# Functions

***functions*** are a key concept in calculus

**Note.** In this part of the course we are concerned with functions having real values.

*Example 1*: $\sqrt{a}$ for real values of $a \geq 0$

- for any number $a \geq 0$
- the function $\sqrt{\cdot}$ 'operates' on $a$
- and provides a value $\sqrt{a} \geq 0$

*Example 2*: $\cos\theta$ for real values of $\theta$

- for any number $\theta \in \mathbb{R}$
- the function $\cos$ 'operates' on $\theta$
- and provides a value $\cos\theta \in [-1, 1]$

*Example 3*: $\sup(A)$ for sets of real numbers $A$

- for any set $A$ of real numbers
- the function $\sup$ 'operates' on $A$
- and provides a value $\sup(A) \in \mathbb{R} \cup \{\infty\}$

e.g. $\sup\{1, 2\} = 2$, $\sup[0, \pi) = \pi$, $\sup(\mathbb{N}) = \infty$



# What is a Function?

Any function is associated with two important sets:

***Domain.*** the function is able to operate meaningfully on all elements in a set called its 'domain'

***Range.*** when the function operates on an element in its 'domain', it returns an element in a set called its 'range'

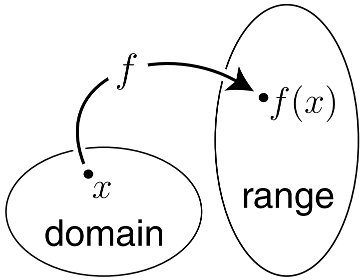
Hence: $\sqrt{\cdot}$ has domain $[0, \infty)$ and range $[0, \infty)$

$\cos$ has domain $\mathbb{R}$ and range $[-1, 1]$

$\sup$ has range $\mathbb{R} \cup \{\infty\}$. Its domain is the set of all non-empty subsets of $\mathbb{R}$

We can say that

a function 'maps' any element in its domain to an element in its range.




**Note that:**

- a function is ***defined for all elements*** in its domain.
- a function maps any one element in its domain to ***only one element*** in its range.

# Notation



Any function can be written symbolically as $f$, meaning

if $x$ is in the domain of the function
then $f(x)$ takes a value in the range of the function

The function can also be written as $f(\cdot)$ with the dot $(\cdot)$ indicating where the "argument" of the function is to appear.

*Examples.*

| | |
|---|---|
| sine function : | $\sin(\cdot)$ |
| exponential : | $e^{\cdot}$ |
| square root : | $\sqrt{\cdot}$ |
| modulus : | $\lvert \cdot \rvert$ |

**Note.** a function is the rule that assigns a value to any argument in the domain.

Thus, $\sin(x)$ is not strictly a function

– it is the value of the function $\sin$ when applied to a number $x$ in its domain

The function $\sin$ is the same in each of

$$\sin(x),\ \sin(t),\ \sin(\tfrac{\pi}{2}),\ \sin(10^4),\ \text{etc.}$$

but these only represent values of the function at different points, not (strictly) functions themselves.

# Loose Notation (Defining Functions)



We will often use the loose terminology

'the function $\sin(x)$' or 'the function $\sqrt{1-t^2}$'.

Strictly, this is an abbreviation for the correct definition.

*For example.* $\sqrt{1-t^2}$ stands for:

a function $f$ defined such that $f(t) = \sqrt{1-t^2}$ for values of $t$ where this formula makes sense

it also suggests that the symbol $t$ is expected to be used as the argument of the function.

The definition can also be written in the form

$$f \ : \ t \ \mapsto \ \sqrt{1-t^2}$$

meaning the function $f$ maps a number $t$ to $\sqrt{1-t^2}$

(for values of $t$ where $\sqrt{1-t^2}$ is meaningful)

**Note.** This defines the same function $f(\cdot)$ whatever symbol is used in place of $t$

*Exercise*: State the domain and range of 'the function' $\sqrt{r(2-r)}$

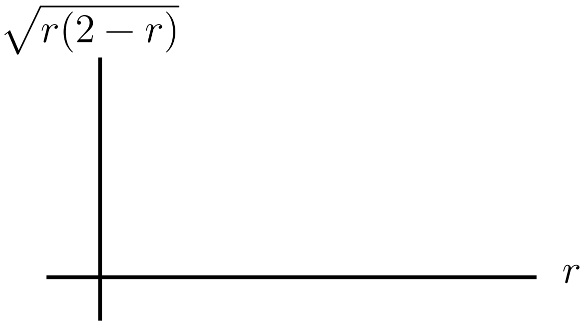
# Graphs of Functions



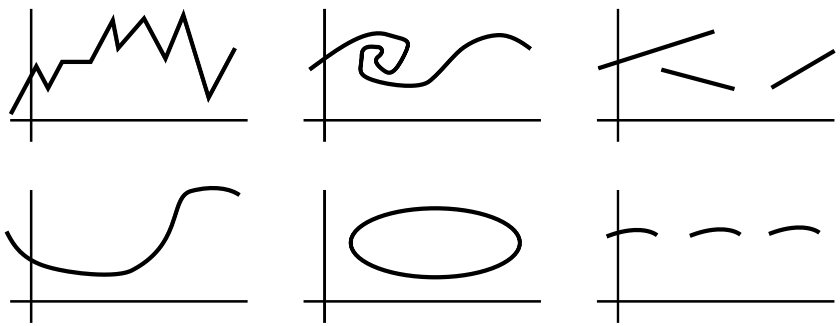
Real-valued functions of real-valued arguments can (usually) be drawn graphically.

Conventionally, the domain is on a horizonatal axis.

*Example 1.* sketch a graph of the function $\sqrt{r(2-r)}$




*Example 2.* Which of these curves can represent a function?



**Note.** Because a function can only be single-valued, the graph of a function cannot intersect a vertical line more than once

# Combining Functions



Functions can be added, subtracted, multiplied and divided to create new functions

If $f$ and $g$ are functions with domains $A$ and $B$, we can define $f+g$, $f-g$, $fg$ and $f/g$ such that

$$\begin{aligned}
(f+g)(x) &= f(x)+g(x) \quad \text{for } x \in A \cap B \\
(f-g)(x) &= f(x)-g(x) \quad \text{for } x \in A \cap B \\
(fg)(x) &= f(x)g(x) \quad \text{for } x \in A \cap B \\
(f/g)(x) &= f(x)/g(x)
\end{aligned}$$

In the latter case, the domain must exclude points where $g(x)=0$. Thus, the domain is

$$\{x \mid x \in A \cap B \text{ and } g(x) \neq 0\}$$

*Exercise.* Add, subtract, multiply and divide the functions $\sqrt{x(2-x)}$ and $1-x^2$. In each case, what is the domain?

***Composition.*** A 'composite' function $f \circ g$ can be defined such that

$$(f \circ g)(x) = f\big(g(x)\big)$$

*Exercise.* If $f(x)=\sqrt{x}$ and $g(x)=1-x^2$, what are the functions $f \circ g$ and $g \circ f$ and what are their domains?

# Inverse Functions

If a function $f$ maps $x$ to $f(x)$, the inverse process would map $f(x)$ back to $x$.

A function that does this reverse mapping is called the 'inverse' of the function $f$, written as $f^{-1}$.

The domain/range of $f^{-1}$ is the range/domain of $f$.

The composition of $f$ and $f^{-1}$ simply maps $x$ to itself:

$$f^{-1}\big(f(x)\big) = x \qquad f\big(f^{-1}(x)\big) = x$$

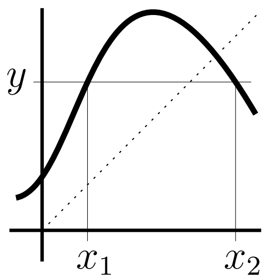
$$\text{or} \quad (f^{-1} \circ f)(x) = x \qquad (f \circ f^{-1})(x) = x$$

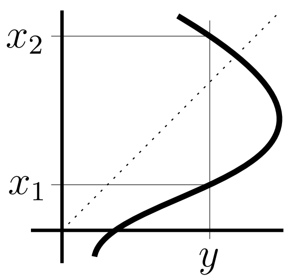
*Example.* What is the inverse function of $\sqrt{x}$ ?

**Note.** An inverse function cannot be constructed if different points in the domain map to the same point in the range.

*Sketch*:

note reflection about a line at $45^\circ$




If $y = f(x_1) = f(x_2)$ for $x_1 \neq x_2$ then we would have $f^{-1}(y) = x_1$ and $f^{-1}(y) = x_2$.



# Inverse Functions

How to find the inverse of a function $f(x)$:

suppose $y = f(x)$ and solve for $x = f^{-1}(y)$

*Example 1.* Find the inverse of the function $\sqrt[3]{z^5 + 1}$. What are its domain and range?

*Example 2.* Limit the domain of $\sqrt[4]{t(8 - t)}$ to an interval (including $t = 0$) in which the inverse function can be found.

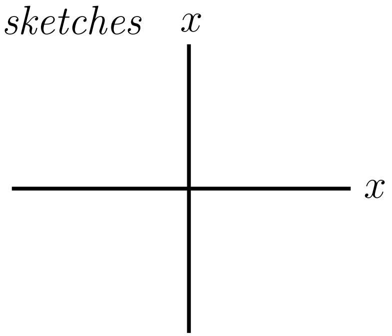
What is the inverse function and what are its domain and range?

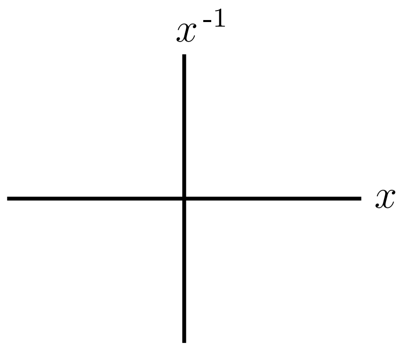
# Standard Functions

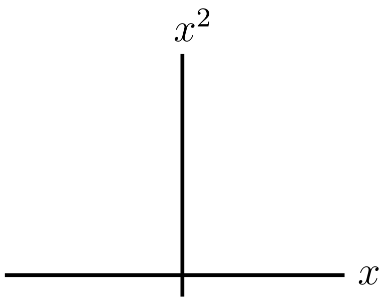
Many functions are well-known:

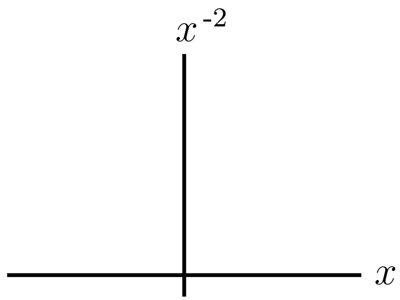
**Powers,** e.g. $x$, $x^2$, $x^3$, $x^{-1}$ $x^{-2}$, $x^{-3}$ etc.

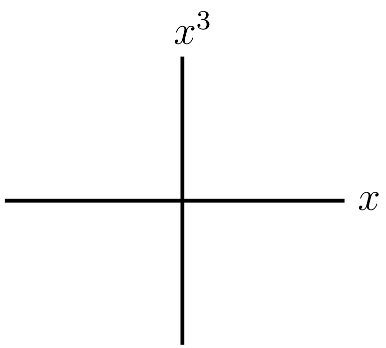
*sketches*

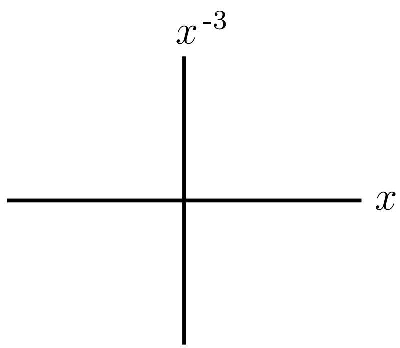



**Polynomials,** e.g. $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

The 'order' of a polynomial is the highest power

*we also say*:

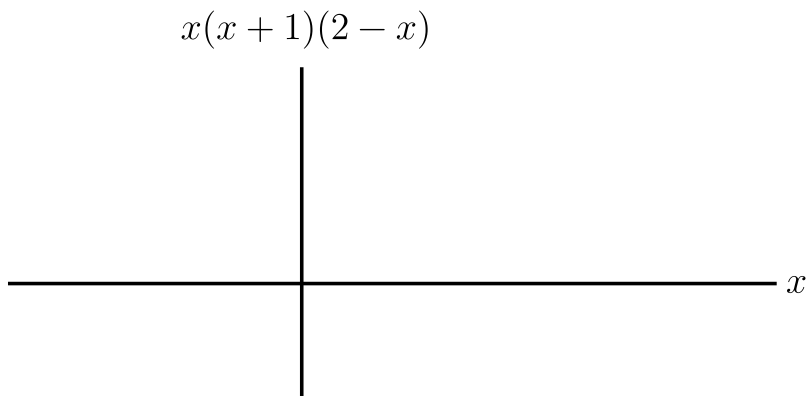
| | | |
|---|---|---|
| 'constant' | for order 0 | $a$ |
| 'linear' | for order 1 | $ax + b$ |
| 'quadratic' | for order 2 | $ax^2 + bx + c$ |
| 'cubic' | for order 3 | $ax^3 + bx^2 + cx + d$ |

sketches for polynomials depend very much on the order and the values of the coefficients ($a_0$, $a_1$, etc.)

*useful features include*
- the term with the highest power
- where the polynomial is zero
- positions of maxima and minima

**Note.** the term with the highest power gives the behaviour when $|x|$ is large

*Example.* sketch the function $x(x+1)(2-x)$



**Rational Functions,** e.g. $\dfrac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$

Rational functions are ratios of polynomials.

| | |
|---|---|
| *useful* | • coefficients of the highest powers |
| *features* | • where the polynomials are zero |
| *include* | • positions of maxima and minima |

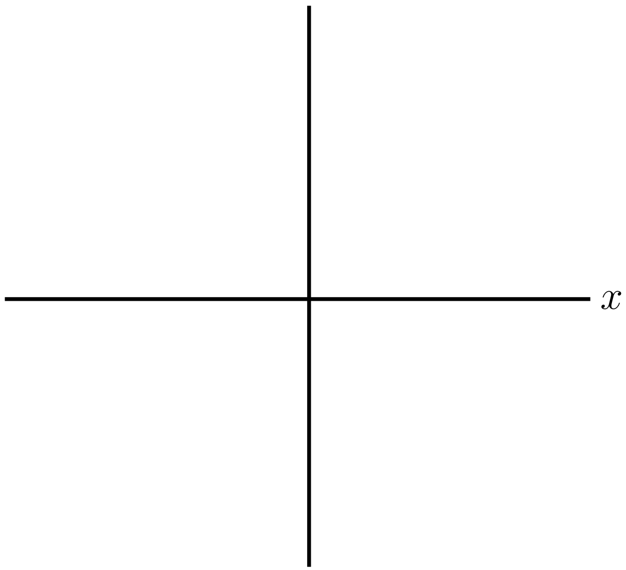
**Notes.** the ratio $\dfrac{a_n x^n}{b_m x^m}$ dominates for large $|x|$

zeros arise at zeros of the numerator, and

singularities arise at zeros of the denominator

(except if numerator and denominator are both zero)

*Example.* sketch the function $\dfrac{4(x^2-1)(x^2-9)}{(1+x^2)(2x+3)(x-2)}$






**Algebraic Functions.** e.g. $\sqrt[3]{z^2 - 2}$

Algebraic functions can be written using the usual algebra of adding, subtracting, multiplying, dividing, powers and roots.

*Examples*: $\sqrt{1-t^4}$, $\dfrac{1-x}{1+x^2}$, $|z|$, $\sqrt[3]{1-\sqrt{y}+y^2}$

**Transcendental Functions**

Any function that cannot be written as an algebraic function is called a transcendental function.

transcendental functions include

| | | | |
|---|---|---|---|
| $\sin$ | $\cos$ | $\tan$ | $\cot$ |
| $\sec$ | $\mathrm{cosec}$ | $\sin^{-1}$ | $\cos^{-1}$ |
| $\exp$ | $\ln$ | $\log$ | $\cosh$ |
| $\sinh$ | $\sinh^{-1}$ | $\cosh^{-1}$ | |

and many more ...

## Exponential and Logarithmic Functions

Exponential functions have the form $a^x$
for some number $a > 0$ (normally with $a \neq 1$)

The inverse function is written as $\log_a(\cdot)$
for the same number $a > 0$ (if $a \neq 1$)

Hence, if $y = a^x$ then $x = \log_a y$

**Notes.** $a^x$ has domain $\mathbb{R}$ and range $(0, \infty)$
$\log_a x$ has domain $(0, \infty)$ and range $\mathbb{R}$

$$a^{-x} = (\tfrac{1}{a})^x \qquad a^x a^y = a^{x+y}$$

$$\log_a \tfrac{1}{x} = -\log_a x \qquad \log_a(xy) = \log_a x + \log_a y$$

*Sketch*

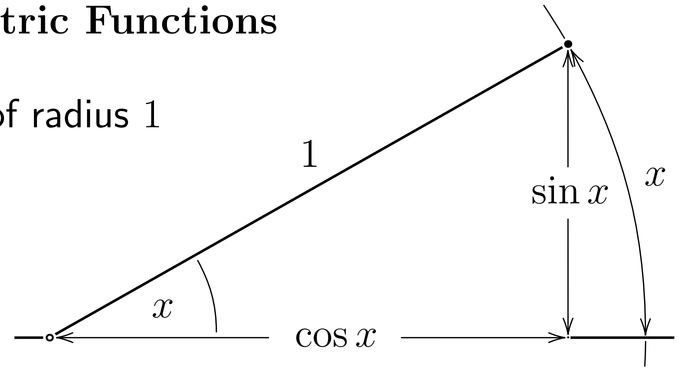
The base $e$ of natural logarithms is the value of $a$
for which the graph of $a^x$ has slope $1$ at $x = 0$

This has a special notation:

$$\exp x = e^x \qquad \ln x = \log_e x$$

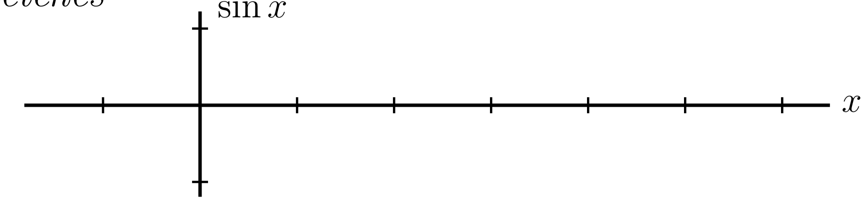
## Trigonometric Functions

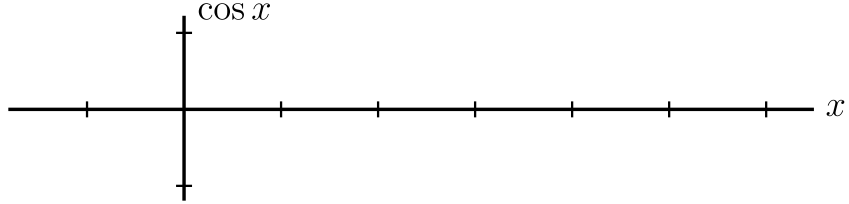
For a circle of radius $1$




- angle $x$ is length along the circumference
- $\sin x$ and $\cos x$ are the lengths shown
- $x$ increases by $2\pi$ around one full circle
- $\sin x$ and $\cos x$ repeat as $x$ increases by $2\pi$
- $\sin$ and $\cos$ have domain $\mathbb{R}$ and range $[-1, 1]$

*sketches*



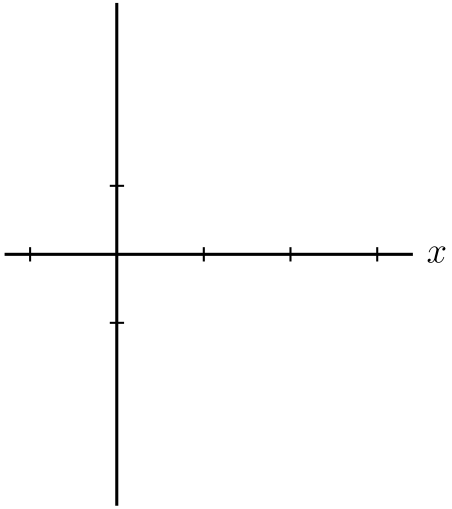

We also define

$$\tan = \frac{\sin}{\cos}, \quad \cot = \frac{\cos}{\sin}, \quad \sec = \frac{1}{\cos}, \quad \mathrm{cosec} = \frac{1}{\sin}$$

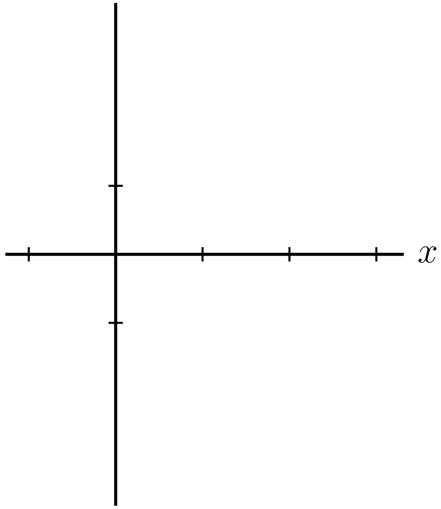
## Trigonometric Functions

*more sketches*

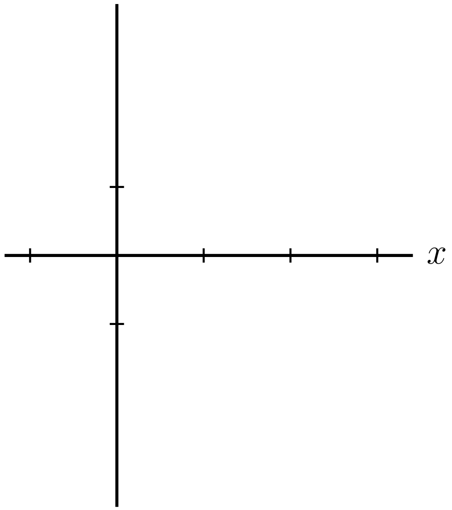
$\tan x$



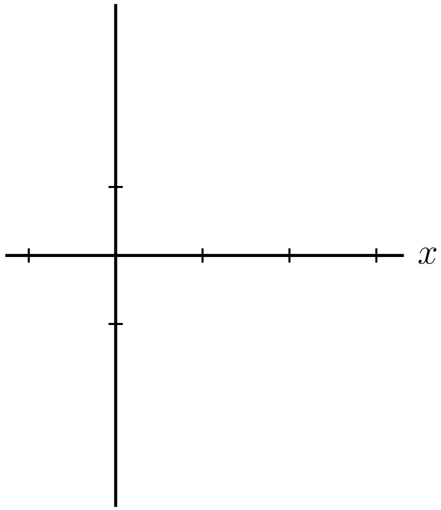

$\cot x$




$\sec x$




$\operatorname{cosec} x$

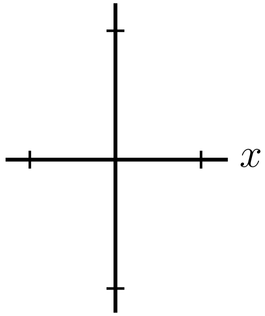




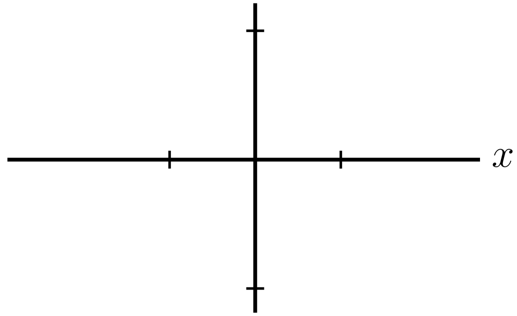

## Inverse Trigonometric Functions

for inverses of $\sin$, $\cos$, $\tan$ and $\cot$, the domains are restricted so that the functions are single-valued

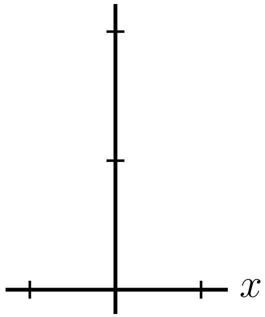
$\sin^{-1} x$



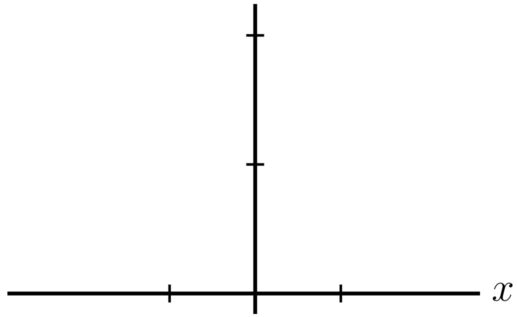

$\tan^{-1} x$




$\cos^{-1} x$




$\cot^{-1} x$




What are the domain and range of $\sin^{-1}(\sin x)$ ?

What are the domain and range of $\sin(\sin^{-1} x)$ ?

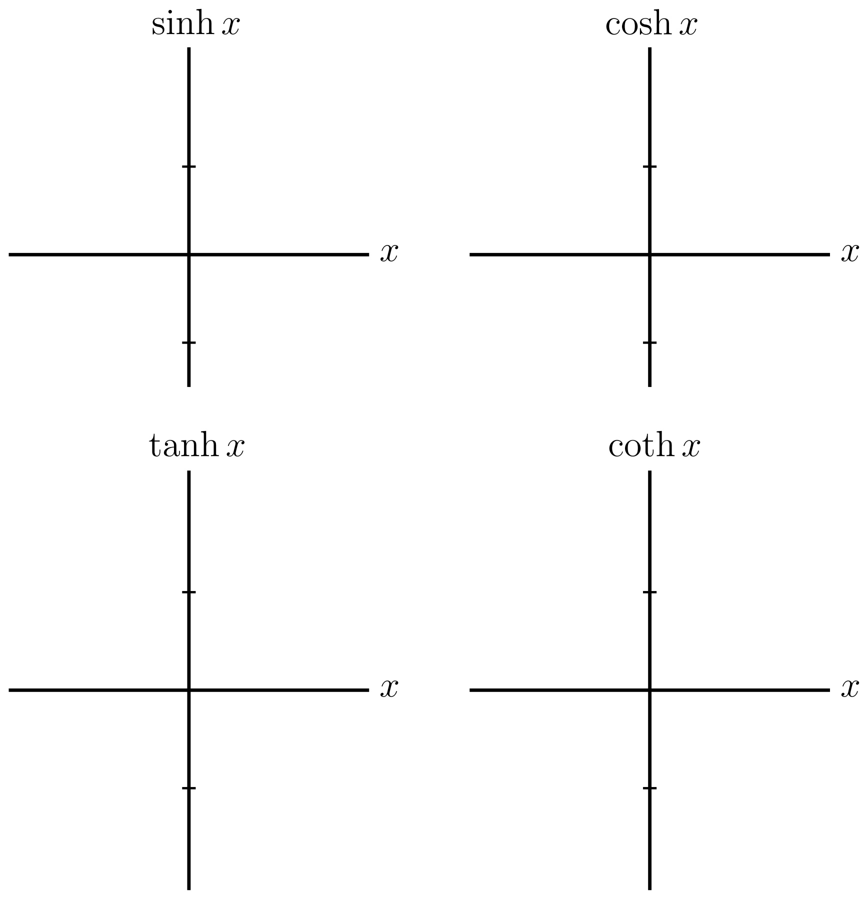
## Hyperbolic Functions

'hyperbolic' sine, cosine, tangent and cotangent are

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x}$$

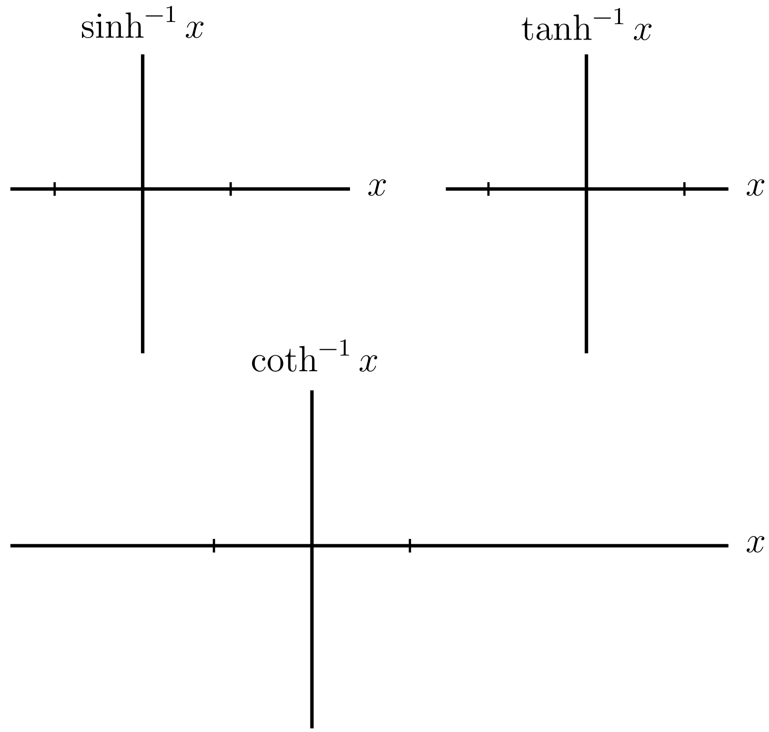
all defined using the exponential function $e^x$



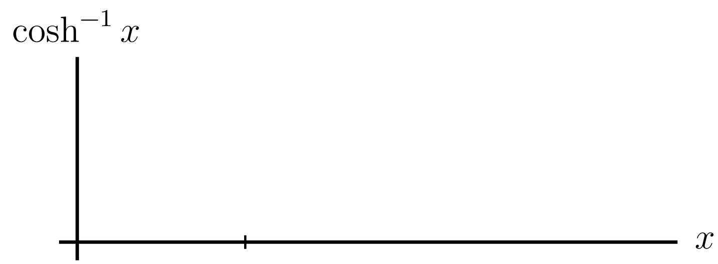



## Inverse Hyperbolic Functions

sinh, tanh and coth are single-valued, so that their domains are not restricted in constructing inverses




For $\cosh^{-1}$, the domain of $\cosh$ is restricted to $[0, \infty)$



## Simple Transformations

A curve $(x, y)$ satisfying $y = f(x)$ is easily shifted or stretched, horizontally or vertically

the curve $(x, y)$ satisfying $y = a + f(x - b)$ is simply

- shifted upwards by $a$
- shifted to the right by $b$

the curve $(x, y)$ satisfying $y = cf\left(\frac{x}{d}\right)$ is simply

- stretched vertically by $c$
- stretched horizontally by $d$

*Examples.* Find an equation for the parabola $y = cx^2$ when it is stretched horizontally by a factor of $3$ and vertically by a factor of $9$?

Sketch the function $1 + \cos(x - \frac{\pi}{3})$



## Change of Variable

Sometimes an expression can be written more simply in terms of different variables.

*Example*: an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes $u^2 + v^2 = 1$ (a circle with unit radius) in terms of new variables defined by $x = au$, $y = bv$.

*Example 1.* Find a change of variables for which the formula $9x^2 + 36x - 4y^2 + 8y - 4 = 0$ can be rewritten as $s^2 - t^2 = 1$

Sketch both expressions.

*Example 2.* Can an hyperbola $\frac{x^2}{a^2} - \frac{x^2}{b^2} = 1$ be transformed into a circle using only real variables?

# Symmetry and Periodicity

**symmetry:** A function $f(x)$ is said to be

an ***even*** function if $f(x) = f(-x)$
an ***odd*** function if $f(x) = -f(-x)$

for every value of $x$ and $-x$ in the domain of $f$.

We can also say that $f(x)$ is

***symmetric*** about $x = 0$ if it is even
***anti-symmetric*** about $x = 0$ if it is odd

**periodic:** A function $f(x)$ is

***periodic*** with 'period' $p$ if $f(x + p) = f(x)$

for every value of $x$ in the domain of $f$.

*Example.* What is the minimum period of each of the functions $\sin$, $\cos$, $\tan$ and $\cot$?

Are any of these functions even or odd?

Are their inverse functions even or odd?



# Increasing and Decreasing

**increasing or decreasing:** A function $f(x)$ is

***increasing*** in an interval if $f(x_1) > f(x_2)$
***decreasing*** in an interval if $f(x_1) < f(x_2)$

for every choice of $x_1$ and $x_2$ in the interval satisfying $x_1 > x_2$.

*Example.* Identify all intervals between $-\pi$ and $\pi$ where $\tan$ and $\cot$ are increasing.

***Monotonic*** is another term that you may encounter. A function is 'monotonic' in an interval if it is not both increasing and decreasing in different parts of the interval, although it may include constant sections.

A function is 'strictly monotonic' in an interval if it is either increasing or decreasing (and never constant).

*Example.* True or False?
a strictly monotonic functions has an inverse

the inverse of a function is monotonic