

## The Moment of a Force

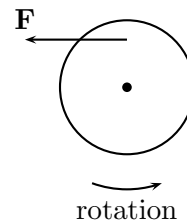
When we consider cases where forces act on a body of non-zero size (i.e. not a particle), the main new aspect that we need to take account of is that such a body can *rotate*, as well as move through space, under the action of forces.

(The complete treatment of cases like this requires more theory of vectors than we have covered, so we shall consider only two dimensional (2-D) cases. The bodies themselves do not have to be 2-D; the important point is that all the forces (more precisely, the lines of action of the forces) should lie in or be parallel to a single plane. Such forces are said to be *coplanar*.)

If forces act on a finite size body they may produce a *turning effect*. This is measured by the *moment* of the force.

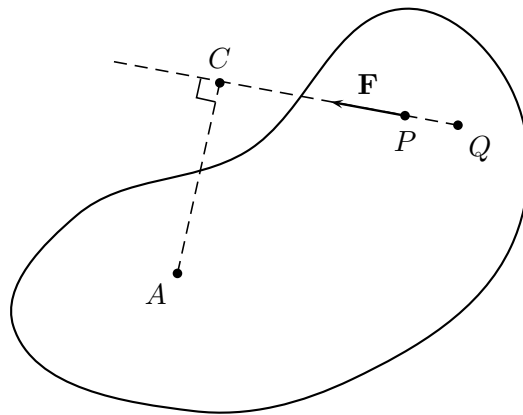
Wheel with fixed centre.

The further away  $F$  is from the centre, the greater the turning effect.



The **moment** of a force  $\mathbf{F}$  (applied at a point  $P$ ) about a point  $A$  is equal to  $\pm Fd$  where:

- $F$  is the magnitude of the force.
- $d$  is the perpendicular distance from  $A$  to the line of action of the force.



$$d = AC$$

The units of moment are newton metres (Nm).

**Note:** If the force is the same  $\mathbf{F}$ , with the same line of action, but applied at a different point  $Q$ , the moment about  $A$  is unchanged.

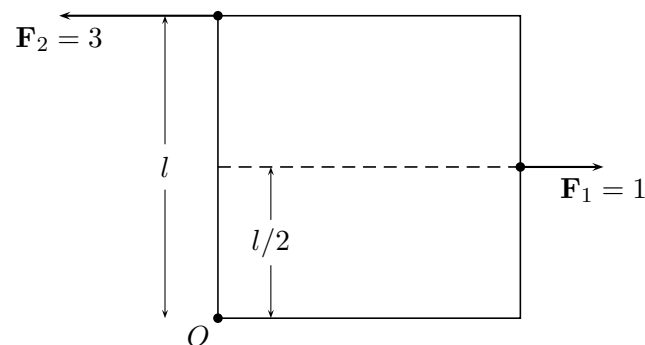
We take:

+ sign for rotation anticlockwise.

– sign for rotation clockwise.

If there are several coplanar forces acting, (perhaps with different lines of action), the total turning effect about a point  $A$  is the *algebraic sum* of the moments about  $A$  (i.e. the signs are included).

**Example:** Forces on a square of side  $l$ :



$$\text{Moment of } \mathbf{F}_1 \text{ about } O = -1 \times \frac{l}{2} = -l/2$$

$$\text{Moment of } \mathbf{F}_2 \text{ about } O = 3 \times l = 3l$$

$$\text{Total moment} = 3l - \frac{l}{2} = \frac{5l}{2}.$$

We can find the moment of a force  $\mathbf{F}$  about a point  $A$  by finding its magnitude  $F$  and the perpendicular distance  $d$  of the line of action from  $A$  and then calculating  $\pm Fd$ , choosing the correct sign for clockwise or anticlockwise.

However, it is usually best to resolve the force into two separate forces,  $\mathbf{F}_1$  parallel to the  $X$ -axis and the  $\mathbf{F}_2$  other parallel to the  $Y$ -axis. We can then easily find the moments of these separately, and the total moment is the sum of these.

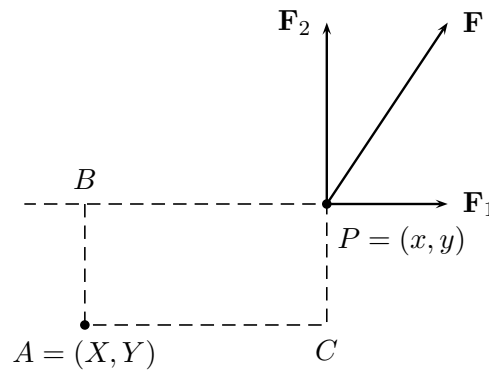
Suppose  $\mathbf{F}$  acts at  $P$ , whose coordinates are  $(x, y)$ . We want the moment about  $A$ , whose coordinates are  $(X, Y)$ .

Let  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j}$ , so  $F_1$  and  $F_2$  are the two components.

Clearly  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

where  $\mathbf{F}_1 = F_1 \mathbf{i}$ , parallel to the  $X$ -axis

and  $\mathbf{F}_2 = F_2 \mathbf{j}$ , parallel to the  $Y$ -axis.



**Note:**

perpendicular distance from  $A$  to line of action of  $\mathbf{F}_1$  is  $AB = y - Y$

perpendicular distance from  $A$  to line of action of  $\mathbf{F}_2$  is  $AC = x - X$

So total moment is  $-F_1(y - Y) + F_2(x - X)$ .

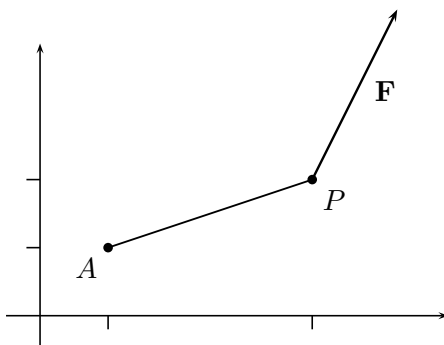
(Note signs:  $\mathbf{F}_1$  acts clockwise ( $-$ ), but  $\mathbf{F}_2$  acts anticlockwise ( $+$ ).)

**Example 1:** Find the moment of force  $\mathbf{F} = 6\mathbf{i} - 3\mathbf{j}$ , applied at  $(4, 5)$  about the point  $(1, 2)$ .

$$\begin{aligned}\text{Here } (x, y) &= (4, 5) \\ (X, Y) &= (1, 2) \\ \mathbf{F} &= (6, -3)\end{aligned}$$

$$\begin{aligned}\text{so moment} &= -6(5 - 2) + (-3)(4 - 1) \\ &= -18 - 9 = -27 \text{ Nm (i.e. clockwise)}\end{aligned}$$

**Example 2:** Find the moment of a force of 6 newtons, applied in the direction of  $\mathbf{i} + 2\mathbf{j}$ , at point  $(4, 2)$ , about  $(1, 1)$ .



**Answer:** Note  $\mathbf{F}$  is parallel to  $\mathbf{i} + 2\mathbf{j}$ . We need to find the components of  $\mathbf{F}$ .

The magnitude of  $\mathbf{i} + 2\mathbf{j}$  is  $\sqrt{1^2 + 2^2} = \sqrt{5}$ ,  
 so a unit vector in the direction of the force is  $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ .

$\mathbf{F}$  has magnitude 6, so  $\mathbf{F} = \frac{6}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$ .

Hence  $F_1 = \frac{6}{\sqrt{5}}$  and  $F_2 = \frac{12}{\sqrt{5}}$ .

Now  $P = (x, y) = (4, 2)$  and  $A = (X, Y) = (1, 1)$

$$\text{so moment} = -\frac{6}{\sqrt{5}}(2 - 1) + \frac{12}{\sqrt{5}}(4 - 1) = \frac{30}{\sqrt{5}} = 6\sqrt{5} \text{ Nm.}$$

**Note:** Special case: the moment of force  $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$  applied at  $(x, y)$  about the origin is  $-F_1y + F_2x$ , since  $X$  and  $Y$  are both zero.

## Resultants of forces on bodies of non-zero size

If different forces act at different points then the total moment about any point  $P$  is the algebraic (i.e. including the signs) sum of each moment about  $P$ .

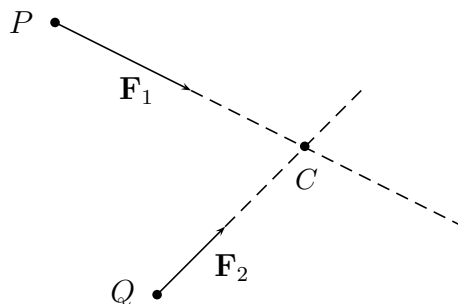
Suppose we have two forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$ . The resultant force  $\mathbf{R}$  is the force which has the same effect as  $\mathbf{F}_1$  and  $\mathbf{F}_2$  combined. Clearly the vector  $\mathbf{R}$  is the vector sum of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  so

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2.$$

However we also need the line of action of  $\mathbf{R}$  so as to produce the correct total moment about any point. There are two cases.

**Case 1**  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are not parallel.

$\mathbf{F}_1$  is applied at  $P$  and  $\mathbf{F}_2$  at  $Q$  as shown.



The moment of  $\mathbf{F}_1$  is the same if it is applied at any point on the line of action. Therefore it is the same as if it is applied at  $C$ , where the lines of action cross.

Likewise, the moment of  $\mathbf{F}_2$  is the same if it is applied at any point on the line of action. Therefore it is the same as if it is applied at  $C$ , where the lines of action cross.

Hence, provided the line of action of  $\mathbf{R}$  passes through  $C$  we get the correct total moment.

**Example 1:** Calculate the resultant and its line of action for the two forces

$\mathbf{F}_1 = 2\mathbf{i} + \mathbf{j}$  N applied at point  $(1, 0)$  and

$\mathbf{F}_2 = -3\mathbf{i} + 3\mathbf{j}$  N applied at point  $(0, 0)$ .

First calculate the vector sum

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (2, 1) + (-3, 3) = (-1, 4)$$

Since vector  $\mathbf{F}_1 = (2, 1)$ , the gradient of its line of action is  $\frac{1}{2}$ .

The equation of the line of action is

$$y = \frac{1}{2}x + c_1 \quad \text{and since it passes through } (1, 0)$$

$$0 = \frac{1}{2} + c_1 \quad \Rightarrow \quad c_1 = -\frac{1}{2} \quad \text{giving the equation}$$

$$y = \frac{1}{2}x - \frac{1}{2} \quad (1)$$

Vector  $\mathbf{F}_2 = (-3, 3)$ , so the gradient of its line of action is  $\frac{3}{-3} = -1$ .

The equation of the line of action is

$$y = (-1)x + c_2 \quad \text{and since it passes through } (0, 0)$$

$$0 = 0 + c_2 \Rightarrow c_2 = 0 \quad \text{giving the equation}$$

$$y = -x \quad (2)$$

The point of intersection of (1) and (2) is when

$$\begin{aligned} -x &= \frac{1}{2}x - \frac{1}{2} \\ \therefore -2x &= x - 1 \\ \therefore x &= \frac{1}{3} \quad \text{and from (2)} \quad y = -\frac{1}{3} \end{aligned}$$

Vector  $\mathbf{R} = (-1, 4)$ , so the gradient of its line of action is  $\frac{4}{-1} = -4$ .

The equation of its line of action is

$$y = -4x + c_3 \quad \text{and since it passes through } \left(\frac{1}{3}, -\frac{1}{3}\right)$$

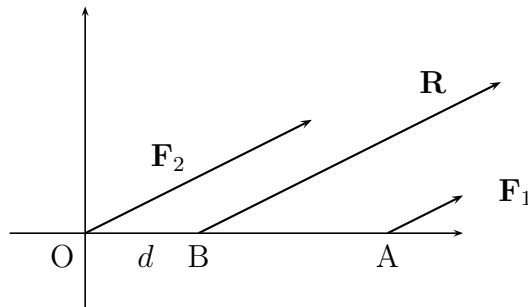
$$-\frac{1}{3} = -4\frac{1}{3} + c_3 \Rightarrow c_3 = 1$$

giving the equation

$$y = -4x + 1 \quad \text{which is the line of action of the resultant } \mathbf{R}.$$

**Case 2** If the forces  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are parallel their lines of action do not cross. The vector resultant  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$  as before. We can find the line of action by taking moments about a point. Do by means of an example.

**Example 2:**  $\mathbf{F}_1 = 2\mathbf{i} + \mathbf{j}$  N is applied at  $A(2, 0)$ .  $\mathbf{F}_2 = 6\mathbf{i} + 3\mathbf{j}$  N is applied at  $O(0, 0)$ . Note that  $\mathbf{F}_2$  is parallel to  $\mathbf{F}_1$ . Find the resultant  $\mathbf{R}$  and the point  $B$  where its line of action crosses the  $X$ -axis. Hence find the equation of the line of action.



$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = 8\mathbf{i} + 4\mathbf{j}$  which is also parallel to  $\mathbf{F}_1$  and  $\mathbf{F}_2$ .  
Its magnitude is  $\sqrt{80}$  N.

Let the line of action of  $\mathbf{R}$  cross the  $X$ -axis at  $B(d, 0)$ . Consider moments about  $O$ .

1. At  $A(2, 0)$ ,  $\mathbf{F}_1$  consists of force  $2\mathbf{i}$  which has moment  $2 \times 0 = 0$  Nm about  $O$ , together with  $1\mathbf{j}$  which has moment  $1 \times 2 = 2$  Nm about  $O$ .  
The total moment of  $\mathbf{F}_1$  about  $O$  is thus 2 Nm.

2. The line of action of  $\mathbf{F}_2$  passes through  $O$  so the moment about  $O$  is zero.

3. At  $B(d, 0)$ ,  $\mathbf{R}$  consists of force  $8\mathbf{i}$  which has moment  $8 \times 0 = 0$  Nm about  $O$ , together with  $4\mathbf{j}$  which has moment  $4 \times d = 4d$  Nm about  $O$ . The total moment of  $\mathbf{R}$  about  $O$  is thus  $4d$  Nm.

The total moment of  $\mathbf{R}$  must equal the total moment of  $\mathbf{F}_1$  and  $\mathbf{F}_2$  so

$$4d = 2 + 0 \Rightarrow d = 0.5$$

Thus the line of action of  $\mathbf{R}$  passes through  $(0.5, 0)$ .

Since  $\mathbf{R} = 8\mathbf{i} + 4\mathbf{j}$  its line of action must have gradient  $\frac{4}{8} = \frac{1}{2}$ . The equation of the line of action is thus

$$y = \frac{1}{2}x + c$$

and since it goes through  $(0.5, 0)$ ,  $0 = 0.5 \times 0.5 + c \Rightarrow c = -0.25$ .

the equation of the line of action is thus  $y = 0.5x - 0.25$ .

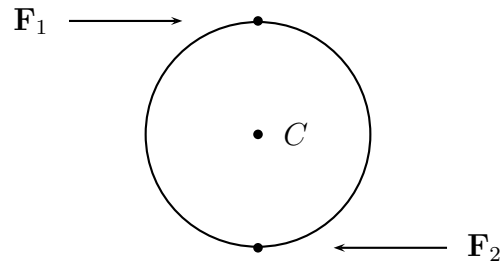
More than 2 forces Now we know how to combine any two forces into a resultant and how to calculate the line of action of the resultant we can repeat the process to find the resultant of 3 or more forces and to find its line of action. (N.B. We assume here that the forces are coplanar – effectively in 2D only.)



### Special case – Couples

If we have  $\mathbf{F}_2 = -\mathbf{F}_1$  then the line of action of  $\mathbf{F}_2$  is in the opposite direction (antiparallel) to that of  $\mathbf{F}_1$ . Provided the lines of action are not the same, but parallel to each other, then we have a turning effect even though the resultant is zero

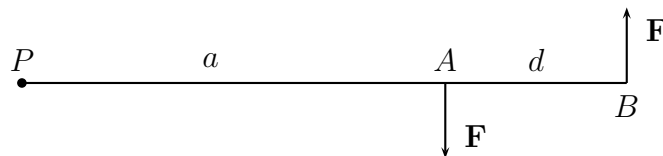
$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = 0$$



This situation, where there is a turning effect but with resultant equal to zero, is called a couple. Clearly we cannot use a single force to represent this.

When a couple acts upon a body there is no change in position but there may be a rotation.

Clearly the couple has a moment, and we can find the magnitude of this as follows.



Here we have equal and opposite forces of magnitude  $F$  with lines of action through points  $A$  and  $B$  which are a distance  $d$  apart. The line  $PAB$  is perpendicular to the lines of action. The moment about point  $P$  is

$$-Fa + F(a + d) = Fd$$

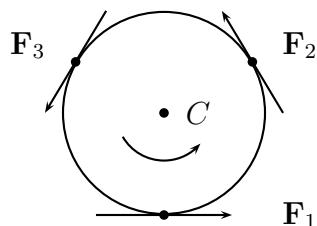
and this is the magnitude of the couple.

Note: The magnitude of the couple depends on  $F$  and  $d$  only, but is independent of the distance  $a$ .

Therefore the turning effect of the couple is the same about all points, so if we find the moment about any point we have its magnitude.

Generalisation to more than two forces. If we have more than two forces then we can have a resultant which is zero even if the lines of action are not parallel or antiparallel. If the moment about any point is not equal to zero then we have a couple.

e.g. consider three forces with the same magnitude, arranged as shown, with their lines of action at angles of  $120^\circ$  to each other, around a circle.



The resultant  $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$ , since the force polygon is a closed equilateral triangle. Clearly there is a moment about the centre of the circle  $C$ . Therefore we have a couple.

## 1.1 Equilibrium of Coplanar forces

Coplanar means that the lines of action of all the forces are in a single plane (or are parallel to the plane) i.e. in 2D.

A body will be in equilibrium under the action of coplanar forces if :-

(i) The resultant is zero.

AND (ii) They do not reduce to a couple.

This is really three conditions since we need both the  $X$ -component and the  $Y$ -component of the resultant to be zero.

Now since the moment of a couple is independent of the point about which it is taken, we can write these conditions as :-

(i) The resultant is zero.

AND (ii) The moment about any point is zero.

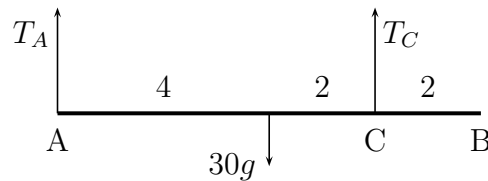
Hence we always need to check three conditions.

**Example 1:**

A uniform plank,  $AB$ , is 8 m long and has mass 30 kg.

It is supported in equilibrium in a horizontal position by two vertical inextensible ropes. One of the ropes is attached to the plank at  $A$  and the other rope to the point  $C$ , where  $BC = 2$  m.

Find the tension in each rope.



Resolve vertically (up)

$$T_A + T_C - 30g = 0 \quad (1)$$

Moments about A

$$6T_C - 30g \times 4 = 0 \quad (2)$$

N.B. No need to resolve horizontally since there are no horizontal forces.

From (2)

$$\Rightarrow T_C = \frac{30 \times 9.81 \times 4}{6} = 196.2 \text{ N}$$

From (1)

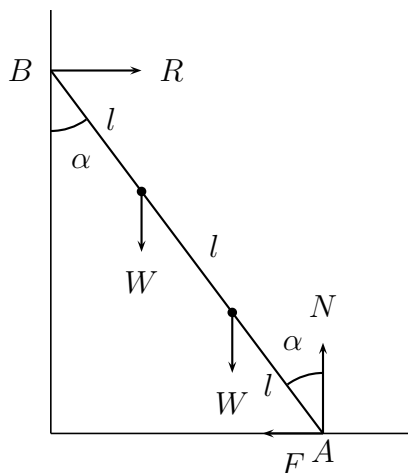
$$T_A = 30g - T_C = 98.1 \text{ N}$$

In fact we could get  $T_A$  slightly quicker by taking moments about C

$$30g \times 2 - 6T_A = 0 \Rightarrow T_A = \frac{60 \times 9.81}{6} = 98.1 \text{ N}$$

so sometimes it is useful to take moments about two points.

**Example 2a:** A light rod of length  $3l$  leans against a smooth vertical wall and a rough horizontal floor at an angle  $\alpha$  to the downward vertical as shown.



Weights  $W$  are placed at  $1/3$  and  $2/3$  of the way along the rod. Find the friction force needed for equilibrium.

Let the normal reaction at the base of the rod  $A$  be  $N$ .

Let the normal reaction at the top of the rod  $B$  be  $R$ .

Let the friction force at the base of the rod be  $F$ .

The friction force acts horizontally to the left since if slipping occurs the base would move horizontally to the right. The two normal reactions are perpendicular to the two surfaces.

1. Resolve vertically:

$$N - 2W = 0 \quad (1)$$

2. Resolve horizontally:

$$R - F = 0 \quad (2)$$

3. Take moments about  $A$ :

$$W(l \sin \alpha) + W(2l \sin \alpha) - R(3l \cos \alpha) = 0 \quad (3)$$

From (3)

$$3Wl \sin \alpha - 3Rl \cos \alpha = 0 \Rightarrow R = W \tan \alpha$$

and from (2)

$$F = R = W \tan \alpha$$

Note that (1) was not used here. It would be needed if we wanted to calculate  $N$ , and we shall use it in the following example.

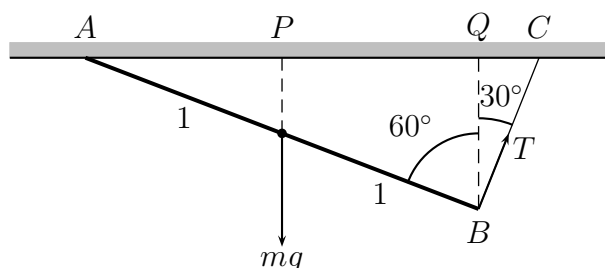
**Example 2b:** Verify that the moment about  $B$  is also zero.

$$\text{Moment about } B = -W(l \sin \alpha) - W(2l \sin \alpha) + N(3l \sin \alpha) - F(3l \cos \alpha)$$

Now use  $N = 2W$  from (1) and the result of Example 1a:  $F = W \tan \alpha$

$$\begin{aligned} \text{Moment about } B &= -3Wl \sin \alpha + 6Wl \sin \alpha - W \tan \alpha(3l \cos \alpha) \\ &= 3Wl \sin \alpha - 3Wl \sin \alpha = 0 \end{aligned}$$

**Example 3:** A uniform rod of mass 3 kg and length 2 m has one end attached by a smooth hinge to a ceiling at  $A$ . The other end  $B$  is attached to a light inelastic string. The other end of the string is attached to the ceiling at  $C$ . The rod makes an angle of  $30^\circ$  with the horizontal and the string an angle of  $60^\circ$  to the horizontal, as shown. Find the tension in the string.



For the uniform rod we shall take the force of gravity as acting at the mid point of the rod. (This is the centre of gravity – see later).

Draw vertical lines through  $B$  and the centre of the rod up to the ceiling as shown.

Note that the angle  $\hat{A}BC = 60^\circ + 30^\circ = 90^\circ$ .

$$\text{Length } AP = 1 \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

The line of action of the tension is  $BC$  which is perpendicular to  $AB$ . Thus the perpendicular distance of the line of action of tension  $T$  is the length  $AB = 2$  m.

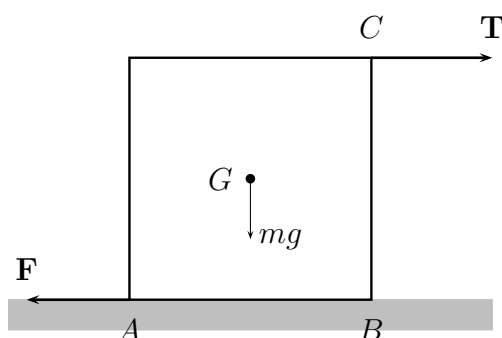
Take moments about  $A$ .

$$\begin{aligned}
 -mg \times AP + T \times AB &= 0 \\
 -3 \times 10 \times \frac{\sqrt{3}}{2} + T \times 2 &= 0 \\
 \therefore T &= \frac{3 \times 10 \times \sqrt{3}}{4} \\
 &= 12.99 \text{ N.}
 \end{aligned}$$

Note 1. There will be horizontal and vertical forces on the rod at the hinge  $A$ . By taking moments about  $A$  we avoided having to calculate these.

Note 2. The hinge in this problem is smooth. If it were (very) rough then the rod could be supported without any string by the moment of the hinge at  $A$ . We assume the hinge is smooth so that there is no moment at  $A$  due to the hinge itself.

**Example 4:** A cube of side 2 m and mass  $m$  kg rests on a rough horizontal surface as shown.



The cube is pulled in a horizontal direction by a light inextensible string attached at  $C$ . The coefficient of static friction is  $\mu$ . The tension in the string  $\mathbf{T}$  is gradually increased from 0.

Show that if  $\mu = 0.6$  the cube will tilt before sliding.

Show that if  $\mu = 0.2$  the cube will slide before tilting.

Answer

Clearly if it does tilt then it will rotate about  $B$ .

For the cube to tilt the total moment about  $B$  must be greater than 0.

The total moment about  $B$  is  $mg \times 1 - T \times 2$ ,  
so at the point it is just about to tilt

$$T = \frac{1}{2}mg \text{ N}$$

Also at this point of there is no vertical or horizontal motion so

Resolving vertically (up):  $R - mg = 0$  where  $R$  is the normal reaction.

Resolving horizontally (right):  $T - F = 0$

$$\therefore R = mg \quad \text{and} \quad F = T = 0.5mg$$

The maximum friction available is  $F = \mu R = \mu mg$ .

Thus if  $\mu$  is greater than 0.5 the cube will tilt first.

If it is less than 0.5 it will slide first.