Definition of a Derivative

The rate of change $f'(t)$ of a function $f(t)$, as its argument t changes, can be expressed as the limit

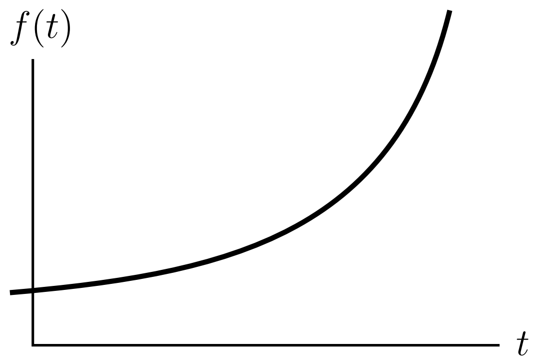
$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

provided this limit exists.

The function f' is called ***the derivative*** of f
Calculating f' from f is called ***differentiation***

Sketch:

$f'(t)$ gives the slope of the tangent at $\big(t, f(t)\big)$



Note. A subtle point is that we temporarily introduce a new variable h in the expression

$$\frac{f(t+h) - f(t)}{h}$$

which is a function of ***two*** variables t and h. We treat t as a constant in calculating $\lim_{h \to 0}$ which, then, eliminates h, so that $f'(t)$ depends only on t.

Alternative forms

Several different notations can be used to represent the derivative of $f(t)$, namely

$$f'(t) = f^{(1)}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \frac{\mathrm{d}f}{\mathrm{d}t}(t) = \mathrm{D}f(t) = f_t(t) = \dot{f}(t)$$

The definition can also be written in an alternative (but equivalent) way, simply by setting $h = s - t$.

It follows that

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) = \lim_{s \to t} \frac{f(s) - f(t)}{s - t}$$

Higher derivatives

If the function $f'(t)$ is differentiated, the result is the 'second' derivative of f, written as $f''(t)$.

Repeated differentiation gives third, fourth, fifth, etc., derivatives, $f'''(t)$, $f''''(t)$, $f'''''(t)$, etc.

It is convenient to write the n^{th} derivative of $f(t)$ as

$$f^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f = \frac{\mathrm{d}^n f}{\mathrm{d}t^n} = \mathrm{D}^n f$$

The first three of these notations are the ones that are most commonly used

Derivative of an Inverse function

If a function f has the inverse f^{-1} and we write $u = f(x)$ and $w = f(z)$, then

$$\frac{f(z) - f(x)}{z - x} = \frac{w - u}{f^{-1}(w) - f^{-1}(u)}$$

It follows that

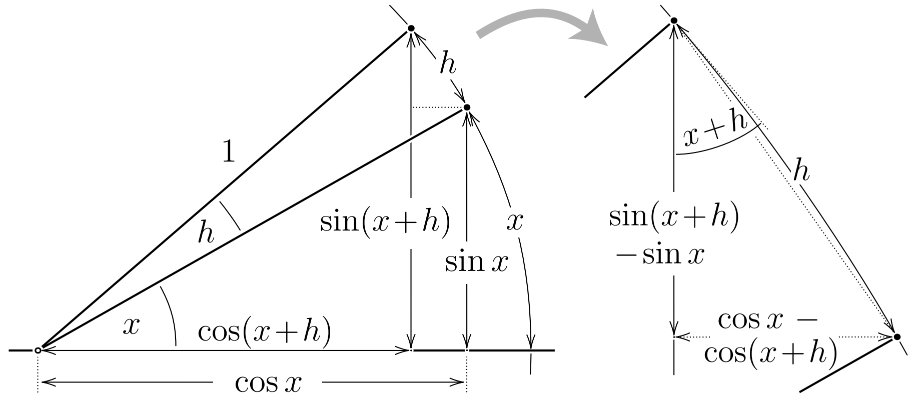
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} f^{-1}(u) &= \lim_{w \to u} \frac{f^{-1}(w) - f^{-1}(u)}{w - u} \\ &= \lim_{z \to x} \frac{z - x}{f(z) - f(x)} \\ &= 1 \Big/ \frac{\mathrm{d}}{\mathrm{d}x} f(x) \end{aligned}$$

unless $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = 0$, in which case $\frac{\mathrm{d}}{\mathrm{d}u} f^{-1}(u)$ is not defined.

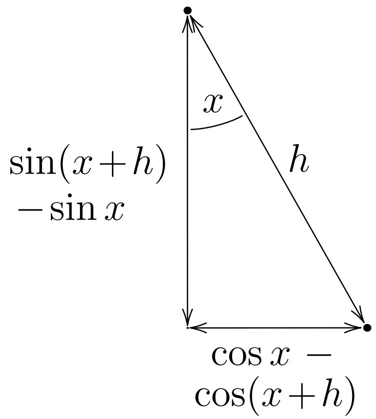
Example. Given that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$, for $n \in \mathbb{N}$, what is $\frac{\mathrm{d}}{\mathrm{d}x} \sqrt[n]{x}$, for $x \geq 0$?

Differentiating $\sin$ and $\cos$

Increasing the angle x by h in the diagram



we can see that, as $h \to 0$, the enlarged diagram approaches this triangle $\longrightarrow$ (stretched by $\frac{1}{h}$)



which demonstrates that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = -\sin x$$

Differentiating e^x and $\ln x$

Recall that the number e is chosen such that the slope of the tangent to the graph of e^x at $x = 0$ is exactly 1.

This means that

$$\lim_{h\to 0} \frac{e^{0+h} - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

We can easily use this to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\, e^x = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x \, \frac{e^h - 1}{h} = e^x$$

since e^x is constant in taking the limit $\lim\limits_{h\to 0}$

If we write $z = e^x$, so that the inverse function gives $x = \ln z$, we know that

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln z = 1 \Big/ \frac{\mathrm{d}}{\mathrm{d}x}\, e^x = 1/e^x = 1/z$$

So we can write

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\, e^x = e^x$$

Differentiating x^r

We can write

$$(x+h)^r = x^r + \tfrac{r}{1} h x^{r-1} + \tfrac{r\times(r-1)}{1\times 2} h^2 x^{r-2} + \cdots$$

using the binomial theorem, for *any* value of r.

If $r \notin \mathbb{N}$ then this results in an infinite power series in h (a topic to be covered in more detail later).

This can be rearranged to give

$$(x+h)^r = x^r + r h x^{r-1} + \mathbf{O}(h^2) \quad \text{as} \quad h \to 0$$

so that

$$\frac{(x+h)^r - x^r}{h} = \frac{r h x^{r-1} + \mathbf{O}(h^2)}{h} = r x^{r-1} + \mathbf{O}(h)$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\, x^r = \lim_{h\to 0} \big(r x^{r-1} + \mathbf{O}(h) \big) = r x^{r-1}$$

More on Order Notation

If $f(h) = \mathbf{O}(h^n)$ as $h \to 0$, with $n > 0$, then

$$\lim_{h\to 0} \left| \frac{f(h)}{h^n} \right| \le C \quad \Longrightarrow \quad \lim_{h\to 0} \left| \frac{f(h)}{h^n} \right| |h^n| \le \lim_{h\to 0} C |h^n|$$

and so

$$\lim_{h\to 0} |f(h)| \le 0 \quad \Longrightarrow \quad \lim_{h\to 0} f(h) = 0$$

Sums and Products

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)+g(x)\big)=f'(x)+g'(x)}$$

derivative of a sum

Example. 1^{st} and 2^{nd} derivatives of $\sin x+\cos x-x^3$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)g(x)\big)=f(x)g'(x)+f'(x)g(x)}$$

derivative of a product

first times derivative of second
plus second times derivative of first

Example. differentiate $e^x \sin x \cos x$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\big(cf(x)\big)=cf'(x)}$$

derivative with a constant factor

Example. differentiate $7e^x \sin x \cos x$

Quotients and Chain Rule

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\,\frac{f(x)}{g(x)}=\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}}$$

derivative of a quotient

bottom times derivative of top minus top times
derivative of bottom, over bottom squared

Example. Find $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\dfrac{3t^2-2}{t^2+1}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\,f\big(g(x)\big)=f'\big(g(x)\big)g'(x)}$$

chain rule
or function of a function rule

derivative of the outer function
times derivative of the inner function

Example. If $y=(x^3+2)^5$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Example. Find $\dfrac{\mathrm{d}}{\mathrm{d}t}\,f(at+b)$

More Trigonometric Derivatives

$\boxed{\tan x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

this also equals $1 + \tan^2 x$ (why?)

$\boxed{\cot x}$

$\boxed{\operatorname{cosec} x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\cot x \operatorname{cosec} x$$

$\boxed{\sec x}$

Inverse Trigonometric Derivatives

$\boxed{\sin^{-1} x}$

$\boxed{\cos^{-1} x}$

Let $y = \cos^{-1} x$ for $x \in [-1, 1]$, $y \in [0, \pi]$
so $x = \cos y$ giving $\frac{\mathrm{d}x}{\mathrm{d}y} = -\sin y$
so $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = \frac{\mathrm{d}y}{\mathrm{d}x} = 1/\frac{\mathrm{d}x}{\mathrm{d}y} = -1/\sin y$
Using $\cos^2 y + \sin^2 y = 1$ and $\sin y \geq 0$
(for $y \in [0, \pi]$) we can solve for $\sin y$:
$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$, and so
$\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -1/\sqrt{1 - x^2}$ for $x \in (-1, 1)$.

$\boxed{\tan^{-1} x}$

Derivatives of $\sinh$ and $\cosh$

$\boxed{\sinh x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$\boxed{\cosh x}$

$\boxed{\sinh^{-1} x}$

$\boxed{\cosh^{-1} x}$

Let $y = \cosh^{-1} x$ for $x \in [1, \infty)$, $y \in [0, \infty)$
so $x = \cosh y$ giving $\frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y$
so $\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{\mathrm{d}y}{\mathrm{d}x} = 1/\frac{\mathrm{d}x}{\mathrm{d}y} = 1/\sinh y$

Using $\cosh^2 y - \sinh^2 y = 1$ and $\sinh y \geq 0$ (for $y \in [0, \infty]$) we can solve for $\sinh y$:

$\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$, and so $\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = 1/\sqrt{x^2 - 1}$ for $x \in (1, \infty)$.

More Hyperbolic Derivatives

$\boxed{\tanh x}$

$\boxed{\coth x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\cosh x}{\sinh x} = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\mathrm{cosech}^2 x$$

this also equals $1 - \coth^2 x$ (why?)

$\boxed{\mathrm{sech}\, x}$

$\boxed{\mathrm{cosech}\, x}$

Derivatives of $\tanh^{-1}$ and $\coth^{-1}$

$\tanh^{-1} x$

$\coth^{-1} x$

l'Hôpital's Rule

The definition of derivative helps to make it clear why l'Hôpital's rule works.

We will show why it works for functions $f(x)$ and $g(x)$ that are continuous at $x = a$, with $f(a) = g(a) = 0$

Since $f(a) = 0$ we must have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a}$$

Likewise $g'(a) = \lim_{x \to a} \dfrac{g(x)}{x - a}$ (since $g(a) = 0$)

and so

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x)}{x - a} \frac{x - a}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Given that l'Hôpital's rule applies as $x \to a$, for functions tending to either zero or plus or minus infinity, it is easy to demonstrate that the rule applies for functions that tend to either 0 or $\pm\infty$, as $x \to \infty$

Let $x = 1/z$ then, using l'Hôpital's rule and the chain rule

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{z \to 0^+} \frac{f(1/z)}{g(1/z)} = \lim_{z \to 0^+} \frac{-f'(1/z)/z^2}{-g'(1/z)/z^2}$$

$$= \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

Parametric Differentiation

Curves defined parametrically involve two functions, such as $x = t^3 - t$, $y = t^2$

Sketch:

In general we would have $x = f(t)$, $y = g(t)$ for two functions f and g.

The slope of the curve at any point $\big(f(t), g(t)\big)$ is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)}{f'(t)}$$

Example. What is the slope of the curve (x, y) given that $x = t^3 - t$ and $y = t^2$?
Hence show that the curve is multivalued for $x \in [-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}]$.

Implicit Differentiation

Curves of (x, y) that are defined through a single relation between x and y can also be differentiated to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, using the procedure

- treat y as the function $y(x)$
- differentiate with respect to x
- solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$

Example 1. If $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Note. The derivative is often found in terms of both x and y and is only meaningful at points on the curve.

Example 2. If $u + v^2 = \sin(u^2 + v)$ find $\dfrac{\mathrm{d}u}{\mathrm{d}v}$

Logarithmic Differentiation

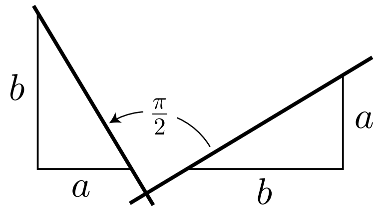
Differentiating a function consisting of positive terms multiplied or divided together can often be made easier by first taking logarithms

Example 1. $y = \dfrac{x(1+x^2)\sqrt{x^2-1}}{x-2}$ for $x > 2$

Example 2. $v = t^t$ for $t > 0$

Example 3. $w = \dfrac{e^s(1-s)}{2-s^2}$ for $s > \sqrt{2}$

Curves at Right Angles



If a line has slope a/b
then a line rotated by $\frac{\pi}{2}$
has slope $-b/a$.
Product of the slopes is -1

Generally:

If curves A and B have slopes $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_A$ and $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_B$ and $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_A \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_B = -1$ where the curves intersect, then the curves intersect at right angles ($\frac{\pi}{2}$ or $90°$)

Example. The formulae $x^2+y^2 = r^2$ and $y = x\tan\theta$ define two curves for constant r or θ (r and θ represent polar coordinates)

Sketch:

For $x^2 + y^2 = r^2$ we have $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_r = -x/y$.
For $y = x\tan\theta$ we have $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta = \tan\theta = y/x$.
So $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_r \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta = -1$ where the curves intersect.
i.e.: r and θ offer an 'orthogonal coordinate system'

Turning Points and Critical Points

A function $f(t)$ has a ***critical point*** where $f'(t) = 0$

A critical point a (where $f'(a) = 0$) is a

'***maximum***' if $f(t) < f(a)$ for all t close to a

'***minimum***' if $f(t) > f(a)$ for all t close to a

If $f(t) < f(a)$ on one side of a and $f(t) > f(a)$ on the other (for all t close enough to a) then a is called a

'***point of inflection***'.

Sketch:

At a critical point a, if

$f''(a) > 0$ then the point is a minimum

$f''(a) < 0$ then the point is a maximum

$f''(a) = 0$ then further information is needed

In fact, if the *first* non-zero derivative is of odd order, then a is a point of inflection. If it is of even order, then it has the same effect as $f''(a)$ above.

Derivatives in Curve Sketching

Knowing the location and type (max, min, inflection) of critical points is valuable additional information that can be used in curve sketching.

Example 1. Sketch the function $6\dfrac{1+x^2}{3+x^4}$.

Example 2. Sketch the function $\dfrac{t(t-2)}{t^2-1}$

(even knowing there are no turning points is useful)