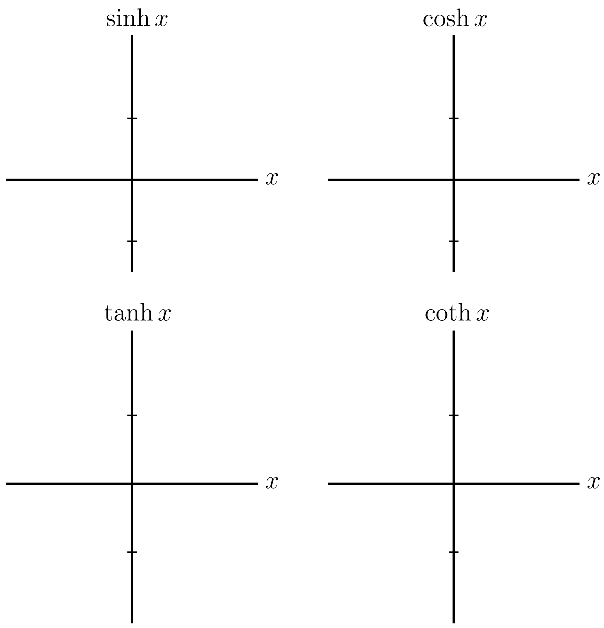
What are the domain and range of $\sin(\sin^{-1} x)$?

Hyperbolic Functions

'hyperbolic' sine, cosine, tangent and cotangent are

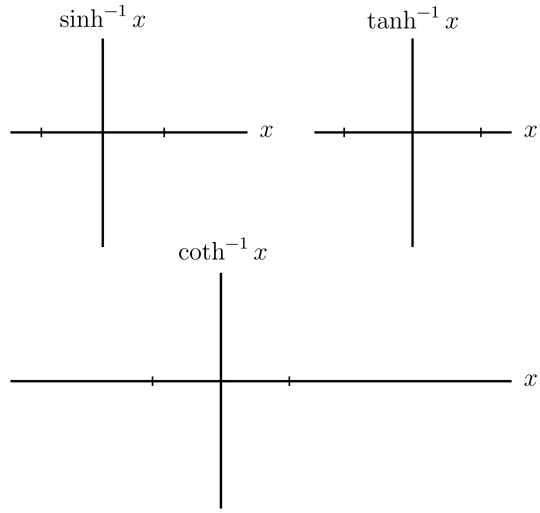
$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x}$$

all defined using the exponential function e^x

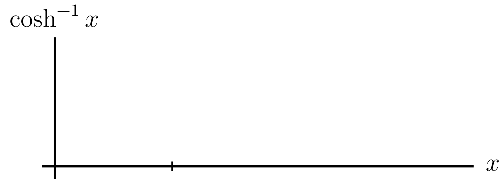


Inverse Hyperbolic Functions

$\sinh$, $\tanh$ and $\coth$ are single-valued, so that their domains are not restricted in constructing inverses



For $\cosh^{-1}$, the domain of $\cosh$ is restricted to $[0, \infty)$



Simple Transformations

A curve (x, y) satisfying $y = f(x)$ is easily shifted or stretched, horizontally or vertically

the curve (x, y) satisfying $y = a + f(x - b)$ is simply

- shifted upwards by a
- shifted to the right by b

the curve (x, y) satisfying $y = cf\left(\frac{x}{d}\right)$ is simply

- stretched vertically by c
- stretched horizontally by d

Examples. Find an equation for the parabola $y = cx^2$ when it is stretched horizontally by a factor of 3 and vertically by a factor of 9?

Sketch the function $1 + \cos(x - \frac{\pi}{3})$

Change of Variable

Sometimes an expression can be written more simply in terms of different variables.

Example: an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $u^2 + v^2 = 1$ (a circle with unit radius) in terms of new variables defined by $x = au$, $y = bv$.

Example 1. Find a change of variables for which the formula $9x^2 + 36x - 4y^2 + 8y - 4 = 0$ can be rewritten as $s^2 - t^2 = 1$

Sketch both expressions.

Example 2. Can an hyperbola $\frac{x^2}{a^2} - \frac{x^2}{b^2} = 1$ be transformed into a circle using only real variables?

Symmetry and Periodicity

symmetry: A function $f(x)$ is said to be

an ***even*** function if $f(x) = f(-x)$
an ***odd*** function if $f(x) = -f(-x)$

for every value of x and $-x$ in the domain of f.

We can also say that $f(x)$ is

symmetric about $x = 0$ if it is even
anti-symmetric about $x = 0$ if it is odd

periodic: A function $f(x)$ is

periodic with 'period' p if $f(x + p) = f(x)$

for every value of x in the domain of f.

Example. What is the minimum period of each of the functions $\sin$, $\cos$, $\tan$ and $\cot$?

Are any of these functions even or odd?

Are their inverse functions even or odd?

Increasing and Decreasing

increasing or decreasing: A function $f(x)$ is

increasing in an interval if $f(x_1) > f(x_2)$
decreasing in an interval if $f(x_1) < f(x_2)$

for every choice of x_1 and x_2 in the interval satisfying $x_1 > x_2$.

Example. Identify all intervals between $-\pi$ and π where $\tan$ and $\cot$ are increasing.

Monotonic is another term that you may encounter. A function is 'monotonic' in an interval if it is not both increasing and decreasing in different parts of the interval, although it may include constant sections.

A function is 'strictly monotonic' in an interval if it is either increasing or decreasing (and never constant).

Example. True or False?
a strictly monotonic functions has an inverse

the inverse of a function is monotonic