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The thick flame asymptotic limit and Damköhler's hypothesis

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Abstract
We derive analytical expressions for the burning rate of a flame propagating in a prescribed steady parallel flow whose scale is much smaller than the laminar flame thickness. In this specific context, the asymptotic results can be viewed as an analytical test of Damköhler's hypothesis relating to the influence of the small scales in the flow on the flame; the increase in the effective diffusion processes is described. The results are not restricted to the adiabatic equidiffusional case, which is treated first, but address also the influence of non-unit Lewis numbers and volumetric heat losses. In particular, it is shown that non-unit Lewis number effects become insignificant in the asymptotic limit considered. It is also shown that the dependence of the effective propagation speed on the flow is the same as in the adiabatic equidiffusional case, provided it is scaled with the speed of the planar non-adiabatic flame.

1. Introduction

Current views on premixed flame propagation in turbulent flow fields[1] are, to a large extent, still based on two influential hypotheses proposed by Damköhler in 1940 [1].

According to the first hypothesis, the large flow scales wrinkle the flame without a significant change in its structure. The increase in the effective propagation speed U_T is thus associated with increased flame area, with local normal propagation speed and flame thickness being those of the laminar planar flame U_L and δ_L, say; these are given by $U_L = \sqrt{D/\tau}$ and $\delta_L = \sqrt{D\tau}$, where D and τ are the thermal diffusion coefficient and the chemical time characteristic of the reactive mixture, respectively. We shall not be concerned in this paper with this turbulent combustion mode known as the flamelet regime, see e.g. [3]. We simply note that a large amount of work has been devoted to it due to its importance in applications, which seems to be in line with Damköhler's original view[2]. Examples of analytical contributions useful in

[1] See the monograph by Peters for an up-to-date account on turbulent combustion, [2].
[2] It should be mentioned, however, that an important poorly understood issue, particularly relevant in the flamelet regime, is related to the manifestation of intrinsic flame instabilities under turbulence (see [2, 4–6]).

the flamelet regime include Clavin–Williams' formula [7, 8], which provides a relation between U_{T} and the turbulence intensity u', the G-equation type studies [9, 10], which model the flame as an interface advancing relative to the combustible gas with normal speed U_{L} (possibly with a stretch-correction), and the renormalization method studies which yield analytical expressions for U_{T} in terms of u' [11, 12]. A serious limitation shared by such approaches, however, is that they do not extend to strongly turbulent situations for which the flow scales become comparable to or smaller than δ_{L}.

In such situations, the small scales in the flow, according to Damköhler's second hypothesis, do not cause any significant flame wrinkling but do change the flame structure by enhancing the diffusive processes; the normal propagation speed and flame thickness are the same as in the laminar case, but with D replaced by an effective thermal diffusivity D^*, i.e. $U_{\mathrm{L}}^* = \sqrt{D^*/\tau}$ and $\delta_{\mathrm{L}}^* = \sqrt{D^*\tau}$. However, unlike the first hypothesis, this second one seems to have received little support, especially as far as analytical work is concerned. A good summary of reservations against it is given in Williams' book [7 p 438]; questions arise concerning the legitimacy of using the laminar flame chemical time τ in U_{L}^*, possible extinction phenomena caused by the small scales, and the fact that the true flame structure is not yet known at high turbulent intensities. Notwithstanding these reservations, the hypothesis remains a legitimate starting point to account for the effect of small flow scales, as used, for example, by Ronney and Yakhot in [13] for extending Yakhot's turbulent flame speed formula [11] to highly turbulent situations.

Clearly, this hypothesis needs serious examination before it can be accepted as well founded. For this purpose, analytical results that are valid in the limit of small flow scales would provide valuable insight into the dependence of the effective propagation speed on the scale and intensity of the flow, even if only available in the simplest flow configurations. Surprisingly, such results seem to be unavailable. Investigations based on the stagnation flow configuration [2–7] are of limited help since the flow involved is characterized by a single parameter, the strain rate, rather than two independent ones for the scale and intensity of the velocity field. More suitable for our purpose are the two-parameter flow models which have been successfully used in the literature to describe some features of turbulent combustion, see e.g. [10] and [14–18]. These studies, however, are generally restricted to the flamelet regime, since they rely on the eikonal equation or on a slowly varying flame approximation [16, 17]. They do not, therefore, address Damköhler's second hypothesis, although the types of flow used would be suitable candidates at small enough scales.

In the present investigation, we perform a test of Damköhler's second hypothesis in the framework of an arbitrarily prescribed parallel steady flow. This is probably the simplest possible choice if two independent flow parameters are to be retained. The analysis will be carried out in the asymptotic limit where the flow scale l (relative to δ_{L}) tends to zero, while the flow intensity is an arbitrary O(1) quantity. In this specific context, our main objectives are to

(a) determine the effective flame speed (which will be denoted by U_{T}) and its dependence on the flow,

(b) assess non-unit Lewis number effects, and

(c) clarify the influence of volumetric heat losses.

It is worth emphasizing that the results must be considered only as a first step towards clarifying a difficult and as yet controversial problem. Additional important features not considered in this paper and which will be investigated in the future will be discussed briefly in the final section.

The paper is structured as follows. We begin by formulating the problem within the constant density approximation. An asymptotic solution is then derived in the equidiffusional adiabatic case, in order to illustrate the salient features of the approach with maximum simplicity. The analysis is then extended to account for non-unit Lewis numbers and non-zero heat losses. Finally a synthesis and discussion of the main results, with answers to the three objectives outlined above, is given.

2. Formulation

We consider a two-dimensional flame propagating against a steady parallel flow in the x-direction. Within the thermo-diffusive approximation (with constant density and constant transport properties), a relevant model consists of the equations

$$[U + u(y)]Y_x = Le^{-1}(Y_{xx} + Y_{yy}) - \omega, \tag{1}$$

$$[U + u(y)]T_x = T_{xx} + T_{yy} + \omega - \frac{\kappa}{\beta}T, \tag{2}$$

and the boundary conditions

$$Y = 1, \qquad T = 0 \qquad \text{as } x \to -\infty, \tag{3}$$

$$Y_x = T_x = 0 \qquad \text{as } x \to +\infty, \tag{4}$$

$$Y_y = T_y = 0 \qquad \text{at } y = 0 \text{ and } y = l. \tag{5}$$

The equations above are written in a frame attached to the flame whose propagation speed relative to the laboratory is U, with $U > 0$ indicating a propagation to the left. The velocity of the parallel flow along the positive x-direction relative to the laboratory is denoted by $u(y)$. T and Y are the (scaled) temperature and mass fraction of the fuel which is assumed to limit the reaction. Le is the Lewis number and ω is the reaction rate which is taken to be of the Arrhenius form

$$\omega = \frac{\beta^2}{2Le} Y \exp\{\beta(T - 1)\},$$

where β is the Zeldovich number. A sink term of strength κ/β is included in the formulation to account for volumetric heat losses (see [7–19]).

The units for speed and length chosen for non-dimensionalization correspond to the propagation speed U_L and the thickness δ_L of the adiabatic unstretched planar flame (more precisely to the asymptotic values of these as $\beta \to \infty$).

The boundary conditions (3) and (4) correspond to a frozen mixture with prescribed temperature and composition upstream, and uniform properties far downstream. The boundary conditions (5) are based on the assumption that all profiles have zero slope at $y = 0$ and $y = l$. Here, broadly speaking, l represents a characteristic (transverse) length of the flow (measured with planar flame thickness δ_L). For example, for a parallel flow in a channel with adiabatic walls, l can be taken as the channel width; (5) then expresses the adiabaticity and impenetrability of the walls located at $y = 0$ and $y = l$. If the flow is periodic in the y-direction then l may be viewed as equal to the period, and the origin of the y-axis is to be chosen so that the flame is vertical at $y = 0$ and $y = l$.

The solution of the problem thus formulated must yield, in particular, the propagation eigenvalue U. By integration of (1) over the whole domain, taking into account (5), and assuming the fuel to be totally depleted far downstream we obtain the relation

$$U_\mathrm{T} \equiv U + \bar{u} = \frac{1}{l}\int_{-\infty}^{\infty}\int_0^l \omega \,\mathrm{d}y\,\mathrm{d}x,$$

where $\bar{u}$ represents the mean flow speed. This implies that the quantity $U + \bar{u}$ appears as an effective propagation speed U_T (measured against U_L) as conventionally defined in turbulent combustion; U_T is also the flame propagation speed relative to an observer moving with the mean flow. It makes sense to choose the reference frame of the laboratory such that $\bar{u}$ is now zero, in which case $U_T = U$. Accordingly, we shall use equations (1) and (2) with U replaced by U_T, and hence will have the constraint

$$\bar{u} = \frac{1}{l}\int_0^l u \, \mathrm{d}y = 0 \qquad \text{or} \qquad \int_0^1 u \, \mathrm{d}\eta = 0 \tag{6}$$

in terms of the scale $\eta = y/l$.

Our aim is to determine U_T in the asymptotic limit of small flow scale $l \to 0$ and large β (with $\beta^{-1} \ll l$).

3. The adiabatic equidiffusional case

In the limit $\beta \to \infty$ adopted in this study, the reaction is confined to a thin sheet, given by $x = f(y)$ say. We begin with the equidiffusional adiabatic case ($Le = 1$, $\kappa = 0$) for which we need only solve for temperature, since the equations and boundary conditions imply that $Y + T = 1$. Using the transverse scale $\eta = y/l$ and the longitudinal coordinate $\zeta = x - f(y)$, and writing $f(y) = l^2 F(\eta)$, the problem becomes

$$T \equiv 1 \qquad \text{for } \zeta > 0,$$

$$[U_T + u(\eta) + F'']T_\zeta = (1 + l^2 F'^2)T_{\zeta\zeta} + l^{-2}T_{\eta\eta} - 2F'T_{\zeta\eta} \qquad \text{for } \zeta < 0, \tag{7}$$

$$T = 0 \qquad \text{as } \zeta \to -\infty, \tag{8}$$

$$T = 1, \qquad T_\zeta = (1 + l^2 F'^2)^{-1/2} \qquad \text{at } \zeta = 0^-, \tag{9}$$

$$T_\eta = F' = 0 \qquad \text{at } \eta = 0 \text{ and } \eta = 1. \tag{10}$$

Note that the fuel is assumed to be depleted behind the flame so that T is identically equal to one for $\zeta > 0$. In the unburnt gas $\zeta < 0$, T is governed by (7) subject to the upstream condition (8), the jump conditions (9) (see e.g. [20]), and the zero-slope conditions (10).

In addition, we may impose for convenience that $F(0) = 0$, since translational invariance in the x-direction allows the origin on the x-axis to be freely chosen. We shall seek an asymptotic solution of the problem, thus formulated, in the limit $l \to 0$. We begin by writing straightforward expansions in the form

$$T = T_0 + lT_1 + l^2T_2 + \cdots, \qquad U_T = U_0 + lU_1 + l^2U_2 + \cdots,$$
$$F = F_0 + lF_1 + \cdots,$$

which we substitute into (7)–(10).

To $\mathcal{O}(l^{-2})$ we find $T_{0\eta\eta} = 0$ which, when used with the zero-slope conditions (10), implies that T_0 must be a function of ζ only, $T_0 = T_0(\zeta)$. To $\mathcal{O}(l^{-1})$ we have similarly $T_{1\eta\eta} = 0$ and $T_1 = T_1(\zeta)$.

To $\mathcal{O}(1)$ we obtain

$$[U_0 + u(\eta) + F_0''(\eta)]T_{0\zeta} - T_{0\zeta\zeta} = T_{2\eta\eta}, \tag{11}$$

which we integrate with respect to η from 0 to 1, taking into account (6) and (10). This yields the ordinary differential equation

$$U_0T_{0\zeta} - T_{0\zeta\zeta} = 0,$$

whose solution subject to $T_0(-\infty) = 0$, $T_0(0) = 1$ and $T_{0\zeta}(0) = 1$ is given by

$$U_0 = 1, \qquad T_0 = \exp(\zeta) \qquad (\zeta \leqslant 0). \tag{12}$$

Thus, in a first approximation, the solution corresponds to the laminar planar flame.

We now integrate (11) twice with respect to η from 0 to η using (12). We obtain $T_2 = (S(\eta)+F_0(\eta)) \exp \zeta + \tilde{T}_2(\zeta)$, where $\tilde{T}_2(\zeta)$ is an arbitrary function of integration and the function $S(\eta)$ is defined such that

$$S(\eta) \equiv \int_0^{\eta} \mathrm{d}\eta_2 \int_0^{\eta_2} u(\eta_1)\, \mathrm{d}\eta_1. \tag{13}$$

From the continuity of temperature at the reaction sheet, $T_2 = 0$ at $\zeta = 0$, and the fact that $S(0) = F_0(0) = 0$, it then follows that

$$F_0 = -S(\eta), \tag{14}$$

giving the first approximation to the flame shape. It also follows that T_2 must be a function of ζ only, $T_2 = T_2(\zeta)$, with $T_2(0) = 0$.

To $\mathcal{O}(l)$ we obtain

$$T_{1\zeta} - T_{1\zeta\zeta} = -(U_1 + F_1'')\, \mathrm{e}^{\zeta} + T_{3\eta\eta}, \tag{15}$$

which, integrated with respect to η from 0 to 1 yields $T_{1\zeta} - T_{1\zeta\zeta} = -U_1 \mathrm{e}^{\zeta}$. The auxilliary conditions for this equation being $T_1(-\infty) = T_1(0) = T_{1\zeta}(0) = 0$, the solution is obviously the trivial one

$$U_1 = 0 \qquad \text{and} \qquad T_1 \equiv 0, \tag{16}$$

and implies, when reinjected into (15) and using (10), that T_3 must be of the form $T_3 = \mathrm{e}^{\zeta} F_1(\eta) + \tilde{T}_3(\zeta)$. To $\mathcal{O}(l^2)$ we find

$$T_{2\zeta} - T_{2\zeta\zeta} = -(U_2 + F_2'')\, \mathrm{e}^{\zeta} + F_0'^2\, \mathrm{e}^{\zeta} + T_{4\eta\eta},$$

which we integrate as above from $\eta = 0$ to 1 to get

$$T_{2\zeta} - T_{2\zeta\zeta} = -A\, \mathrm{e}^{\zeta},$$

where

$$A \equiv U_2 - \int_0^1 F_0'^2\, \mathrm{d}\eta.$$

The solution of this equation subject to $T_2(-\infty) = 0$ and $T_2(0) = 0$ is

$$T_2 = A\zeta\, \mathrm{e}^{\zeta}. \tag{17}$$

The constant A is to be determined from the jump in the temperature slope at the reaction sheet, $T_{2\zeta}(0^-) = -F_0'^2/2$. Clearly, this is impossible since the rhs is a function of η and the lhs is not. This suggests that our straightforward expansion should be complemented by an expansion in an inner region near the flame. As we shall show below, the matching between the two expansions implies that T_2 must satisfy the following integral form of the jump condition

$$T_{2\zeta} = -\frac{1}{2} \int_0^1 F_0'^2\, \mathrm{d}\eta \qquad \text{at } \zeta = 0^-, \tag{18}$$

which allows the constant A to be determined in (17), leading to

$$U_2 = \frac{1}{2} \int_0^1 F_0'^2\, \mathrm{d}\eta. \tag{19}$$

Thus, using (12), (13), (14) and (16), a two-term expansion for the propagation speed is given by

$$U_{\mathrm{T}} \sim 1 + \frac{l^2}{2} \int_0^1 \left[\int_0^{\eta} u(\eta_1)\, \mathrm{d}\eta_1 \right]^2 \mathrm{d}\eta, \tag{20}$$

a formula which will be discussed in the final section.

In the remainder of this section, we shall examine the inner expansion and the matching conditions, with the main aim of justifying our use of the jump relationship (18). We denote the straightforward expansion above by

$$T^{\text{outer}} \sim \mathrm{e}^{\zeta} + l^2 T_2(\zeta) + l^3 T_3(\zeta, \eta) + \cdots, \tag{21}$$

where T_0 and T_1 are given by (12) and (17), and write an inner expansion in the form

$$T^{\text{inner}} \sim 1 + l\xi + l^2 \frac{\xi^2}{2} + l^3 \hat{T}_3(\xi, \eta) + \cdots \qquad (\zeta = l\xi). \tag{22}$$

The first three terms of (22) have been given explicitly, using a Taylor expansion of the uniformly valid leading-order solution e^{ζ} as $\zeta \to 0$ and the condition $T_2(0) = 0$. By substitution of (22) into (7)–(10), we find that $\hat{T}_3$ is governed by $\hat{T}_{3\xi\xi} + \hat{T}_{3\eta\eta} = \xi + F_1''(\eta)$, so that

$$\hat{T}_3 = \frac{\xi^3}{6} + F_1(\eta) + \theta(\xi, \eta), \tag{23}$$

where θ satisfies

$$\theta_{\xi\xi} + \theta_{\eta\eta} = 0, \tag{24}$$

and is to be determined subject to the conditions

$$\theta_\eta(\xi, 0) = 0, \qquad \theta_\eta(\xi, 1) = 0, \qquad \theta_\xi(0, \eta) = -\frac{F_0'^2}{2}, \qquad \theta(0, \eta) = -F_1(\eta). \tag{25}$$

Using the method of separation of variables, we can determine θ by solving (24) subject to the first three conditions in (25); the last condition then determines F_1. We thus find (using $F_1(0) = 0$) that

$$\theta = -\frac{\xi}{2} \int_0^1 F_0'^2 \, \mathrm{d}\eta + \sum_{n=1}^{\infty} a_n [\mathrm{e}^{n\pi\xi} \cos(n\pi\eta) - 1] \tag{26}$$

and

$$F_1 = \sum_{n=1}^{\infty} a_n [1 - \cos(n\pi\eta)], \qquad \text{with } a_n = -\frac{1}{n\pi} \int_0^1 F_0'^2 \cos(n\pi\eta) \, \mathrm{d}\eta. \tag{27}$$

Now the matching of the outer and inner expansions (21) and (22) to $\mathcal{O}(l^3)$ imposes the requirement

$$\hat{T}_3(\xi, \eta) \sim \frac{\xi^3}{6} + T_{2\zeta}(0^-)\xi + T_3(0, \eta) \qquad \text{as } \xi \to -\infty, \tag{28}$$

which, together with (26) and (23), implies that

$$T_{2\zeta}(0^-)\xi + T_3(0, \eta) \sim F_1(\eta) - \frac{\xi}{2} \int_0^1 F_0'^2 \, \mathrm{d}\eta + a_0 \tag{29}$$

with $a_0 = -\sum_{n=1}^{\infty} a_n$. From this, (18) follows at once (along with $T_3(0, \eta) = a_0 + F_1(\eta)$, which is useful when carrying the problem to higher order). Thus, our use of (18) is justified.

Finally, it should be noted that (18) can be deduced, without actually solving for θ, from (28) and the relation

$$\int_0^1 \theta_\xi(\xi, \eta) \, \mathrm{d}\eta = -\frac{1}{2} \int_0^1 F_0'^2 \, \mathrm{d}\eta, \tag{30}$$

which must hold, in particular, as $\xi \to -\infty$; this relation is obtained by applying the divergence theorem to the integral of Laplace equation (24) over the rectangular domain $[\xi, 0] \times [0, 1]$.

4. Heat loss and preferential diffusion effects

We now extend the analysis to account for non-zero heat losses and non-unit Lewis numbers. With the additional assumption $le \equiv \beta(Le-1) = \mathrm{O}(1)$ as $\beta \to \infty$, we may reformulate the problem in terms of T^0 and H, where T^0 is the leading-order temperature in an expansion in β^{-1} and H the excess enthalpy defined by $Y + T \sim 1 + \beta^{-1}H$ (see [20]).

Dropping the superscript in T^0, the problem is given by

$$T = 1, \qquad \zeta > 0, \tag{31}$$

$$[U_\mathrm{T} + u(\eta) + F'']T_\zeta = (1 + l^2 F'^2)T_{\zeta\zeta} + l^{-2}T_{\eta\eta} - 2F'T_{\zeta\eta}, \qquad \zeta < 0, \tag{32}$$

$$\begin{aligned}[U_\mathrm{T} + u(\eta) + F'']H_\zeta = {} & ((1 + l^2F'^2)\partial_{\zeta\zeta} + l^{-2}\partial_{\eta\eta} - 2F'\partial_{\zeta\eta})(H + le\,T) \\ & - le\,F''T_\zeta - \kappa T, \qquad \zeta \neq 0\end{aligned} \tag{33}$$

subject to the boundary conditions

$$T = H = 0 \qquad \text{as } \zeta \to -\infty, \tag{34}$$

$$T_\eta = H_\eta = F' = 0 \qquad \text{at } \eta = 0 \text{ and } \eta = 1, \tag{35}$$

and the jump conditions

$$[T] = [H] = 0, \qquad [H_\zeta] + le[T_\zeta] = 0, \qquad [T_\zeta] = -(1 + l^2F'^2)^{-1/2}\mathrm{e}^{H/2} \qquad \text{at } \zeta = 0. \tag{36}$$

In addition, we shall disallow exponentially growing solutions as $\zeta \to \infty$.

We follow the methodology of the previous section, skipping a few similar details. We seek expansions in the form

$$\begin{aligned} T &= T_0 + l^2T_2 + \cdots, \qquad & H &= H_0 + l^2H_2 + \cdots, \\ U_\mathrm{T} &= U_0 + l^2U_2 + \cdots, \qquad & F &= F_0 + lF_1 + \cdots, \end{aligned}$$

which we substitute into (31)–(36). To $\mathcal{O}(l^{-2})$ we find $T_{0\eta\eta} = H_{0\eta\eta} = 0$ which implies that $T_0 = T_0(\zeta)$ and $H_0 = H_0(\zeta)$.

To $\mathcal{O}(1)$ we have

$$[U_0 + u(\eta) + F_0'']T_{0\zeta} - T_{0\zeta\zeta} = T_{2\eta\eta}, \tag{37}$$

$$[U_0 + u(\eta) + F_0'']H_{0\zeta} - H_{0\zeta\zeta} = H_{2\eta\eta} + le(T_{2\eta\eta} + T_{0\zeta\zeta} - F_0''T_{0\zeta}) - \kappa T_0, \tag{38}$$

which we integrate with respect to η from 0 to 1, using (6) and (35). This yields the ODE system

$$U_0T_{0\zeta} - T_{0\zeta\zeta} = 0, \qquad U_0H_{0\zeta} - H_{0\zeta\zeta} = le\,T_{0\zeta\zeta} - \kappa T_0, \tag{39}$$

subject to

$$T_0 = H_0 = 0 \qquad \text{as } \zeta \to -\infty, \tag{40}$$

$$[T_0] = [H_0] = [H_{0\zeta}] + le\,[T_{0\zeta}] = [T_{0\zeta}] + \mathrm{e}^{H_0/2} = 0 \qquad \text{at } \zeta = 0. \tag{41}$$

We note that (39)–(41) describe the propagation of a planar flame under volumetric heat losses. This problem is well known in the literature (see e.g. [7–19]). Its solution (free from exponentially growing terms in the burnt gas) is

$$T_0 = \begin{cases} \mathrm{e}^{U_0\zeta}, \\ 1, \end{cases} \qquad H_0 = \begin{cases} \left[\left(\dfrac{\kappa}{U_0^2} - le\right)U_0\zeta - \dfrac{2\kappa}{U_0^2}\right]\mathrm{e}^{U_0\zeta}, & \zeta \leqslant 0, \\ -\dfrac{\kappa}{U_0}\zeta - \dfrac{2\kappa}{U_0^2}, & \zeta \geqslant 0, \end{cases} \tag{42}$$

where $U_0 = U_0(\kappa)$ is the larger of the two roots of the equation

$$U_0^2 \ln U_0 = -\kappa \tag{43}$$

(the smaller root corresponds to an unstable solution); clearly, solutions exist only if κ is less than an extinction value given by $\kappa_{\text{ext}} = (2\mathrm{e})^{-1}$.

Thus, in a first approximation, the propagation speed is that of the planar non-adiabatic flame, given by (43). We now integrate (37) twice from $\eta = 0$ to η to obtain $T_2 = (S(\eta) + F_0(\eta))\mathrm{e}^{\zeta} + \tilde{T}_2(\zeta)$, where $\tilde{T}_2(\zeta)$ is a function of integration and S is as defined in (13). From the condition $T_2 = 0$ at $\zeta = 0$, we again conclude that

$$F_0 = -S(\eta), \qquad T_2 = T_2(\zeta), \qquad T_2(0) = 0.$$

Thus, F_0 is as in the adiabatic equidiffusional case, and T_2 must be a function of ζ only. It then follows from (38) and (39) that $H_{2\eta\eta} = le\, F_0'' T_{0\zeta}$, and hence (by integrating twice using (35) and (42)) H_2 must be of the form

$$H_2 = \tilde{H}_2(\zeta) + le\, F_0 T_{0\zeta} = \begin{cases} \tilde{H}_2(\zeta) + le\, U_0 F_0(\eta)\mathrm{e}^{U_0\zeta}, & \zeta < 0, \\ \tilde{H}_2(\zeta), & \zeta > 0. \end{cases} \tag{44}$$

To $\mathcal{O}(l^2)$ we find

$$U_0 T_{2\zeta} - T_{2\zeta\zeta} = -(U_2 + F_2'')T_{0\zeta} + F_0'^2 T_{0\zeta\zeta} + T_{4\eta\eta},$$

$$\begin{aligned} U_0 H_{2\zeta} - H_{2\zeta\zeta} = {} & -(U_2 + F_2'')H_{0\zeta} + F_0'^2(H_{0\zeta\zeta} + le\, T_{0\zeta\zeta}) + T_{4\eta\eta} + le\, T_{4\eta\eta} \\ & + le\, T_{2\zeta\zeta} - le\,(F_0'' T_{2\zeta} + F_2'' T_{0\zeta}) - 2F_0' H_{2\zeta\eta} - \kappa T_2, \end{aligned}$$

which we integrate with respect to η from 0 to 1.

For T_2 we obtain

$$U_0 T_{2\zeta} - T_{2\zeta\zeta} = -A\mathrm{e}^{U_0\zeta}, \qquad \zeta < 0,$$

where

$$A \equiv U_0 U_2 - U_0^2 \int_0^1 F_0'^2 \,\mathrm{d}\eta,$$

to be solved with $T_2(-\infty) = 0$ and $T_2(0) = 0$. Thus

$$T_2 = \begin{cases} \dfrac{A}{U_0}\zeta\mathrm{e}^{U_0\zeta}, & \zeta < 0, \\ 0, & \zeta > 0. \end{cases} \tag{45}$$

For $\tilde{H}_2$ we obtain similarly

$$U_0 \tilde{H}_{2\zeta} - \tilde{H}_{2\zeta\zeta} = -U_2 H_{0\zeta} + (H_{0\zeta\zeta} - le\, T_{0\zeta\zeta}) \int_0^1 F_0'^2 \,\mathrm{d}\eta + le\, T_{2\zeta\zeta} - \kappa T_2$$

for $\zeta \neq 0$, which, when integrated from $\zeta = -\infty$ to 0^-, yields

$$U_0 \tilde{H}_2(0^-) = \tilde{H}_{2\zeta}(0^-) + U_2\left(le + \frac{3\kappa}{U_0^2}\right) - U_0\left(3le + \frac{2\kappa}{U_0^2}\right)\int_0^1 F_0'^2 \,\mathrm{d}\eta. \tag{46}$$

We also note that in the burnt gas, $\zeta > 0$, $U_0 H_{2\zeta} - H_{2\zeta\zeta} = U_2\kappa/U_0$ so that

$$\tilde{H}_{2\zeta}(0^+) = \frac{U_2\kappa}{U_0^2}, \tag{47}$$

after exponentially growing solutions have been eliminated.

We now turn to the remaining jump conditions to be satisfied by T_2 and H_2, namely

$$[H_2] = 0, \qquad [H_{2\zeta}] + le\,[T_{2\zeta}] = 0, \qquad [T_{2\zeta}] = \frac{U_0}{2}(F_0'^2 - H_2) \qquad \text{at } \zeta = 0.$$

Given that T_2 is independent of η and in view of (44), these cannot be satisfied. This suggests that our straightforward expansion is to be complemented by and matched with the solution in an inner region near the flame. From the matching, we shall show that the outer solutions must obey the following integral form of the jumps

$$[\tilde{H}_2] = le\, U_0 \int_0^1 F_0\, \mathrm{d}\eta, \tag{48}$$

$$[T_{2\zeta}] = \frac{U_0}{2}\left(\int_0^1 F_0'^2\, \mathrm{d}\eta - \tilde{H}_2(0^+)\right), \tag{49}$$

$$[\tilde{H}_{2\zeta}] + le\,[T_{2\zeta}] = le\, U_0^2 \int_0^1 F_0\, \mathrm{d}\eta - 2le\, U_0 \int_0^1 F_0'^2\, \mathrm{d}\eta, \tag{50}$$

which we shall now use before justifying them.

From (45), (47) and (50) we get

$$\tilde{H}_{2\zeta}(0^-) = U_2\left(\frac{\kappa}{U_0^2} - le\right) + 3le\, U_0 \int_0^1 F_0'^2\, \mathrm{d}\eta - le\, U_0^2 \int_0^1 F_0\, \mathrm{d}\eta,$$

which substituted into (46) yields

$$\tilde{H}_2(0^+) = \frac{4\kappa}{U_0^3}\left(U_2 - \frac{U_0}{2}\int_0^1 F_0'^2\, \mathrm{d}\eta\right), \tag{51}$$

after making note of (48). Using this result with (45) and (49), we find $\tilde{H}_2(0) = 0$ and

$$U_2 = \frac{U_0}{2}\int_0^1 F_0'^2\, \mathrm{d}\eta. \tag{52}$$

This is the main result we have been seeking, which allows a two-term approximation for U_T to be written as

$$\frac{U_\mathrm{T}}{U_0} = 1 + \frac{l^2}{2}\int_0^1 \left[\int_0^\eta u(\eta_1)\, \mathrm{d}\eta_1\right]^2 \mathrm{d}\eta, \qquad \text{with } U_0^2 \ln U_0 = -\kappa. \tag{53}$$

Thus, U_T is the same as in the equidiffusional adiabatic case, provided it is scaled with the propagation speed $U_0(\kappa)$ of the non-adiabatic planar flame. In particular, U_T does not depend on the Lewis number at least for nearly equidiffusive Lewis numbers, $Le - 1 = \mathrm{O}(\beta^{-1})$.

The rest of this section will be devoted to the analysis of the solution in an inner region near the flame; the main aim is to justify the integral form of the jumps (48)–(50).

The straightforward expansion above provides an outer expansion

$$T^{\text{outer}} \sim T_0(\zeta) + l^2 T_2(\zeta) + l^3 T_3(\zeta, \eta) + \cdots, \tag{54}$$

$$H^{\text{outer}} \sim H_0(\zeta) + l^2 H_2(\zeta) + l^3 H_3(\zeta, \eta) + \cdots, \tag{55}$$

where T_0, H_0 are given by (42), T_2 by (45) and H_2 by (44).

Inner expansions in terms of $\xi = \zeta/l$ are sought in the form

$$T^{\text{inner}} \sim \hat{T}_0(\xi, \eta) + l\hat{T}_2(\xi, \eta) + l^2\hat{T}_2(\xi, \eta) + l^3\hat{T}_3(\xi, \eta) + \cdots, \tag{56}$$

$$H^{\text{inner}} \sim \hat{H}_0(\xi, \eta) + l\hat{H}_1(\xi, \eta) + l^2\hat{H}_2(\xi, \eta) + l^3\hat{H}_3(\xi, \eta) + \cdots. \tag{57}$$

In preparation for the matching, we note that

$$\text{as } \zeta \to 0^-, \qquad T^{\text{outer}} \sim 1 + lU_0\xi + l^2U_0^2\frac{\xi^2}{2} + l^3\left[\frac{U_0^3\xi^3}{6} + T_{2\zeta}(0^-)\xi + T_3(0^-,\eta)\right], \tag{58}$$

$$H^{\text{outer}} \sim -\frac{2\kappa}{U_0^2} - l\left[\frac{\kappa}{U_0^2} + le\right]U_0\xi + l^2[\tilde{h}_2(0^-) + le\,U_0F_0 - le\,U_0^2\xi^2],$$
$$+ l^3\left[\left(\frac{\kappa}{U_0^2} - 3le\right)\frac{U_0^3\xi^3}{6} + le\,U_0^2F_0\xi + \tilde{H}_{2\zeta}(0^-)\xi + h_3(0^-,\eta)\right], \tag{59}$$

and

$$\text{as } \zeta \to 0^+, \qquad H^{\text{outer}} \sim -\frac{2\kappa}{U_0^2} - l\frac{\kappa\xi}{U_0} + l^2\tilde{H}_2(0^+) + l^3[\tilde{H}_{2\zeta}(0^+)\xi + H_3(0^+,\eta)]. \tag{60}$$

Since the leading-order solution (42) is uniformly valid and $T_2(0) = 0$, we have, in view of (58)–(60),

$$\hat{T}_0 = 1, \qquad \hat{T}_1 = U_0\xi, \qquad \hat{T}_2 = U_0^2\frac{\xi^2}{2}, \qquad \xi < 0,$$

and

$$\hat{H}_0 = -\frac{2\kappa}{U_0^2}, \qquad \hat{H}_1 = \begin{cases} -\left(\dfrac{\kappa}{U_0^2} + le\right)U_0\xi, & \zeta < 0, \\ -\dfrac{\kappa}{U_0}\xi, & \zeta > 0. \end{cases}$$

For $\hat{H}_2$ we have to solve

$$\hat{H}_{2\xi\xi} + \hat{H}_{2\eta\eta} = \begin{cases} le\,U_0F_0'' - 2le\,U_0^2, & \zeta < 0, \\ 0, & \zeta > 0, \end{cases}$$

subject to the boundary and jump conditions

$$\hat{H}_{2\eta}(\xi,0) = \hat{H}_{2\eta}(\xi,1) = 0, \qquad [\hat{H}_2] = [\hat{H}_{2\xi}] = 0 \qquad \text{at } \xi = 0,$$

and the matching requirements

$$\hat{H}_2 \sim \tilde{H}_2(0^-) + le\,U_0F_0 - le\,U_0^2\xi^2 \qquad \text{as } \xi \to -\infty,$$
$$\hat{H}_2 \sim \tilde{H}_2(0^+) \qquad \text{as } \xi \to +\infty.$$

Thus, writing

$$\hat{H}_2 = \phi(\xi,\eta) + \begin{cases} \tilde{H}_2(0^-) + le\,U_0F_0 - le\,U_0^2\xi^2, & \zeta < 0, \\ \tilde{H}_2(0^+), & \zeta > 0, \end{cases} \tag{61}$$

it is seen that ϕ satisfies the Laplace equation with

$$\phi \sim 0 \qquad \text{as } \xi \to \pm\infty, \qquad \phi_\eta(\xi,0) = 0,$$
$$\phi_\eta(\xi,1) = 0, \qquad [\phi_\xi] = 0, \qquad [\phi] = le\,U_0F_0 - [\tilde{H}_2].$$

The solution subject to the first three conditions is

$$\phi(\xi,\eta) = \begin{cases} \sum_{n=1}^{\infty} a_n\,\mathrm{e}^{n\pi\xi}\cos(n\pi\eta), & \zeta < 0, \\ \sum_{n=1}^{\infty} b_n\,\mathrm{e}^{-n\pi\xi}\cos(n\pi\eta), & \zeta > 0, \end{cases}$$

with $b_n = -a_n$ so as to insure that $[\phi_\xi] = 0$. From the last condition, we then have

$$[\tilde{H}_2] = le\,U_0\int_0^1 F_0\,\mathrm{d}\eta,$$

and

$$b_n = -a_n = le\,U_0\int_0^1 F_0\cos(n\pi\eta)\,\mathrm{d}\eta,$$

legitimizing, in particular, the use of (48).

Finally, we note for later reference that

$$\int_0^1 \sigma_2 \,\mathrm{d}\eta = \tilde{H}_2(0^+) = \tilde{H}_2(0^-) + le\, U_0 \int_0^1 F_0 \,\mathrm{d}\eta \qquad \text{where } \sigma_2 \equiv \hat{H}_2(0,\eta), \tag{62}$$

a relation which follows from (61), written at $\xi = 0$, when integrated with respect to η. The problem for $\hat{T}_3$ is given by

$$\hat{T}_{3\xi\xi} + \hat{T}_{3\eta\eta} = U_0^3\xi + U_0 F_1''(\eta) \qquad (\xi < 0),$$

with

$$\hat{T}_{3\eta}(\xi,0) = 0, \qquad \hat{T}_{3\eta}(\xi,1) = 0, \qquad \hat{T}_{3\xi}(0^-,\eta) = U_0\left(\frac{\sigma_2}{2} - \frac{F_0'^2}{2}\right), \qquad \hat{T}_3(0,\eta) = 0.$$

Proceeding as in the equidiffusional adiabatic case, we find that

$$\hat{T}_3 = \frac{U_0^3\xi^3}{6} + U_0 F_1 + \frac{U_0\xi}{2}\int_0^1 (\sigma_2 - F_0'^2)\,\mathrm{d}\eta + U_0 \sum_{n=1}^{\infty} a_n[\mathrm{e}^{n\pi\xi}\cos(n\pi\eta) - 1],$$

where F_1 and the coefficients a_n are as in (27), but with $F_0'^2$ replaced by $F_0'^2 - \sigma_2$. The matching requirement

$$\hat{T}_3(\xi,\eta) \sim \frac{U_0^3\xi^3}{6} + T_{2\zeta}(0^-)\xi + T_3(0,\eta) \qquad \text{as } \xi \to -\infty,$$

then leads to the jump condition (49).

Finally, the problem for $\hat{H}_3$ is found to be

$$\hat{H}_{3\xi\xi} + \hat{H}_{3\eta\eta} = \begin{cases} \left(\dfrac{\kappa}{U_0^2} - le\right)U_0^3\xi - \left(\dfrac{\kappa}{U_0} + le\, U_0\right)F_1'' + le\, U_0^2 F_0''\xi, & \zeta < 0, \\ +U_0\hat{H}_{2\xi} + 2F_0'\hat{H}_{2\xi\eta}, & \\ -\dfrac{\kappa}{U_0}F_1'' + U_0\hat{H}_{2\xi} + 2F_0'\hat{H}_{2\xi\eta}, & \zeta > 0, \end{cases} \tag{63}$$

with the boundary and jump conditions

$$\hat{H}_{3\eta}(\xi,0) = 0, \qquad \hat{H}_{3\eta}(\xi,1) = 0, \qquad [\hat{H}_{3\xi}] + le\,[\hat{T}_{3\xi}] = 0, \qquad [\hat{H}_3] = 0$$

and the matching requirements

$$\hat{H}_3(\xi,\eta) \sim \left(\frac{\kappa}{U_0^2} - 3le\right)\frac{U_0^3\xi^3}{6} + (\tilde{H}_{2\zeta}(0^-) + le\, U_0^2 F_0)\xi + H_3(0^-,\eta) \qquad \text{as } \xi \to -\infty,$$

$$\hat{H}_3(\xi,\eta) \sim \tilde{H}_{2\zeta}(0^+)\xi + H_3(0^+,\eta) \qquad \text{as } \xi \to +\infty.$$

Applying the divergence theorem to the double integral of equation (63) over the right rectangular domain from $\xi = 0^+$ to ξ large and positive, and using the boundary and matching conditions, we find

$$\tilde{H}_{2\zeta}(0^+) - \int_0^1 \hat{H}_{3\xi}(0^+,\eta)\,\mathrm{d}\eta = U_0\tilde{H}_2(0^+) - U_0\int_0^1 \hat{H}_2(0^+,\eta)\,\mathrm{d}\eta - 2\int_0^1 F_0'\hat{H}_{2\eta}(0^+,\eta)\,\mathrm{d}\eta.$$

Similarly, by integration over the left rectangular domain from ξ large and negative to $\xi = 0^-$, we find

$$\tilde{H}_{2\zeta}(0^-) - \int_0^1 \hat{H}_{3\xi}(0^-,\eta)\,\mathrm{d}\eta = U_0\tilde{H}_2(0^-) - U_0\int_0^1 \hat{H}_2(0^-,\eta)\,\mathrm{d}\eta$$
$$-2\int_0^1 F_0'\hat{H}_{2\eta}(0^-,\eta)\,\mathrm{d}\eta + 2le\,U_0\int_0^1 F_0'^2\,\mathrm{d}\eta.$$

Subtracting the last two equations, we get

$$[\tilde{H}_2] - \int_0^1 [\hat{H}_{3\xi}]\,\mathrm{d}\eta = U_0[\tilde{H}_2] - 2\int_0^1 F_0'[\hat{H}_{2\eta}]\,\mathrm{d}\eta - 2le\,U_0\int_0^1 F_0'^2\,\mathrm{d}\eta$$

The jump (50) then follows by noticing that $[\hat{H}_{2\eta}] = 0$ since $[\hat{H}_2] = 0$ and using $[\hat{H}_{3\xi}] = -le\,[\hat{T}_{3\xi}] = -le\,U_0(F_0'^2/2 - \sigma_2/2)$ and (49). Thus our use of (50) is justified.

In the next section we shall summarize and discuss the main results.

5. Concluding remarks

We have carried out an analytical test of Damköhler's second hypothesis in the framework of a prescribed steady parallel flow. The work has exploited the distinguished limit when the scale l of the flow goes to zero with its intensity being of order unity. The main result is given in (53), namely

$$\frac{U_\mathrm{T}}{U_0} = 1 + \frac{l^2}{2}\int_0^1 \left[\int_0^\eta u(\eta_1)\,\mathrm{d}\eta_1\right]^2 \mathrm{d}\eta,$$

with

$$U_0^2 \ln U_0 = -\kappa.$$

This formula demonstrates, in the limit considered, that there is an increase in U_T caused by the flow; this increase is found even in the presence of heat losses, provided these do not exceed the critical extinction value of the non-adiabatic planar flame, i.e. κ must be less than $\kappa_\mathrm{ext} = (2\mathrm{e})^{-1}$. U_T is seen to depend quadratically on both the scale and intensity of the flow. These conclusions are in line with the results of [21], describing flame propagation in Poiseuille flow under adiabatic conditions, which provide a partial numerical verification of the formula above. Finally, it is worth noting the non-dependence of U_T on the Lewis number (for nearly equidiffusive values) and the simple way in which heat losses affect the burning rate: U_T is the same as in the equidiffusional adiabatic case, provided it is scaled with the propagation speed $U_0(\kappa)$ of the non-adiabatic planar flame.

It should be emphasized that several aspects which are important in clarifying more fully the effect of small flow scales on the flame have not been accounted for. These include flow unsteadiness, the use of other distinguished limits for scale and intensity, the analysis of more complex flows, and the effect of the finite activation energy of the reaction. These aspects and others will be addressed in future studies. The findings presented here do, however, provide a first step in examining Damköhler's second hypothesis.

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