



Hydrodynamic instabilities of propagating interfaces under Darcy's law

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The hydrodynamic instabilities of propagating interfaces in Hele-Shaw channels or porous media under the influence of an imposed flow and gravitational acceleration are investigated within the framework of Darcy's law. The stability analysis pertains to an interface between two fluids with different densities, viscosities, and permeabilities, which can be susceptible to Darrieus-Landau, Saffman-Taylor, and Rayleigh-Taylor instabilities. A theoretical analysis, treating the interface as a hydrodynamic discontinuity, yields a simple dispersion relation between the perturbation growth rate s and its wave number k in the form $s = (ak - bk^2)/(1 + ck)$, where a , b , and c are constants determined by problem parameters. The constant a characterizes all three hydrodynamic instabilities, which are long wave in nature. In contrast, b and c , which characterize the influences of local curvature and flow strain on interface propagation speed, typically provide stabilization at short wavelenghts comparable to the interface's diffusive thickness. The theoretical findings for Darcy's law are compared with a generalization of the classical work by Joulin and Sivashinsky, which is based on a Euler-Darcy model. The comparison provides a conceptual bridge between predictions based on Darcy's law and those on Euler's equation and offers valuable insights into the role of confinement on interface instabilities in Hele-Shaw channels. Numerical analyses of the instabilities are carried out for premixed flames using a simplified chemistry model and Darcy's law. The numerical results corroborate with the explicit formula with a reasonable accuracy. Time-dependent numerical simulations of unstable premixed flames are carried out to gain insights into the nonlinear development of these instabilities. The findings offer potential strategies for control of interface instabilities in Hele-Shaw channels or porous media. Special emphasis is given to the critical role played by the imposed flow in destabilizing or stabilizing the interface, depending on its direction relative to the interface propagation.

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I. INTRODUCTION

Premixed flames have long been known to exhibit two familiar hydrodynamic instabilities: the Darrieus-Landau (DL) instability and the Rayleigh-Taylor (RT) instability. The DL instability [1,2] is experienced by a premixed flame, or any interface, propagating towards a denser fluid, while the RT instability [3,4] occurs when gravity points away from the denser fluid. In

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addition to these instabilities, flames propagating in narrow geometries such as a Hele-Shaw channel or in a porous medium are prone to another hydrodynamic instability, namely, the Saffman-Taylor (ST) instability. First described by Saffman and Taylor [5] for a material liquid interface, the ST instability is encountered when a less viscous fluid displaces a more viscous fluid.

The role of Saffman-Taylor instability for a propagating interface such as a premixed flame was first studied by Joulin and Sivashinsky [6]. Their analysis, as well as subsequent ones in combustion theory, e.g., Refs. [7–9], are not based on Darcy’s law as in Saffman-Taylor’s original analysis, but use the so-called Euler-Darcy equation, which combines Darcy’s law with the inertial terms of the Euler equation. Such models have been proposed based on heuristic arguments and approximation, see, e.g., [10–12] and references therein. They do not have, however, a solid foundation based on a consistent asymptotic derivation, unlike the case of Darcy’s law which can be derived, in a Hele-Shaw channel for example, from the Navier-Stokes equation in the asymptotic limit of zero channel width. In other words, Darcy’s law without the additional ad hoc terms used in Euler-Darcy type equations, is the appropriate equation to use for an asymptotically correct, leading-order description of the flow field in a narrow channel. This observation is supported by recent combustion investigations [13–16] featuring Darcy’s law. In this paper we shall revisit the theoretical analysis developed in [6] using the more appropriate Darcy’s law as well as relaxing some of the restrictions adopted therein. As we shall demonstrate, this will lead to a more transparent description of the interface hydrodynamic instabilities in narrow channels or porous media.

Another important motivation for this study is to clarify the critical role played by an imposed flow in destabilizing or stabilizing the propagating interface, depending on the flow direction relative to the direction of propagation. The effect of the direction of a parallel flow relative to that of flame propagation on the effective propagation speed has been addressed in [17] both for wide and narrow channels. Its effect on hydrodynamic flame stability has been studied recently in [9] based on the Euler-Darcy model, leading to a dispersion relation generalizing that in [6]. Outside the field of combustion, the stability of chemical fronts propagating in Hele-Shaw cells or porous media using Darcy’s law has been addressed in several studies [18–21]. These have been conducted, however, for freely propagating interfaces, that is, in the absence of an imposed flow, which has a crucial role when considering the Saffman-Taylor instability, as we shall demonstrate. Therefore, it is important to distinguish between the stability of freely propagating interfaces, such as the flames investigated in the Hele-Shaw channel experiments of Ronney, Almarcha, and others [13,22–25], from interfaces subjected to an imposed flow. Failure to appreciate this distinction may lead to erroneous interpretation of experimental and numerical findings when comparing with theoretical results. It is worth pointing out, for example, that the stability dispersion relation derived by Joulin and Sivashinsky corresponds in a Hele-Shaw channel to a single specific value of the mean flow, the value which allows an undisturbed (depth-averaged) planar flame front to be stationary with respect to the channel walls. Strictly speaking, their results are not applicable to freely propagating flames, which are commonly considered in the literature [16,26]. In this paper we shall provide a simple dispersion relation combining the effect of the various hydrodynamic instabilities aforementioned and accounting for the presence of an imposed flow.

The paper is structured as follows. A theoretical analysis is developed in Sec. II, accounting for the interaction of the three hydrodynamic instabilities of a propagating interface in the presence of an imposed flow and gravitational acceleration. In Sec. III, a brief review of the results based on the Euler-Darcy model is carried out for the purpose of comparison with the pioneering work by Joulin and Sivashinsky [6] and the recent work by Miroschnichenko *et. al.* [9]. In principle, the content of Secs. II and III is of general validity and not restricted only to combustion applications. Section IV presents a numerical study based on Darcy’s law focusing on premixed flame instabilities. Numerical simulations are reported, including eigenvalue computations and direct numerical simulations, which are used to validate and extend the analytical results.

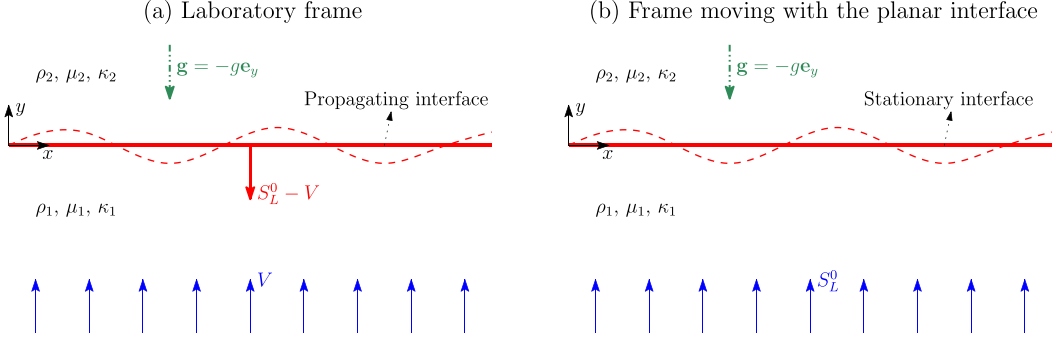


FIG. 1. (a) Schematic illustration, in the laboratory frame, of an unperturbed planar interface (solid line), which separates two media with different physical properties and propagates with a speed S_L^0 (with respect to upstream fluid 1). With respect to the laboratory, the speed of the planar interface is $S_L^0 - V$. Dashed line indicates perturbed interface, which may propagate at a different speed due to kinematic and curvature-induced effects. (b) The corresponding illustration of the situation in the frame moving with the planar interface.

II. DARRIEUS-LANDAU AND SAFFMAN-TAYLOR INSTABILITIES BASED ON DARCY'S LAW: THEORETICAL ANALYSIS

A. Formulation

Consider an interface propagating with respect to the fluid below it (fluid 1) with local normal propagation speed S_L , as illustrated Fig. 1(a). The fluid is assumed to be flowing upwards from $y = -\infty$ with velocity $\mathbf{u} = V\mathbf{e}_y$, in the laboratory frame. The density ρ , viscosity μ , and permeability κ are constant and equal to ρ_1, μ_1, κ_1 in the lower fluid and ρ_2, μ_2, κ_2 in the upper. For Hele-Shaw channels, $\kappa_1 = \kappa_2 = h^2/12$, where h is the channel width. The flow field \mathbf{u} in the laboratory frame is assumed to follow Darcy's law, so that on each side of the interface,

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} = -\frac{\kappa}{\mu} \nabla(p + \rho gy), \quad (1)$$

from which it follows that the pressure field p satisfies Laplace's equation, $\nabla^2 p = 0$.

In the unperturbed state, the interface is flat and $S_L = S_L^0$, where S_L^0 is assumed to be a known constant. In the laboratory frame, the interface therefore propagates downwards with constant speed $S_L^0 - V$. It is convenient to choose a moving frame in which the unperturbed planar interface is stationary, assumed to be located at $y = 0$, as illustrated in Fig. 1(b). In this frame the flow field is given by $\mathbf{v} = \mathbf{u} + (S_L^0 - V)\mathbf{e}_y$, and we shall write $\mathbf{v} = (u, v)$ to define its components. In the moving frame, the governing equations are given by

$$\nabla \cdot \mathbf{v} = 0, \quad \mathbf{v} = -\frac{\kappa}{\mu} \nabla(p + \rho gy) + (S_L^0 - V)\mathbf{e}_y, \quad \nabla^2 p = 0 \quad (2)$$

on both sides of the interface, and $\mathbf{v} = S_L^0 \mathbf{e}_y$, as $y \rightarrow -\infty$ as indicated in Fig. 1(b).

Let the (perturbed) interface be described by the equation $y = f(x, t)$ with unit normal \mathbf{n} pointing towards fluid 2. By definition, the local propagation speed S_L (with respect to fluid 1) is given by $S_L = (\mathbf{v} - \mathbf{U}) \cdot \mathbf{n}|_{y=f^-}$, where \mathbf{U} is the local interface velocity in the moving frame. In terms of f , we have

$$\mathbf{n} = \frac{(-f_x, 1)}{(1 + f_x^2)^{\frac{1}{2}}}, \quad \mathbf{U} \cdot \mathbf{n} = \frac{f_t}{(1 + f_x^2)^{\frac{1}{2}}}. \quad (3)$$

At the interface, the continuity of mass flux and pressure and the definition of S_L require

$$[[\mathbf{v} \cdot \mathbf{n}]] = \left(\frac{\rho_1}{\rho_2} - 1 \right) S_L, \quad [[p]] = 0, \quad S_L = \frac{v - u f_x - f_t}{(1 + f_x^2)^{\frac{1}{2}}} \Bigg|_{y=f^-}, \quad (4)$$

with the notation $[[\varphi]] \equiv \varphi|_{y=f^+} - \varphi|_{y=f^-}$. The first condition follows from the conservation of mass requirement $[[\rho(\mathbf{v} - \mathbf{U}) \cdot \mathbf{n}]] = 0$ and the definition of S_L . As for the second condition, this is obtained upon integrating, across the interface, Darcy's equation (which is also valid within the interface).

To close the problem formulation, it is important to define the local propagation speed S_L of the curved interface appropriately. Following Markstein [27], we shall assume that the deviation of S_L from S_L^0 (the propagation speed of the unperturbed planar interface) has a linear dependence on the local interface curvature of the form

$$\frac{S_L}{S_L^0} = 1 + \mathcal{L} \nabla \cdot \mathbf{n} \quad (5)$$

involving a Markstein length \mathcal{L} . Such dependence on curvature has not been taken into account in [6], where it was assumed that $S_L = S_L^0$ and where the physical meaning of the planar value S_L^0 is discussed, at least in the context of flame propagation in a Hele-Shaw channel. As emphasized in this reference, S_L^0 need not be the usual so-called laminar flame speed but rather an effective (depth-averaged) value thereof, which accounts for flame curvature along the Hele-Shaw channel wall's normal direction, as well as heat losses. In principle, S_L^0 may also depend on the effective (depth-averaged) flow, V in our notation, an assumption which was adopted in [9]. The latter dependence is, however, negligible in the asymptotic limit of zero channel width and hence will not be adopted in our study.

More importantly, we note that the Markstein model (5), although phenomenological in nature, is highly attractive due to its simplicity and its ability to realistically capture the effect of the interface curvature on its propagation speed. Therefore, this model will be adopted in most parts of this investigation. It must be emphasized, however, that other somewhat more complex models may be used which are based on asymptotic analysis accounting for the inner structure of the interface as well as local flow nonuniformities. This significant aspect is discussed in Sec. IID.

B. Linear stability of the planar interface

The basic solution $(\mathbf{v}, p, f) = (\bar{\mathbf{v}}, \bar{p}, \bar{f})$ corresponds to a stationary flat interface with $\bar{f} = 0$ and

$$\bar{\mathbf{v}} = S_L^0 \mathbf{e}_y, \quad -\bar{p} = \rho_1 g y + \frac{\mu_1}{\kappa_1} V y \quad \text{for } y < 0, \quad (6)$$

$$\bar{\mathbf{v}} = \frac{\rho_1}{\rho_2} S_L^0 \mathbf{e}_y, \quad -\bar{p} = \rho_2 g y + \frac{\mu_2}{\kappa_2} \left[V + \left(\frac{\rho_1}{\rho_2} - 1 \right) S_L^0 \right] y \quad \text{for } y > 0, \quad (7)$$

which satisfies clearly the governing equations and auxiliary conditions. To the basic solution $(\bar{\mathbf{v}}, \bar{p}, \bar{f})$ we add small perturbations such that $(\mathbf{v}, p, f) = (\bar{\mathbf{v}}, \bar{p}, \bar{f}) + (\mathbf{v}', p', f')$, where the primed quantities satisfy the linearized governing equations

$$\nabla \cdot \mathbf{v}' = 0, \quad \mathbf{v}' = -\frac{\kappa}{\mu} \nabla p', \quad \nabla^2 p' = 0.$$

In terms of p' and f' , the linearized interfacial conditions at $y = 0$ can be written as

$$\left[\left[\frac{\kappa}{\mu} p'_y \right] \right] = \left(\frac{\rho_1}{\rho_2} - 1 \right) S_L^0 \mathcal{L} f'_{xx}, \quad [[p']] = \alpha f', \quad f'_t = -\frac{\kappa_1}{\mu_1} p'_y \Bigg|_{y=0^-} + S_L^0 \mathcal{L} f'_{xx}, \quad (8)$$

where

$$\alpha = \left(\frac{\mu_2}{\kappa_2} - \frac{\mu_1}{\kappa_1} \right) V + (\rho_2 - \rho_1) g + \frac{\mu_2}{\kappa_2} \left(\frac{\rho_1}{\rho_2} - 1 \right) S_L^0. \quad (9)$$

The final linear stability problem is given by $\nabla^2 p' = 0$, to be solved in the domains $y > 0$ and $y < 0$, subject to the interfacial conditions (8) at $y = 0$ and the boundary condition $p' \rightarrow 0$ as $y \rightarrow \pm\infty$.

Since the stability problem does not depend explicitly on x and t , we look for normal modes in the form

$$p' = \hat{p}(y) \exp(st + ikx) \quad \text{and} \quad f' = C \exp(st + ikx),$$

where s is in general complex, k real, and C a constant. Then the equation $\nabla^2 p' = 0$ implies that $\hat{p}_{yy} - k^2 \hat{p} = 0$, and hence, using the requirement $p' = 0$ as $y \rightarrow \pm\infty$, we have

$$\hat{p} = \begin{cases} Ae^{-ky} & \text{for } y > 0, \\ Be^{ky} & \text{for } y < 0, \end{cases}$$

where A and B are constants and the wave number k is assumed positive. Upon imposing the three interfacial conditions (8), we obtain

$$\begin{bmatrix} k\kappa_2/\mu_2 & k\kappa_1/\mu_1 & -(\rho_1/\rho_2 - 1)S_L^0 \mathcal{L}k^2 \\ 1 & -1 & -\alpha \\ 0 & k\kappa_1/\mu_1 & s + S_L^0 \mathcal{L}k^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \mathbf{0}. \quad (10)$$

The solvability of this system of homogeneous equations yields the required dispersion relation:

$$s = \frac{\alpha k}{\mu_1/\kappa_1 + \mu_2/\kappa_2} - \frac{(\mu_2/\kappa_2)(\rho_1/\rho_2) + \mu_1/\kappa_1}{\mu_1/\kappa_1 + \mu_2/\kappa_2} S_L^0 \mathcal{L}k^2, \quad (11)$$

where α is as given in (9). A nondimensional form of this dispersion relation is obtained by multiplying all terms by δ_L^0/S_L^0 , where δ_L^0 is the diffusive thickness of the flat interface, and introducing the nondimensional growth rate and wave number $\tilde{s} = s\delta_L^0/S_L^0$ and $\tilde{k} = k\delta_L^0$, as well as the parameters

$$r = \frac{\rho_1}{\rho_2}, \quad m = \frac{\mu_1/\kappa_1}{\mu_2/\kappa_2}, \quad \mathcal{V} = \frac{V}{S_L^0}, \quad G = \frac{\rho_2 g \kappa_2}{S_L^0 \mu_2}, \quad \mathcal{M} = \frac{\mathcal{L}}{\delta_L^0}. \quad (12)$$

We thus obtain, dropping the tildes, the nondimensional dispersion relation

$$s = ak - bk^2, \quad \text{where} \quad a = \frac{r-1}{1+m} + \frac{1-m}{1+m} \mathcal{V} + \frac{1-r}{1+m} G \quad \text{and} \quad b = \frac{r+m}{1+m} \mathcal{M}. \quad (13)$$

C. Implication of the dispersion relation

We are now able to discuss the stability of the interface implied by the dispersion relations (11) or (13). Since the ratios r and m are positive numbers, the stability is determined only by the two nondimensional parameters a and \mathcal{M} , or equivalently, by the dimensional parameters α and \mathcal{L} . Clearly, instability occurs when $a > 0$ ($\alpha > 0$) or when $\mathcal{M} < 0$ ($\mathcal{L} < 0$). The second instability condition, $\mathcal{M} < 0$, corresponds to a Turing-type instability known as the diffusive-thermal instability in combustion. This instability, whose investigation requires the analysis of the inner diffusive zone of the interface, will not be considered in this paper. Therefore, we shall assume henceforth that the Markstein length \mathcal{L} and the Markstein number \mathcal{M} are positive. It follows that the necessary and sufficient condition for instability is that $a > 0$ ($\alpha > 0$), with the instability being hydrodynamic in nature. Specifically, the interface is prone to three hydrodynamic instabilities corresponding to the three terms in a (or α). The first term leads to the Darrieus-Landau instability when the density ratio $r > 1$, the second term to the Saffman-Taylor instability when $(1-m)\mathcal{V} > 0$, and the third term to the Rayleigh-Taylor instability when $(r-1)G < 0$. It is interesting to note that these three instabilities combine in a simple and transparent manner in the expression of a . When $a > 0$, the range of unstable modes is $k \in (0, b/a)$, with the most unstable mode having growth rate $s = a^2/4b$ and wave number $k = a/2b$. A graphical illustration is provided in Fig. 2 for selected values of a . It is worth specializing the discussion to the case of flame propagation in Hele-Shaw channels which

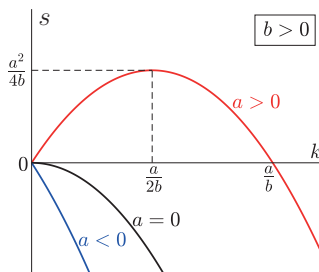


FIG. 2. Schematic illustration of the formula $s = ak - bk^2$ for $a > 0$, $a = 0$, and $a < 0$, all with $b > 0$.

have been actively investigated recently [13,22,24,25]. In this configuration, $\kappa_1 = \kappa_2 = h^2/12$, where h is the channel width, and the dispersion relation (11) takes the form

$$s = \left[\frac{\mu_2(\rho_1 - \rho_2)S_L^0}{\rho_2(\mu_2 + \mu_1)} + \frac{(\mu_2 - \mu_1)V}{\mu_2 + \mu_1} + \frac{h^2(\rho_2 - \rho_1)g}{12(\mu_2 + \mu_1)} \right] k - \left(\frac{\rho_1\mu_2 + \rho_2\mu_1}{\rho_2\mu_1 + \rho_2\mu_2} \right) S_L^0 \mathcal{L} k^2. \quad (14)$$

Again the combined effect of the three hydrodynamic instabilities is encapsulated in the coefficient of k . In premixed flames, since the density of the unburnt gas ρ_1 is larger than that of the burnt gas ρ_2 , the first term in the square bracket is always destabilizing, which is at the root of the DL instability. For the same reason, the third term is stabilizing if $g > 0$ and destabilizing otherwise, which is at the root of the RT instability. Typically, this instability is comparatively weak in small channels due to the factor h^2 then being small. As for the second term, this is seen to be stabilizing if $V < 0$ and destabilizing if $V > 0$, since $\mu_2 > \mu_1$ in flames, given that the dynamic viscosity μ is an increasing function of temperature in gases. This term is at the root of the ST instability. When this term is large, the ST instability is dominant compared to the DL instability. For V less than a critical value V_c , the flame is stable. This critical value is given, using (12) with $m = \mu_1/\mu_2$ and $G = \rho_2gh^2/12S_L^0\mu_2$ in this case, by

$$\frac{V_c}{S_L^0} = \mathcal{V}_c = -\frac{r-1}{1-m}(1-G). \quad (15)$$

For zero gravity, $G = 0$, and the typical values $r = 6$ and $m = 0.3$ for flames, we have stability when $\mathcal{V} < \mathcal{V}_c = -7.14$. Therefore, premixed flames with $\mathcal{M} > 0$ can be stabilized if \mathcal{V} is sufficiently negative in this case.

It is also worth noting from (13) that the growth rates of the DL and RT instabilities, as well as that of the ST instability obviously, depend on the viscosity ratio m . When $m = 1$, corresponding to flame models where the viscosity is assumed constant, $\mu_2 = \mu_1$, the ST instability represented by the second term in (13) is, of course, absent. It is also absent when $\mathcal{V} = 0$, that is, for freely propagating flames. This is so even when viscosity variations are taken into account, i.e., $m \neq 1$, with their effects being still felt through the terms representing the DL and RT instabilities. Furthermore, it is worth emphasizing that the analysis of Joulin and Sivashinsky [6] was carried out for the specific case $\mathcal{V} = 1$, where the dimensional imposed (mean) flow V is equal to S_L^0 . Their results are often used in the literature for freely propagating flames for which $\mathcal{V} = 0$. The difference in the growth rates between these two cases is highlighted in the following formulas:

$$\mathcal{V} = 0 : \quad s = \frac{(r-1)(1-G)}{1+m} k - \frac{r+m}{1+m} \mathcal{M} k^2 \quad (16)$$

$$\mathcal{V} = 1 : \quad s = \frac{r-m-(r-1)G}{1+m} k - \frac{r+m}{1+m} \mathcal{M} k^2, \quad (17)$$

which coincide when the assumption of constant viscosity, $m = 1$, is made.

D. Improved model for the local propagation speed

In this section we revisit the linear stability analysis by highlighting the differences brought about by adopting propagation speed models other than the simple phenomenological Markstein model (5). Such models include those considered in flame studies accounting for the dependence of the flame propagation speed, not only on its curvature, $\nabla \cdot \mathbf{n}$, as first introduced by Markstein [27], but also on the stretching, $-\mathbf{nn} : \nabla \mathbf{v}$, induced by the local flow field [[28], p. 132], as advocated by Karlovitz *et. al.* [29], Eckhaus [30] and Markstein [31]. Therefore, in general, one may anticipate an interface propagation model of the form

$$\frac{S_L}{S_L^0} = 1 + \mathcal{L}_c \nabla \cdot \mathbf{n} + \mathcal{L}_s \mathbf{nn} : \frac{\nabla \mathbf{v}}{S_L^0} \Big|_{y=f^-}, \quad (18)$$

involving two Markstein lengths, namely, \mathcal{L}_c associated with the interface curvature and equal to \mathcal{L} in formula (5), and \mathcal{L}_s associated with the flow strain. It is worth mentioning, however, that the effect of curvature and flow stretching have been shown, for flames modeled by one-step chemistry [32,33], to combine into a single quantity known as the flame stretch, defined as the fractional rate of change of a flame area element [34–36] and equal to $-S_L^0 \nabla \cdot \mathbf{n} - \mathbf{nn} : \nabla \mathbf{v}$. Under such circumstances, formula (18) is to be applied with $\mathcal{L}_c = \mathcal{L}_s = \mathcal{L}$.

It is important to point out that although model (18) is more general than model (5) and that it has been derived rigorously using multiscale analysis at least for premixed flames [37], its applicability to our problem is questionable. This is because it has been derived for flows obeying the Navier-Stokes equation rather than Darcy's equation. It is imperative, therefore, to examine the applicability of model (18) in the context of a Darcy's flow. To this end we have carried out a dedicated multiple-scale analysis for premixed flames with one step chemistry, following the approach of [32,33,38–40]. This rather lengthy analysis, which is presented elsewhere [41], demonstrates that formula (18) is valid, but in marked contrast to the classical analyses of [32,33], $\mathcal{L}_c \neq \mathcal{L}_s$ in general. Furthermore, \mathcal{L}_c in Darcy's model may also depend on \mathcal{V} and G . In view of this finding, it is useful to revisit the linear stability analysis using (18) instead of (5). The analysis proceeds exactly as above, except for the following modifications. Specifically, the linearized interfacial conditions (8) at $y = 0$ now read

$$\begin{aligned} \left[\left[\frac{\kappa}{\mu} p'_y \right] \right] &= \left(\frac{\rho_1}{\rho_2} - 1 \right) \left(S_L^0 \mathcal{L}_c f'_{xx} + \frac{\kappa_1}{\mu_1} \mathcal{L}_s p'_{yy} \Big|_{y=0^-} \right), \\ [[p']] &= \alpha f', \quad f'_t = \frac{\kappa_1}{\mu_1} (\mathcal{L}_s p'_{yy} - p'_y) \Big|_{y=0^-} + S_L^0 \mathcal{L}_c f'_{xx}, \end{aligned} \quad (19)$$

which leads to the dispersion relation

$$s = \frac{\alpha(k + \mathcal{L}_s k^2) - [(\mu_2/\kappa_2)(\rho_1/\rho_2) + \mu_1/\kappa_1] S_L^0 \mathcal{L}_c k^2}{\mu_1/\kappa_1 + \mu_2/\kappa_2 + \mathcal{L}_s (r-1) k \mu_2/\kappa_2}, \quad (20)$$

replacing (11). Introducing the Markstein numbers $\mathcal{M}_c = \mathcal{L}_c/\delta_L^0$ and $\mathcal{M}_s = \mathcal{L}_s/\delta_L^0$, the dispersion relation takes the nondimensional form

$$\begin{aligned} s &= \frac{ak - bk^2}{1 + ck}, \quad \text{where} \quad a = \frac{r-1}{1+m} + \frac{1-m}{1+m} \mathcal{V} + \frac{1-r}{1+m} G, \\ b &= \frac{r+m}{1+m} \mathcal{M}_c + a \mathcal{M}_s, \quad c = \frac{r-1}{1+m} \mathcal{M}_s, \end{aligned} \quad (21)$$

which reduces to (13) when $\mathcal{M}_s = 0$ and $\mathcal{M}_c = \mathcal{M}$.

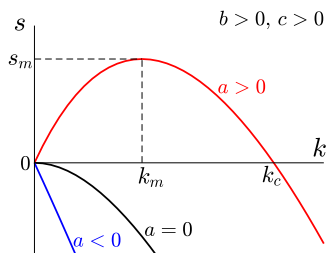


FIG. 3. Schematic illustration of the formula $s = (ak - bk^2)/(1 + ck)$ for $a > 0$, $a = 0$, and $a < 0$, all with $b > 0$ and $c > 0$. Here, $k_c = a/b$, $k_m = (\sqrt{1 + ac/b} - 1)/c$, and $s_m = bk_m^2$.

A brief discussion of the main implications of the new dispersion relation (21) is now provided. We first note the asymptotic behaviors

$$s = ak - k^2(b + ac) + \dots \quad \text{as } k \rightarrow 0 \quad \text{and} \quad s = -\frac{b}{c}k + \frac{1}{c^2}(b + ac) + \dots \quad \text{as } k \rightarrow \infty, \quad (22)$$

which indicates that the parabolic behavior of the original dispersion relation (13) is retained for small values of k , while a linear behavior is approached for larger values. We also note that the parameters b and c have to be both non-negative, which we shall assume, since otherwise $s(k)$ would be unbounded from above for $k \in [0, \infty)$, leading to an ill-posed problem. A schematic illustration of the dispersion curve $s(k)$ is provided in Fig. 3 for positive values of b and c . Since $b > 0$ and $c > 0$, the necessary and sufficient condition for instability corresponds to $a > 0$ as found earlier, with the instability being hydrodynamic in nature. The maximum of the dispersion curve for the unstable cases is reached at $(k, s) = (k_m, s_m)$, where

$$k_m = \frac{1}{c} \left(\sqrt{1 + \frac{ac}{b}} - 1 \right), \quad s_m = bk_m^2. \quad (23)$$

III. REVISITING JOULIN-SIVASHINSKY ANALYSIS BASED ON THE EULER-DARCY MODEL

A. Analysis and discussion

The pioneering analysis by Joulin and Sivashinsky [6] was based on a Euler-Darcy model addressing the stability of a premixed flame. This has been extended recently by Miroshnichenko *et al.* [9] who included curvature effects characterized by a Markstein number \mathcal{M} while accounting for the presence of an imposed flow, characterized in our notation by the nondimensional parameter \mathcal{V} . In the analysis of [9] gravity effects were neglected and the unburnt to burnt density ratio (r in our notation) and viscosity ratio (m) were lumped together into a single parameter, which somewhat obscures their relative contributions. We shall briefly revisit the Euler-Darcy model relaxing the assumptions made in [6,9] to match those of the Darcy's model of the previous section. In this section we shall adopt, for simplicity, the Markstein model (5) and provide in this context a short derivation of the dispersion relation along with a discussion of its implications. For completeness, the dispersion relation corresponding to the improved propagation speed model (18) will be provided in the following section.

The analysis follows closely that of the previous section. The governing equations in the laboratory frame, written on both sides of the interface, are now given by

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \frac{d\mathbf{u}}{dt} = -\nabla(p + \rho gy) - \frac{\mu}{\kappa} \mathbf{u}, \quad (24)$$

instead of Darcy's model (1). In the moving frame introduced above and illustrated in Fig. 1(b), the equations, in terms of $\mathbf{v} = \mathbf{u} + (S_L^0 - V)\mathbf{e}_y$, take the form

$$\nabla \cdot \mathbf{v} = 0, \quad \rho \frac{d\mathbf{v}}{dt} = -\nabla(p + \rho gy) - \frac{\mu}{\kappa}[\mathbf{v} - (S_L^0 - V)\mathbf{e}_y]. \quad (25)$$

These are subject to the boundary condition $\mathbf{v} = S_L^0 \mathbf{e}_y$ as $y \rightarrow -\infty$, and the following familiar interfacial conditions [42] at $y = f(x, t)$:

$$\begin{aligned} [[\mathbf{v} \cdot \mathbf{n}]] &= \left(\frac{\rho_1}{\rho_2} - 1\right)S_L, \quad [[p]] = -\left(\frac{\rho_1}{\rho_2} - 1\right)\rho_1 S_L^2, \\ [[\mathbf{v} \times \mathbf{n}]] &= 0, \quad S_L = \frac{v - uf_x - f_t}{\sqrt{1 + f_x^2}} \Big|_{y=f^-}. \end{aligned} \quad (26)$$

These conditions replace those in (4) and differ by the presence of the third condition, which expresses the continuity of the tangential component of \mathbf{v} , a requirement which is not needed or enforceable for Darcy's law. Furthermore, there is a jump in pressure across the interface characterized by the second condition in (26), associated with the presence of the inertial term on the left-hand side of (25); this jump is absent in (4).

The basic solution $(\mathbf{v}, p, f) = (\bar{\mathbf{v}}, \bar{p}, \bar{f})$ corresponding to a stationary flat interface is given by $\bar{f} = 0$ and

$$\bar{\mathbf{v}} = S_L^0 \mathbf{e}_y, \quad -\bar{p} = \rho_1 gy + \frac{\mu_1}{\kappa_1} V y \quad \text{for } y < 0, \quad (27)$$

$$\bar{\mathbf{v}} = \frac{\rho_1}{\rho_2} S_L^0 \mathbf{e}_y, \quad -\bar{p} = \rho_2 gy + \frac{\mu_2}{\kappa_2} V y + \left(\frac{\rho_1}{\rho_2} - 1\right) \left(\frac{\mu_2}{\kappa_2} V y + \rho_1 S_L^0\right) S_L^0 \quad \text{for } y > 0. \quad (28)$$

As shown in Appendix, the stability of this solution is found to be governed by the quadratic dispersion relation,

$$\begin{aligned} (\rho_1 + \rho_2)s^2 + \left[\frac{\mu_1}{\kappa_1} + \frac{\mu_2}{\kappa_2} + 2\rho_1 S_L^0 (1 + \mathcal{L}k)k \right] s - \alpha k \\ + \left[\left(\frac{\rho_1}{\rho_2} \frac{\mu_2}{\kappa_2} + \frac{\mu_1}{\kappa_1} \right) \mathcal{L} - \left(\frac{\rho_1}{\rho_2} - 1 \right) \rho_1 S_L^0 \right] S_L^0 k^2 + \frac{2\rho_1^2 S_L^0{}^2 \mathcal{L}}{\rho_2} k^3 = 0, \end{aligned} \quad (29)$$

where α is as given in (9). A nondimensional form of this dispersion relation is obtained by multiplying all terms by $\delta_L^2 / \rho_1 S_L^0{}^2$ and introducing the nondimensional growth rate and wave number $\tilde{s} = s\delta_L^2 / S_L^0$ and $\tilde{k} = k\delta_L^2$ and using the parameters defined in (12). Dropping the tildes, we thus obtain the nondimensional dispersion relation

$$\frac{r+1}{r} s^2 + [2(1 + \mathcal{M}k)k + \varphi]s - a\varphi k + (b\varphi + 1 - r)k^2 + 2r\mathcal{M}k^3 = 0, \quad (30)$$

where a and b are as defined in (13) and

$$\varphi = \frac{12\text{Pr}(1+m)}{\epsilon^2 m} \quad \text{with} \quad \epsilon = \frac{\sqrt{12\kappa_1}}{\delta_L^0} \quad \text{and} \quad \text{Pr} = \frac{\mu_1}{\rho_1 D_1}. \quad (31)$$

We note that for a Hele-Shaw channel, $\kappa_1 = h^2/12$ so that $\epsilon = h/\delta_L^0$. In the narrow channel limit $\epsilon \rightarrow 0$, terms containing φ are dominant so that dropping all other terms implies $s = ak - bk^2$, which is the nondimensional dispersion relation (13) obtained above using Darcy's law. In the wide-channel limit, $\epsilon \rightarrow \infty$, $\varphi \rightarrow 0$, leading to a classical result first obtained by Markstein [27] for flames freely propagating in unconfined geometries. In the latter context, more accurate dispersion relations are available, accounting for temperature dependence of all transport coefficients [42,43], but these are not directly applicable in narrow confined channels which are our main focus.

Furthermore, in wide channels the dependence of (the effective speed) S_L in (5) on V cannot, strictly speaking, be ignored [9,17] as we assumed herein [44].

To close this section, we derive now a simple stability criterion based on (30), assuming as before that the Markstein number $\mathcal{M} > 0$. Then the coefficients of s^2 and s in (30) are clearly positive. Therefore, by the Routh-Hurwitz criterion for the second-order polynomial in s , instability occurs if and only if the constant coefficient of the quadratic equation is negative. This condition is equivalent to finding whether the minimum value of the parabola,

$$g(k) = 2r\mathcal{M}k^2 + (b\varphi + 1 - r)k - a\varphi, \quad (32)$$

in the interval $k \in [0, \infty)$ is negative. The minimum occurs at $k = k_* \equiv (r - 1 - b\varphi)/4r\mathcal{M}$ and its value is $g = g_* \equiv -a\varphi - 2r\mathcal{M}k_*^2$. For stability, we then require $g_* > 0$, i.e., $a\varphi < -2r\mathcal{M}k_*^2$, and this is applicable as long as $k_* > 0$. When $k_* < 0$, the condition for stability is simply given by $g(0) > 0$, which requires $a < 0$. This second stability criterion coincides with the stability criterion for Darcy's law, whereas the first criterion pertains to cases with φ (or $1/\epsilon$) being sufficiently small, leading to k_* being positive when $r > 1$. This suggests that the predictions of the Darcy and Darcy-Euler models agree with each other provided $k_* < 0$, a condition which may be written as $\epsilon < \epsilon_c$, where

$$\epsilon_c^2 = \frac{12\text{Pr} \mathcal{M}(r + m)}{m(r - 1)}. \quad (33)$$

Note that both criteria may be combined to yield the necessary and sufficient condition for stability in the form

$$a < a_c \equiv \begin{cases} 0 & \text{for } \epsilon^2 < \epsilon_c^2, \\ -\frac{b(r-1)}{8r\mathcal{M}} \frac{\epsilon_c^2}{\epsilon^2} \left(1 - \frac{\epsilon_c^2}{\epsilon^2}\right)^2 & \text{for } \epsilon^2 > \epsilon_c^2. \end{cases} \quad (34)$$

Note that the stability criterion is always $a < a_c = 0$ when $r < 1$, since then k_m is clearly negative. When $r > 1$ as for premixed flames, both cases in the stability criterion (34) occur. For premixed flames, we can use the typical values $\text{Pr} = 0.7$, $r = 6$, and $m = 0.3$ to evaluate the coefficients in (34), which implies that $\epsilon_c = 6\sqrt{\mathcal{M}}$ and $b(r - 1)/8r\mathcal{M} = 0.5$. The stability criteria can also be written in terms of the critical velocity \mathcal{V}_c below which the interface is stable. This critical value is given by (15) for Darcy's law, and this remains true for the Euler-Darcy model when $\epsilon < \epsilon_c$, which always holds for $r < 1$. When $\epsilon > \epsilon_c$, \mathcal{V}_c is to be computed using the second case in formula (34). For $r > 1$ we thus find

$$\mathcal{V}_c = \begin{cases} -\frac{r-1}{1-m}(1 - G) & \text{for } \epsilon < \epsilon_c, \\ -\frac{r-1}{1-m} \left[1 - G + \frac{r+m}{8r} \frac{\epsilon_c^2}{\epsilon^2} \left(1 - \frac{\epsilon_c^2}{\epsilon^2}\right)^2\right] & \text{for } \epsilon > \epsilon_c. \end{cases} \quad (35)$$

We note that $\mathcal{V}_c \rightarrow -\infty$ as $\epsilon \rightarrow \infty$, indicating that a flame in an infinitely wide channel is impossible to stabilize by the flow.

B. The dispersion relation for Darcy-Euler with an improved interface-propagation model

The previous section was based on the dispersion relation (29) and its nondimensional form (30), obtained when adopting the Markstein propagation speed model (5). In this section we provide for completeness the dispersion relation corresponding to the improved propagation speed model (18), whose derivation is briefly outlined at the end of Appendix. In dimensional form, the dispersion relation is found to be given by

$$a_2 s^2 + a_1 s + a_0 = 0, \quad (36)$$

where

$$\begin{aligned}
 a_2 &= \rho_1 + \rho_2 + (\rho_1 - \rho_2)\mathcal{L}_s k, \\
 a_1 &= \frac{\mu_1}{\kappa_1} + \frac{\mu_2}{\kappa_2} + \left[2\rho_1 S_L^0 + \frac{\mu_2}{\kappa_2} \left(\frac{\rho_1}{\rho_2} - 1 \right) \mathcal{L}_s \right] k + 2\rho_1 S_L^0 \left[\mathcal{L}_c + \left(\frac{\rho_1}{\rho_2} - 1 \right) \mathcal{L}_s \right] k^2, \\
 a_0 &= -\alpha k + \left[\left(\frac{\rho_1}{\rho_2} \frac{\mu_2}{\kappa_2} + \frac{\mu_1}{\kappa_1} \right) S_L^0 \mathcal{L}_c + \left(1 - \frac{\rho_1}{\rho_2} \right) \rho_1 S_L^{02} + \alpha \mathcal{L}_s \right] k^2 + \frac{2\rho_1^2 S_L^{02}}{\rho_2} \left[\mathcal{L}_c + \left(1 - \frac{\rho_2}{\rho_1} \right) \frac{\mathcal{L}_s}{2} \right] k^3.
 \end{aligned}$$

As done above, a nondimensional form is obtained by multiplying all terms by $\delta_L^{02}/\rho_1 S_L^{02}$ and introducing the nondimensional growth rate and wave number $\tilde{s} = s\delta_L^0/S_L^0$ and $\tilde{k} = k\delta_L^0$. Dropping the tildes, we thus have

$$\begin{aligned}
 \frac{r + 1 + (r - 1)\mathcal{M}_s k}{r} s^2 + [2k\{1 + [\mathcal{M}_c + (r - 1)\mathcal{M}_s]k\} + \varphi(1 + ck)]s \\
 - a\varphi k + (b\varphi + 1 - r)k^2 + [2r\mathcal{M}_c + (r - 1)\mathcal{M}_s]k^3 = 0,
 \end{aligned} \tag{37}$$

where a , b , and c are as defined in (21), and φ , ϵ , Pr as defined in (31). It is worth noting that in the narrow channel limit $\epsilon \rightarrow 0$, terms containing φ are dominant so that dropping all other terms leads to the dispersion relation (21) derived for Darcy's law. On the other hand, as $\epsilon \rightarrow \infty$, the terms containing φ drop, and we obtain exactly the same dispersion relation as the one derived as Eq. (10) in the paper by Creta and Matalon [45] provided $\mathcal{M}_s = \mathcal{M}_c$.

Illustrative results based on the dispersion relation (37) are shown in Fig. 4 for the parameter values $r = 6$, $m = 0.3$, $\text{Pr} = 0.7$, and $G = 0$. The three subfigures in the top row are characterized by the Markstein numbers $\mathcal{M}_c = 1$ and $\mathcal{M}_s = 0$, which correspond to cases using the Markstein model (5) where the effect of strain on the interface propagation speed is ignored. In each subfigure, the value of \mathcal{V} is prescribed for selected values of ϵ . When $\epsilon \rightarrow 0$, the curve corresponding to the dispersion relation (13) based on Darcy's law and dependent on \mathcal{V} is approached, while as $\epsilon \rightarrow \infty$, the curves tend to that of the classical Darrieus-Landau instability, which is independent of \mathcal{V} . The effect of confinement is prominent for moderate and smaller values of ϵ , being destabilizing for larger positive values of \mathcal{V} .

Turning now to the middle row of Fig. 4, the effects of curvature and strain are both retained, corresponding to the improved propagation speed model, Eq. (18), with $\mathcal{M}_c = \mathcal{M}_s = 1$. Comparing with the top row, two observations are in order. First, the range of unstable modes and the maximum growth rates are now significantly reduced. Second, the parabolic shape of the dispersion curves of the first row is now strongly modified for larger values of k , in line with the discussion at the end of Sec. IID. This change in shape is intimately associated with nonzero values of \mathcal{M}_s and cannot be avoided by adjusting the curvature Markstein number \mathcal{M}_c while using the Markstein model, Eq. (5). This is evident in the dispersion curves presented in the bottom row. These curves were calculated for $\mathcal{M}_s = 0$, with \mathcal{M}_c adjusted to maintain the same range of unstable modes observed in the middle row subfigures.

IV. DARRIEUS-LANDAU AND SAFFMAN-TAYLOR INSTABILITIES BASED ON DARCY'S LAW: NUMERICAL RESULTS FOR A PREMIXED FLAME

A. Problem formulation

In the theoretical analysis of the previous sections, the interface is treated as a hydrodynamic discontinuity in the spirit of the original studies of the Darrieus-Landau and Saffman-Taylor instabilities [1–5]. The analysis also accounted for local curvature effects of the interface, following the phenomenological approach of Markstein [27]. Strictly speaking, a more elaborate description of curvature effects is possible, accounting for the internal structure of the interface, as found for premixed flames in [33,39]. In the remainder of the paper we shall present a numerical investigation

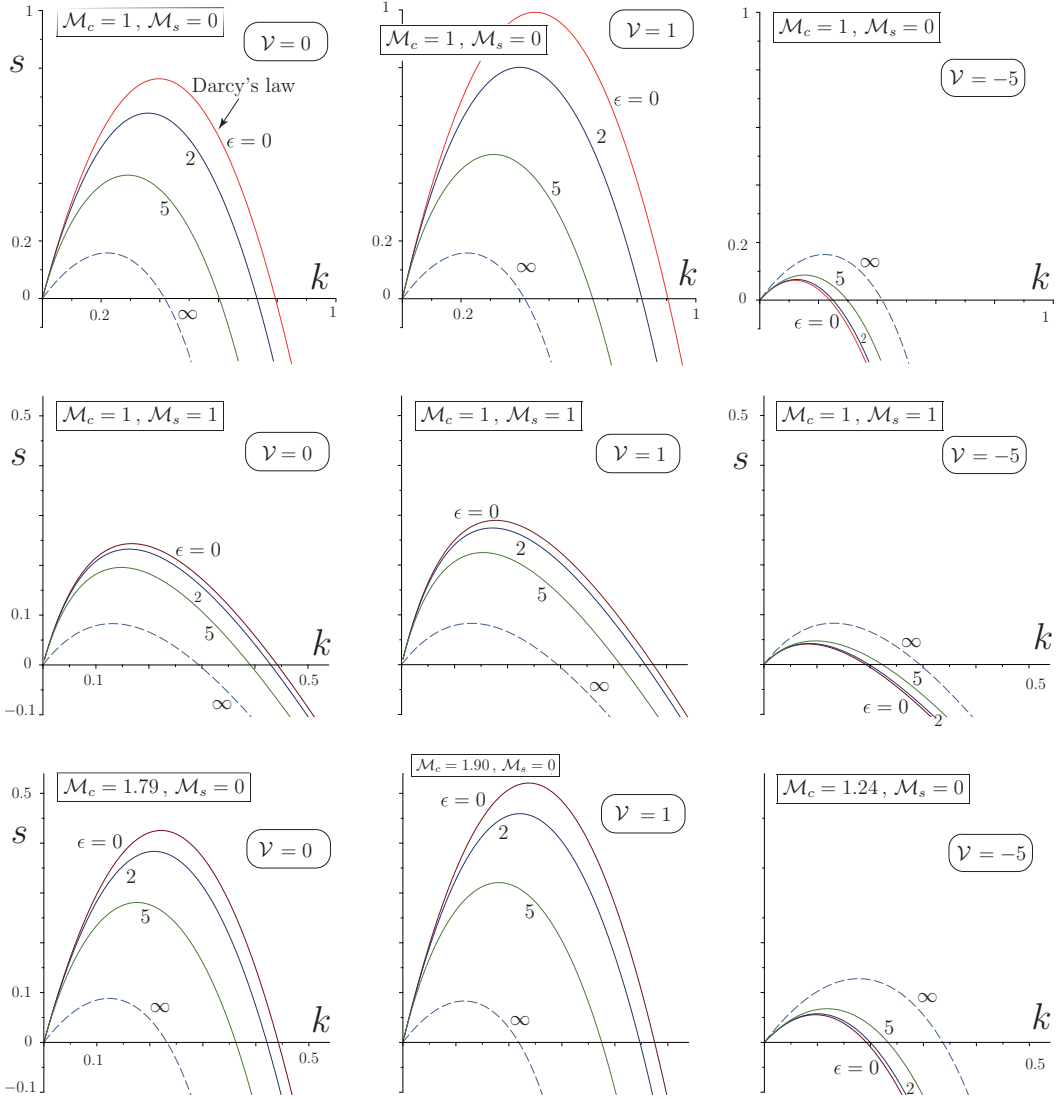


FIG. 4. The growth rate s versus the wave number k based on the dispersion relation (37). All calculations are performed with the parameter values $r = 6$, $m = 0.3$, $G = 0$, and $\text{Pr} = 0.7$. The three cases in the top row have Markstein numbers $\mathcal{M}_c = \mathcal{M}_s = 1$. The cases in the bottom row have all $\mathcal{M}_s = 0$ and \mathcal{M}_c , as indicated in each subfigure. The dispersion curves correspond to three values of \mathcal{V} , namely 0, 1, and -5 , and selected values of ϵ , as indicated. The axes of the subfigures in each row have the same horizontal and vertical length scales and labels.

focusing on the hydrodynamic instabilities of premixed flames in order to complement and validate the theoretical results of the Darcy's law model.

We shall adopt a 2D configuration for our investigation as illustrated in Fig. 1, which may be associated with a depth-averaged description of flames in a Hele-Shaw channel. In this configuration, fluid 1 corresponds to an unburnt reacting mixture and fluid 2 to a burnt gas mixture. For simplicity, the unburnt gas is assumed to be deficient in a reactant which is consumed with a reaction rate per unit volume given by an Arrhenius law $\rho B Y_R e^{-E/RT}$. Here, B is the pre-exponential factor,

Y_R the reactant mass fraction, T the gas temperature, E the activation energy of the reaction, and R the universal gas constant. For this model we may define the adiabatic flame temperature by $T_2 = T_1(1 + q)$, where q quantifies the amount of heat released by the reaction and T_1 the unburnt gas temperature. A nondimensional measure of the activation energy is then given by the Zeldovich number $\beta = E(T_2 - T_1)/RT_2^2$. Further, we shall assume that the thermal diffusivity D is equal to the reactant diffusion coefficient, i.e., a unit Lewis number, so as to eliminate diffusive-thermal (Turing-like) instabilities. We shall also ignore heat losses, whether these are by radiation or by conduction to walls. These assumptions are adopted for simplicity in order to focus on the flame hydrodynamic instabilities. Under these conditions, the temperature T and the mass fraction Y_R are not independent but are related by the equation $Y_R/Y_{R,1} = 1 - (T/T_1 - 1)/q$, where $Y_{R,1}$ is the mass fraction in the unburnt gas. Finally, the density is assumed to depend on temperature according to the ideal gas law, while the transport coefficients are assumed to follow a power-law temperature dependence. Specifically,

$$\rho T = \rho_1 T_1, \quad \frac{\mu}{\mu_1} = \frac{\rho D}{\rho_1 D_1} = \left(\frac{T}{T_1}\right)^n \quad \text{with } n = 0.7.$$

For convenience, we introduce the following nondimensional variables and parameters:

$$\begin{aligned} t^* &= \frac{t S_L^0}{\delta_L}, & \mathbf{x}^* &= \left(\frac{x}{\delta_L}, \frac{y}{\delta_L}\right), & \rho^* &= \frac{\rho}{\rho_1}, & \mathbf{u}^* &= \frac{\mathbf{u}}{S_L^0}, & p^* &= \frac{h^2 p}{12\mu_1 D_1}, & \theta &= \frac{T - T_1}{T_2 - T_1}, \\ \mu^* &= \frac{\mu}{\mu_1}, & \lambda &= \frac{\rho D}{\rho_1 D_1}, & \mathcal{V} &= \frac{V}{S_L^0}, & G &= \frac{\rho_2 g h^2}{12\mu_2 S_L^0}, & r &= \frac{\rho_1}{\rho_2}, & m &= \frac{\mu_1}{\mu_2}, & S &= \frac{S_L^0}{S_{L,\infty}^0}, \end{aligned}$$

where $\delta_L = D_1/S_L^0$ is the laminar flame thickness, and $S_{L,\infty}^0 = [2\beta^{-2}BD_1(\rho_2^2 D_2)/(\rho_1^2 D_1)e^{-E/RT_2}]^{1/2}$ is the asymptotic formula for S_L^0 derived in the limit $\beta \rightarrow \infty$ [[46], p.164]. The value of S , which must approach unity as $\beta \rightarrow \infty$, will be determined [by solving Eq. (48) below] numerically for the typical finite value $\beta = 10$, adopted herein. Dropping the asterisks for t^* , \mathbf{x}^* , ρ^* , \mathbf{u}^* , p^* , and μ^* , the two-dimensional governing equations in the laboratory frame can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (38)$$

$$-\mu \mathbf{u} = \nabla p + \frac{rG}{m} \rho \mathbf{e}_y, \quad (39)$$

$$\rho \frac{\partial \theta}{\partial t} + \rho \mathbf{u} \cdot \nabla \theta = \nabla \cdot (\lambda \nabla \theta) + \omega, \quad (40)$$

$$\rho(1 + q\theta) = 1, \quad \mu = \lambda = (1 + q\theta)^n, \quad (41)$$

where

$$\omega = \frac{\beta^2}{2S^2} (1 + q)^{1-n} \rho (1 - \theta) \exp \left[\frac{\beta(\theta - 1)}{1 + q(\theta - 1)/(1 + q)} \right]. \quad (42)$$

From Eq. (41), it follows that the density ratio $r = 1 + q$ and the viscosity ratio $m = 1/(1 + q)^n$, since $\theta \rightarrow 1$ in the burnt gas as $y \rightarrow +\infty$ and $\theta \rightarrow 0$ in the unburnt gas as $y \rightarrow -\infty$.

B. Linear stability of the planar flame

In the laboratory frame the planar flame propagates in the negative y direction with a constant nondimensional speed $1 - \mathcal{V}$ since $\mathbf{u} = \mathcal{V} \mathbf{e}_y$ as $y \rightarrow -\infty$; see Fig. 1. To study the stability of this planar flame, it is advantageous to shift to a coordinate system moving with the flame front by

using the coordinate transformation $(x, y, t) \mapsto [x, y + (1 - \mathcal{V})t, t]$. Introducing further $\mathbf{v} = \mathbf{u} + (1 - \mathcal{V})\mathbf{e}_y$, we have $\mathbf{v} = \mathbf{e}_y$ as $y \rightarrow -\infty$ and the governing equations may be written as

$$-\frac{1}{q}\nabla \cdot \left(\frac{\nabla p}{\mu} \right) - \frac{rG}{mq} \frac{\partial}{\partial y} \left(\frac{\rho}{\mu} \right) = \nabla \cdot (\lambda \nabla \theta) + \omega, \quad (43)$$

$$-\mu \mathbf{v} + \mu(1 - \mathcal{V})\mathbf{e}_y = \nabla p + \frac{rG}{m} \rho \mathbf{e}_y, \quad (44)$$

$$\rho \frac{\partial \theta}{\partial t} + \rho \mathbf{v} \cdot \nabla \theta = \nabla \cdot (\lambda \nabla \theta) + \omega, \quad (45)$$

$$\rho(1 + q\theta) = 1, \quad \mu = \lambda = (1 + q\theta)^n. \quad (46)$$

Instead of the continuity equation, we have introduced the Poisson equation (43) for the pressure field, which is obtained by combining the continuity equation $\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{v}) = 0$, the temperature equation (45), and the equation of state given in (46).

The basic solution, corresponding to the planar flame, is governed by Eqs. (43)–(46) in which the time and x derivatives are set to zero. Denoted with an overbar, it is given by

$$\bar{\rho} \bar{\mathbf{v}} = \mathbf{e}_y, \quad -\frac{d\bar{p}}{dy} = \bar{\mu} \frac{1 - \bar{\rho}}{\bar{\rho}} + \bar{\mu} \mathcal{V} + \frac{rG}{m} \bar{\rho}, \quad (47)$$

along with $\bar{\rho} = 1/(1 + q\bar{\theta})$ and $\bar{\mu} = \bar{\lambda} = (1 + q\bar{\theta})^n$, where $\bar{\theta}$ satisfies

$$\frac{d\bar{\theta}}{dy} = \frac{d}{dy} \left(\bar{\lambda} \frac{d\bar{\theta}}{dy} \right) + \omega(\bar{\theta}), \quad \bar{\theta}(-\infty) = 0, \quad \bar{\theta}(+\infty) = 1. \quad (48)$$

The temperature field $\bar{\theta}(y)$ and the unknown parameter S which appears in the definition of ω in (42), are independent of \mathcal{V} and G , and are computed numerically for the typical values $n = 0.7$, $q = 5$, and $\beta = 10$; for these values, $S = 0.929$.

The stability of the planar flame is investigated by introducing infinitesimal perturbations such that

$$\begin{bmatrix} \theta \\ p \end{bmatrix} = \begin{bmatrix} \bar{\theta}(y) \\ \bar{p}(y) \end{bmatrix} + \begin{bmatrix} \hat{\theta}(y) \\ \hat{p}(y) \end{bmatrix} e^{st+ikx}, \quad (49)$$

where k is the real wave number of the perturbation and s its growth rate, which is to be obtained as an eigenvalue. Substituting (49) into (43)–(46) and linearizing about the base solutions (47)–(48) results in the linearized system of equations

$$s\bar{\rho}\hat{\theta} + \frac{d\hat{\theta}}{dy} + \bar{\rho}(\hat{v}_y - q\hat{\theta}) \frac{d\bar{\theta}}{dy} = \frac{d}{dy} \left(\bar{\lambda} \frac{d\hat{\theta}}{dy} + \hat{\lambda} \frac{d\bar{\theta}}{dy} \right) - k^2 \bar{\lambda} \hat{\theta} + \hat{\omega}, \quad (50)$$

$$-\frac{1}{q} \frac{d}{dy} \left(\frac{1}{\bar{\mu}} \frac{d\hat{p}}{dy} - \frac{\hat{\mu}}{\bar{\mu}^2} \frac{d\bar{p}}{dy} \right) + \frac{k^2 \hat{p}}{q\bar{\mu}} - \frac{rG}{mq} \frac{d}{dy} \left(\frac{\hat{\rho}}{\bar{\mu}} - \frac{\bar{\rho} \hat{\mu}}{\bar{\mu}^2} \right) = \frac{d}{dy} \left(\bar{\lambda} \frac{d\hat{\theta}}{dy} + \hat{\lambda} \frac{d\bar{\theta}}{dy} \right) - k^2 \bar{\lambda} \hat{\theta} + \hat{\omega}. \quad (51)$$

Here, $\hat{\rho} = -q\bar{\rho}^2 \hat{\theta}$, $\hat{\mu} = nq\bar{\rho} \bar{\mu} \hat{\theta}$, $\hat{\lambda} = nq\bar{\rho} \bar{\lambda} \hat{\theta}$, $-\bar{\mu} \hat{v}_y = d\hat{p}/dy + \hat{\mu}(q\bar{\theta} + \mathcal{V}) + \hat{p}rG/m$, and

$$\hat{\omega} = \frac{\beta^2(1+q)^{1-n}}{2S^2} \left[\hat{\rho}(1 - \bar{\theta}) - \bar{\rho} \hat{\theta} + \frac{\bar{\rho}(1 - \bar{\theta})\beta(1+q)^2 \hat{\theta}}{(1+q\bar{\theta})^2} \right] \exp \left[\frac{\beta(\bar{\theta} - 1)}{1 + q(\bar{\theta} - 1)/(1+q)} \right]. \quad (52)$$

Solving the linear eigenvalue problem given by Eqs. (50)–(51) subject to the boundary conditions $\hat{\theta} = d\hat{p}/dy = 0$ as $y \rightarrow \pm\infty$, yields numerically the required dispersion relation

$$s = s(k; \mathcal{V}, G). \quad (53)$$

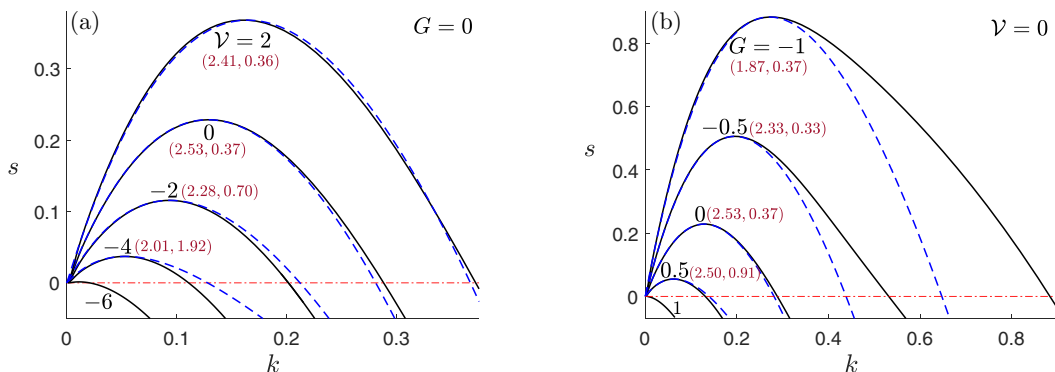


FIG. 5. Dispersion curves $s(k)$ represented as solid lines computed for $q = 5$, $n = 0.7$, and $\beta = 10$ (i.e., $r = 6$ and $m = 0.29$). Left figure (a) corresponds to $G = 0$ and selected values of \mathcal{V} . Right figure (b) corresponds to $\mathcal{V} = 0$ and selected values of G . The dashed lines correspond to the theoretical formula $s = (ak - bk^2)/(1 + ck)$, with a evaluated from equation (21), while b and c are evaluated using the relations $b = s_m/k_m^2$ and $c = a/s_m - 2/k_m$, where (k_m, s_m) corresponds to the maximum point of the numerical dispersion relation. The numbers inside the parentheses for each curve correspond to the resulting values of \mathcal{M}_c and \mathcal{M}_s , extracted from b and c .

C. The dispersion relation computed numerically and its implications

The dispersion relation (53) is computed numerically for the typical values $q = 5$, $n = 0.7$, and $\beta = 10$. The computations are carried out using the eigenvalue solver in comsol multiphysics, as done in our earlier works [47–50]. Illustrative numerical results showing the growth rate s versus the wave number k for selected values of the parameters \mathcal{V} and G are plotted in Fig. 5 as solid lines. Figure 5(a) corresponding to cases without gravity ($G = 0$), shows that the perturbation growth rate increases with increasing values of \mathcal{V} . As \mathcal{V} is decreased, the maximum growth rate decreases and the range of unstable wave numbers shrinks. For $\mathcal{V} = -6$, $s(k) < 0$ except for values of k in a tiny range (difficult to see in the figure) around $k = 0$. For lower values of \mathcal{V} , the growth rate is nonpositive for all wave numbers. This demonstrates the existence of a critical value of \mathcal{V} below which the flame is stable, as predicted theoretically in formula (15) of Sec. II. A similar trend can also be observed in Fig. 5(b), where G is varied while maintaining $\mathcal{V} = 0$, which corresponds to freely propagating flames. Here, as expected, an increase in the value of G is stabilizing, and there exists a critical value of G above which the flame is stable, again in agreement with the theoretical results discussed in Sec. II C.

The dashed lines in Fig. 5 represent the theoretical formula $s = (ak - bk^2)/(1 + ck)$, where a is evaluated using the formula given in (21). As for the parameters b and c , which are also defined in (21), their evaluation requires the values for the two Markstein numbers \mathcal{M}_c and \mathcal{M}_s , which are not readily available from existing theories, as the latter applicability to Darcy’s flows has not been established. The determination of these Markstein numbers, which may also be influenced by \mathcal{V} and G in the context of Darcy’s flows as a preliminary analysis suggests, requires dedicated investigations, say multiple-scale analysis accounting for the flame’s inner structure or numerical/experimental approaches. For the sake of assessing the ability of our theory to predict flame instability, we shall herein evaluate these parameters directly from the numerical dispersion relation itself, as sometimes done in experimental studies [22]. To this end, we shall assume $b = s_m/k_m^2$ and $c = a/s_m - 2/k_m$, with (k_m, s_m) corresponding to the maximum location of the numerical dispersion curve, so that the growth rates from the theoretical formula $s = (ak - bk^2)/(1 + ck)$ and from the numerical dispersion relation (53) coincide at the maximum location. It follows that $\mathcal{M}_s = c(1 + m)/(r - 1)$ and $\mathcal{M}_c = (b - a\mathcal{M}_s)(1 + m)/(r + m)$, and these are reported in Fig. 5 inside the parentheses for each curve. As can be inferred then from Fig. 5, the numerical results corroborate the theoretical

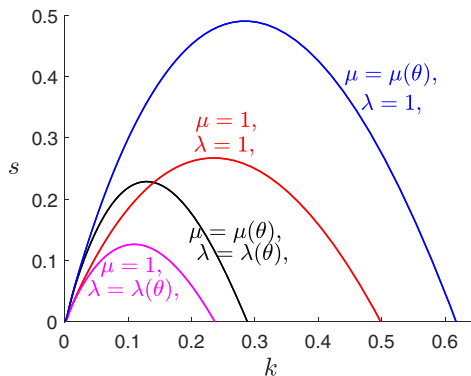


FIG. 6. Dispersion curves $s(k)$ computed for $\mathcal{V} = 0$ and $G = 0$. The curves correspond to different approximations of the functions $\mu(\theta)$ and $\lambda(\theta)$.

formula reasonably well, once the location of the maximum of the numerical and theoretical growth rates are fitted. In particular, the shape of the numerical dispersion curve is well reproduced by the theory. Note, however, that the agreement between the solid and dashed curves appears to be less satisfactory for larger values of k in the two cases of Fig. 5(b) corresponding to $G = -1$ and $G = -0.5$. This deterioration of the agreement is not surprising, given that our modeling of the flame as a hydrodynamic discontinuity is, strictly speaking, valid when $k \ll 1$, that is, for perturbations with wavelength larger than the flame thickness. More importantly, the parameter a , which characterizes the slope of $s(k)$ at $k = 0$, is very well predicted by the theory, and this is so independently of the values of b and c (or of the Markstein numbers). This indicates that our theoretical model correctly captures the key aspects of the three hydrodynamic instabilities, which are the primary focus of this paper.

Before concluding this section, we would like to emphasize the significance of incorporating the temperature-dependent transport coefficients in the numerical dispersion relation. While $\mu(\theta) = \lambda(\theta)$ in typical reacting gaseous mixtures, the two functions play distinct roles. For instance, assuming constant viscosity ($\mu = 1$) eliminates the influence of the mean flow \mathcal{V} , as the term $\mu(1 - \mathcal{V})\mathbf{e}_y = (1 - \mathcal{V})\mathbf{e}_y$ in the governing equation (44) can then be absorbed into the pressure definition. Conversely, $\lambda(\theta)$ affects the diffusive transport within the inner flame zone, as evident from (45); since λ increases with θ , accounting for this increase generally leads to larger flame speed and thickness. Figure 6 illustrates how the dispersion curves are modified by these two functions when $\mathcal{V} = 0$ and $G = 0$. The black curve, computed for temperature-dependent μ and λ , provides the most accurate description, whereas the red curve, computed with $\mu = \lambda = 1$, provides the least accurate description.

D. Illustrative time-dependent numerical simulations

In this section, time-dependent numerical simulations are presented to highlight the role of \mathcal{V} and G on the full development of the hydrodynamic instabilities of premixed flames. The computations are initiated from conditions corresponding to the planar flame solution of Eqs. (47)–(48) upon which small amplitude random disturbances are superimposed. In order to keep the propagating flame within the computational domain, the coordinate system is shifted at each time step, as done in [15,51,52], with the speed $S_T(t)$ defined by

$$S_T(t) = \frac{1}{L_x} \int_0^{L_y} \int_0^{L_x} \omega \, dx dy, \quad (54)$$

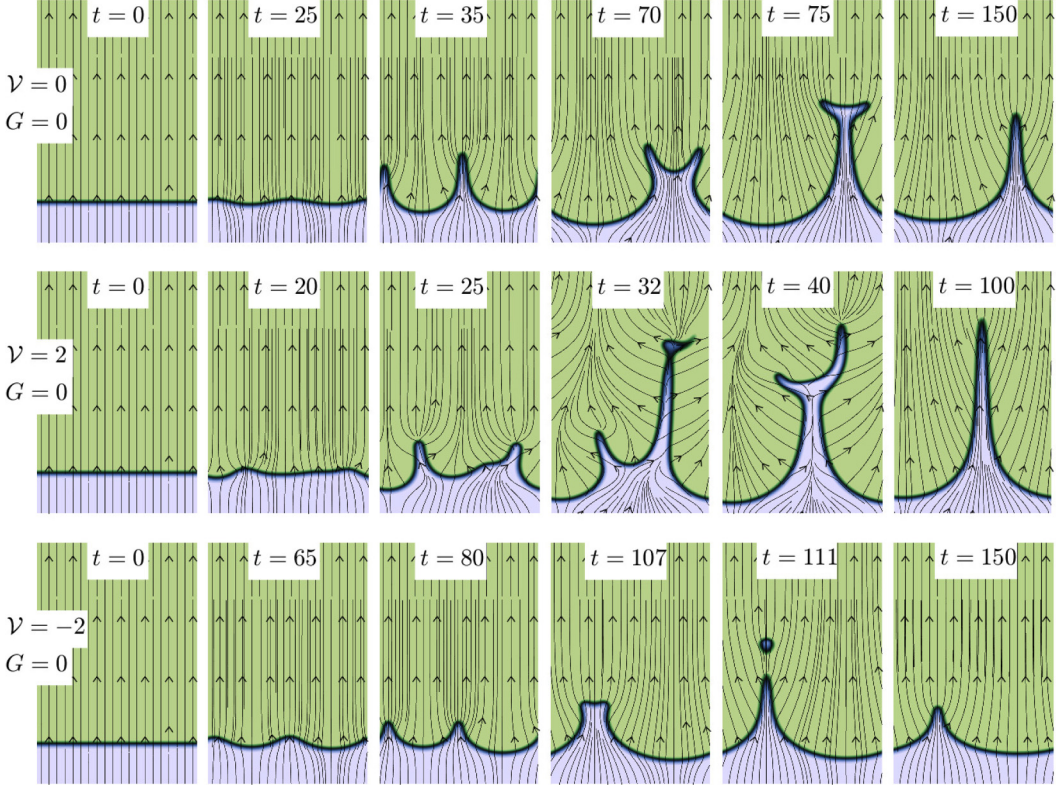


FIG. 7. Instantaneous temperature fields and streamlines, with $G = 0$, seen from a frame moving with the flame speed $S_T(t)$. The upper row of figures correspond to $\mathcal{V} = 0$, the middle row to $\mathcal{V} = 2$, and the lower to $\mathcal{V} = -2$. The horizontal nondimensional domain size shown in each figure is 100 and its vertical extent 180.

where $(0, L_x) \times (0, L_y)$ is the computational domain. We note that $S_T(t)$ is a measure of the global propagation flame speed with respect to fluid 1 at $y = -\infty$ and quantifies the total instantaneous burning rate per unit transverse flame length. Given our nondimensionalization and initial condition corresponding to a planar flame, $S_T(0) = 1$.

In the moving frame adopted, Eqs. (43), (45), and (46) are still valid provided \mathbf{v} is redefined to be $\mathbf{v} = \mathbf{u} + (S_T - \mathcal{V})\mathbf{e}_y$, whereas Eq. (44) is replaced by $-\mu\mathbf{v} + \mu(S_T - \mathcal{V})\mathbf{e}_y = \nabla p + (rG/m)\rho\mathbf{e}_y$. The problem is solved subject to periodic boundary conditions in the x direction along with the conditions

$$\mathbf{v} = S_T(t)\mathbf{e}_y, \quad \theta = 0 \quad \text{as} \quad y \rightarrow -\infty \quad \text{and} \quad p = 0, \quad \frac{\partial\theta}{\partial y} = 0 \quad \text{as} \quad y \rightarrow +\infty. \quad (55)$$

The computations are carried out using comsol multiphysics in a computation domain with size $L_x = 100$ and $L_y = 300$. The initial planar flame is positioned around $y = 100$. More details on the computations can be found in Refs. [15,51,52].

We begin by illustrating the effect of varying the imposed flow speed \mathcal{V} in the absence of gravity, $G = 0$. Shown in Fig. 7 is the temperature field $\theta(\mathbf{x}, t)$ at selected times for three cases pertaining to $\mathcal{V} = 0, 2$, and -2 . The associated global propagation speed $S_T(t)$ is plotted in Fig. 8(a). For all three cases, it is observed that the flame develops two cusplike structures pointing towards the burnt gas, which eventually coalesce into a single cusp, settling apparently into a stable steady state. In Fig. 8(a), the event of coalescence into a single cusp coincides with the occurrence of a peak in $S_T(t)$ before the final asymptotic behavior. Such a behavior is characteristic of the nonlinear

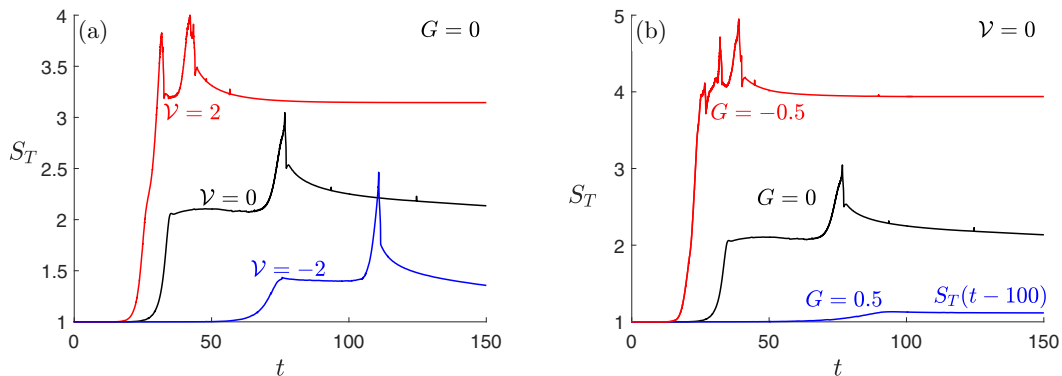


FIG. 8. Global propagation speed S_T vs time t for $G = 0$ with selected values of \mathcal{V} (left figure) and $\mathcal{V} = 0$ with selected values of G (right figure).

development of Darrieus-Landau instability in flames and is qualitatively explainable using the dynamics of pole solutions of the Michelson-Sivashinsky equation [53]; the final steady solution we observe corresponds to the one-pole solution [42]. As for the influence of \mathcal{V} , we first note that that an increase in \mathcal{V} has a destabilizing effect, in agreement with our linear stability analysis; this can be inferred from the instability onsets observed in Fig. 8(a). Furthermore, it is seen that the extent of the cusp structure (or the flame-wrinkling amplitude) increases with \mathcal{V} , a behavior which is typical of Saffman-Taylor instabilities known as viscous fingering [5]. This behavior leads to an increase in the flame surface area with \mathcal{V} , which explains the increase of the corresponding computed steady-state values of S_T . For a better appreciation of the full time evolution of the unstable flames, the reader is referred to the Supplemental Material [54].

We now briefly examine the effect of varying the gravity parameter G for a freely propagating flame, $\mathcal{V} = 0$. The computed results are reported for two values of G , equal to -0.5 (upward flame propagation) and 0.5 (downward flame propagation) in Figs. 9 and 8(b), and can be compared with the gravity free case with $\mathcal{V} = 0$ of Fig. 7. Shown in Fig. 9 is the temperature field $\theta(\mathbf{x}, t)$ at selected times and in Fig. 8(b) the global propagation speed $S_T(t)$. The figures illustrate the expected stabilizing influence of gravity for downward flame propagation, $G > 0$, and destabilizing influence for upward propagation. For both cases, the unstable flames settles ultimately again to single-cusp structures, albeit with markedly different wrinkling amplitudes. We also observe that the onsets of instability for the two cases are quite distinct, with the $G = -0.5$ case being characterized by an earlier instability onset with shorter wavelength wrinkling compared to the $G = 0.5$ case. This is consistent with the linear stability analysis, in particular with the predictions of Fig. 5(b). The reader is referred to [55] for an interesting recent study on the effect of gravity on upward-propagating flames.

It is insightful to examine the vorticity field, $\Omega = \nabla \times \mathbf{v}$, in the context of Darcy's law used herein. This is obtained by taking the curl of the equation $\mathbf{v} - (S_T - \mathcal{V})\mathbf{e}_y = -[\nabla p + (rG/m)\rho\mathbf{e}_y]/\mu$, hence

$$\Omega = \Omega\mathbf{e}_z = \frac{1}{\mu^2}\nabla\mu \times \nabla p - \frac{r}{m}\nabla\left(\frac{\rho}{\mu}\right) \times G\mathbf{e}_y. \quad (56)$$

Since ρ and μ vary spatially only within the interface in our model, where heat loss and preferential diffusion are ignored, the formula indicates that the vorticity is also confined within the interface. Moreover, in the absence of gravity, $G = 0$, vorticity is produced only by viscous baroclinicity associated with the misalignment of pressure and viscosity gradients, $\nabla\mu \times \nabla p \neq 0$. This implies that if viscosity is assumed constant and $G = 0$, but density is allowed to vary so as to induce a Darrieus-Landau instability, then vorticity will still remain identically zero. This is in marked

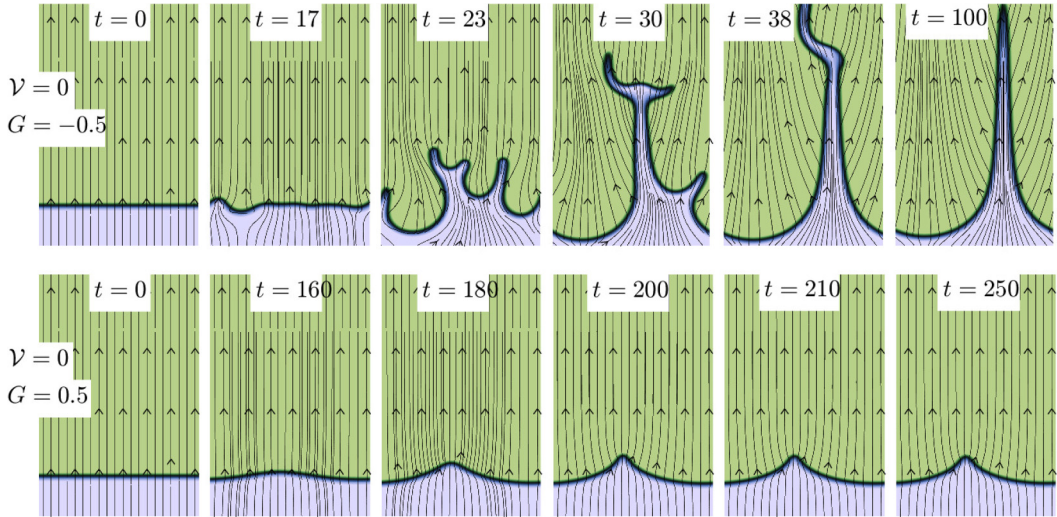


FIG. 9. Instantaneous temperature fields and streamlines, with $\mathcal{V} = 0$, seen from a frame moving with the flame speed $S_T(t)$. The upper row of figures correspond to $G = -0.5$ (upward flame propagation) and the lower to $G = 0.5$ (downward propagation). The horizontal nondimensional domain size shown in each figure is 100 and its vertical extent 180.

contrast with the usual Darrieus-Landau instability for flows governed by the Euler or Navier-Stokes equations, where vorticity is present within the flame and in the burnt gas, and vorticity-based arguments are sometimes proposed as a driving mechanism for the DL instability [42] instead of simpler arguments using mass conservation [[46], p. 354]. Finally, it is worth pointing out that Green's theorem implies

$$\int_0^{L_y} \int_0^{L_x} \Omega \, dx dy = 0, \quad (57)$$

upon using the periodicity boundary condition in the x direction. This explains the alternating regions of positive and negative vorticity within the flame seen in Fig. 10, pertaining to the case $\mathcal{V} = 2$ and $G = 0$ represented in the middle row of Fig. 7. As mentioned, no vorticity is expected when viscosity is assumed constant in this case, as confirmed by computations which are not shown.

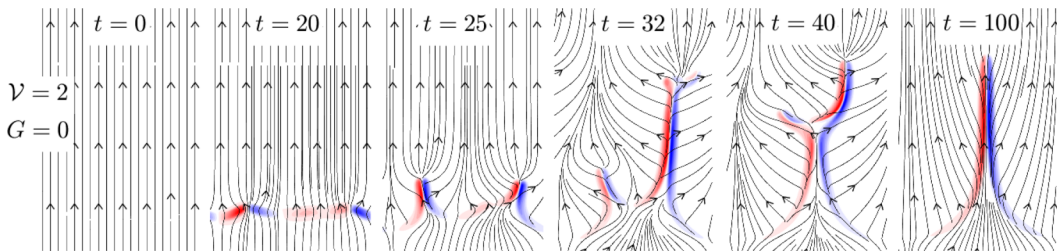


FIG. 10. Instantaneous vorticity fields and streamlines with $G = 0$ and $\mathcal{V} = 2$, seen from a frame moving with the flame speed $S_T(t)$. The horizontal nondimensional domain size shown in each figure is 100 and its vertical extent 180. The figure complements the temperature fields plotted in the middle row of Fig. 7.

V. CONCLUDING REMARKS

Propagating interfaces in Hele-Shaw channels or porous media are typically subject to three hydrodynamic instabilities, namely, the Darrieus-Landau, Saffman-Taylor, and Rayleigh-Taylor instabilities. These instabilities are long wave in nature, while at small wavelengths comparable to the interface diffusive thickness, stabilization often occurs due to the dependence of the interface propagation speed on its local curvature and the flow strain. Using a Darcy's flow model, both the long-wave instabilities and the small-wavelength stabilization are captured by a simple dispersion relation between the perturbation growth rate s and its wave number k of the form $s = (ak - bk^2)/(1 + ck)$ given in Eq. (21). When the effect of the flow strain on the interface propagation speed is ignored, the formula reduces to $s = ak - bk^2$, as given in Eq. (13). The parameter a characterizes the three hydrodynamic instabilities in a transparent way. A key conclusion from the analysis is that a hydrodynamically unstable interface can be fully stabilized by controlling the imposed flow.

In addition to these insightful results associated with Darcy's law, a more complex dispersion relation (30) is obtained within the so-called Euler-Darcy model, generalizing those obtained by Joulin and Sivashinsky [6] and Miroschnichenko *et al.* [9]. The corresponding analysis provides a conceptual bridge between the predictions based on Darcy's law ($\varphi \rightarrow \infty$) and those based on the Euler's equation ($\varphi \rightarrow 0$), through the use of a new parameter φ defined in (31). In Hele-Shaw channels, φ is proportional to the inverse of the channel width squared, and characterizes the effect of confinement. In the limit $\varphi \rightarrow 0$, the effect of confinement vanishes, and so does the Saffman-Taylor instability, as expected.

The theoretical results have been complemented by a numerical stability analysis and time-dependent simulations carried out in the specific case of propagating premixed flames in a Darcy flow. The numerical results are found to be in good agreement with the dispersion relation $s = (ak - bk^2)/(1 + ck)$, in particular, in determining the constant a which encapsulates the effects of the three hydrodynamic instabilities. Furthermore, the numerical simulations revealed the dynamics of unstable flames under the combined influence of DL and ST instabilities, including typical associated phenomena such as viscous fingering, which significantly affects the overall burning rates.

We highlight now important points pertinent to the growth rates of long-wave perturbations ($k \ll 1$) for flames propagating in narrow channels, which are decisive in determining the conditions for the hydrodynamic instabilities onset in the parameter space. Firstly, as we have seen, flames propagating in Hele-Shaw channels can be stabilized by an imposed flow directed from the burnt gas towards the unburnt gas ($V < 0$), and destabilized in the opposite case corresponding to $V > 0$. Secondly, we reemphasize the distinction between a freely propagating flame for which $V = 0$ and a flame opposed by an imposed flow with $V = S_L^0$. The distinction is important because the Saffman-Taylor instability which arises when $V(\mu_2 - \mu_1) \neq 0$ is absent for a freely propagating planar flame. However, viscosity can still affect the perturbation growth rate under these conditions. For instance, in the absence of gravity and for large wavelengths ($k \ll 1$), the growth rates for the two cases are given by

$$\frac{s}{S_L^0 k} = \frac{\mu_2(\rho_1 - \rho_2)}{\rho_2(\mu_2 + \mu_1)} \quad \text{when } V = 0 \quad \text{and} \quad \frac{s}{S_L^0 k} = \frac{\rho_1 \mu_2 - \rho_2 \mu_1}{\rho_2(\mu_2 + \mu_1)} \quad \text{when } V = S_L^0.$$

The second case ($V = S_L^0$), originally studied by Joulin and Sivashinsky [6], is often mistakenly assumed to apply for a freely propagating flame ($V = 0$) in the combustion literature.

As a final point, it is worth comparing the growth rate s_{Darcy} based on Darcy's law, which is applicable under strong confinement, with the growth rate s_{DL} corresponding to the classical Darrieus-Landau analysis, which is applicable for unconfined flames. For $k \ll 1$ and in the absence

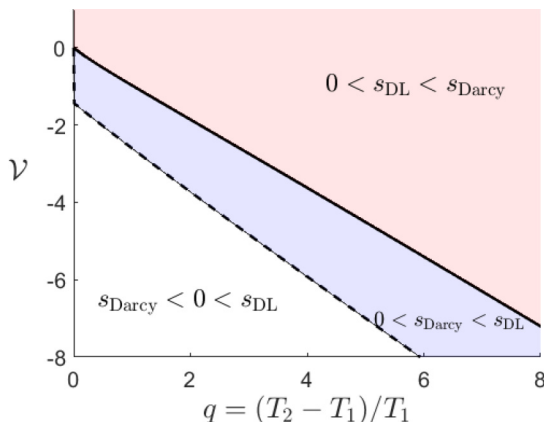


FIG. 11. Regions in the q - \mathcal{V} plane characterizing the effect of confinement on the flame hydrodynamic instabilities in the absence of gravity. Note that s_{DL} is always positive and independent of \mathcal{V} , while s_{Darcy} changes with \mathcal{V} and becomes negative in the unshaded region.

of gravity, the two growth rates are given by

$$\frac{s_{Darcy}}{S_L^0 k} = \frac{\rho_1 \mu_2 - \rho_2 \mu_1 \mathcal{V} + \rho_2 \mu_2 (\mathcal{V} - 1)}{\mu_1 + \mu_2} \quad \text{and} \quad \frac{s_{DL}}{S_L^0 k} = \frac{\rho_1}{\rho_1 + \rho_2} \left(\sqrt{1 + \frac{\rho_1^2 - \rho_2^2}{\rho_1 \rho_2}} - 1 \right),$$

where $\mathcal{V} = V/S_L^0$. The comparison of s_{Darcy} with s_{DL} produces Fig. 11, which has a simple yet important implication. Specifically, in the region below the solid curve, the effect of confinement (attributable to momentum loss due to friction) has a stabilizing influence compared to the unconfined case and a destabilizing influence in the region above the curve. Below the dashed curve, stable flames are encountered under sufficiently strong confinement.

ACKNOWLEDGMENTS

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APPENDIX: LINEAR STABILITY ANALYSIS BASED ON THE EULER-DARCY MODEL

The main aim of this Appendix is to provide a short derivation of the dispersion relation (29) for the convenience of the reader. To this end, to the basic solution $(\bar{\mathbf{v}}, \bar{p}, \bar{f})$ given by Eqs. (27) and (28) and $\bar{f} = 0$, we add small perturbations such that $(\mathbf{v}, p, f) = (\bar{\mathbf{v}}, \bar{p}, \bar{f}) + (\mathbf{v}', p', f')$. The primed quantities satisfy on both sides of the interface the linearized form of Eqs. (25), namely,

$$\nabla \cdot \mathbf{v}' = 0, \quad \rho \left(\frac{\partial \mathbf{v}'}{\partial t} + \bar{v} \frac{\partial \mathbf{v}'}{\partial y} \right) = -\nabla p' - \frac{\mu}{\kappa} \mathbf{v}', \quad (\text{A1})$$

from which it follows that $\nabla^2 p' = 0$. We note that (A1) is a set of three independent scalar equations for the components of $\mathbf{v}' = (u', v')$ and p' , where one of the equations may be replaced by $\nabla^2 p' = 0$, such that

$$p'_{xx} + p'_{yy} = 0, \quad u'_x + v'_y = 0, \quad \rho(v'_t + \bar{v}v'_y) = -p'_y - \frac{\mu}{\kappa} v'. \quad (\text{A2})$$

The equations (A2) are subject to the interfacial conditions (26) linearized at $y = 0$, namely,

$$\begin{aligned} [[v']] &= \left(1 - \frac{\rho_1}{\rho_2}\right) S_L^0 \mathcal{L} f'_{xx}, \quad [[p']] = \alpha f' + 2 \left(\frac{\rho_1}{\rho_2} - 1\right) \rho_1 S_L^{0^2} \mathcal{L} f'_{xx}, \\ [[u']] &= \left(1 - \frac{\rho_1}{\rho_2}\right) S_L^0 f'_x, \quad f'_t = v'|_{y=0^-} + S_L^0 \mathcal{L} f'_{xx}, \end{aligned} \quad (\text{A3})$$

where α is given in (9), and use has been made of the assumption (5) which implies that $S_L = S_L^0(1 - \mathcal{L} f'_{xx} + \dots)$. The linear stability problem is therefore given by Eq. (A2), to be solved in the domains $y > 0$ and $y < 0$, subject to the interfacial conditions (A3) at $y = 0$ and the boundary conditions $p' = u' = v' = 0$ as $y \rightarrow \pm\infty$.

Since the problem does not depend explicitly on x and t , we look for normal modes in the form

$$(f', p', u', v') = [\hat{f}, \hat{p}(y), \hat{u}(y), \hat{v}(y)] \exp(st + ikx),$$

where s is in general complex, k real, and $\hat{f} = C$ a constant. Then, the stability problem reduces to

$$\hat{p}_{yy} - k^2 \hat{p} = 0, \quad ik\hat{u} + \hat{v}_y = 0, \quad \rho(s\hat{v} + \bar{v}\hat{v}_y) = -\hat{p}_y - \frac{\mu}{\kappa} \hat{v}, \quad (\text{A4})$$

applicable in the domains $y > 0$ and $y < 0$, subject to the interfacial conditions

$$\begin{aligned} [[\hat{v}]] &= \left(\frac{\rho_1}{\rho_2} - 1\right) S_L^0 \mathcal{L} k^2 \hat{f}, \quad [[\hat{p}]] = \left[\alpha - 2 \left(\frac{\rho_1}{\rho_2} - 1\right) \rho_1 S_L^{0^2} \mathcal{L} k^2\right] \hat{f}, \\ [[\hat{u}]] &= i \left(1 - \frac{\rho_1}{\rho_2}\right) S_L^0 k \hat{f}, \quad (s + S_L^0 \mathcal{L} k^2) \hat{f} = \hat{v}(0^-) \end{aligned} \quad (\text{A5})$$

at $y = 0$, and the boundary conditions $\hat{p} = \hat{u} = \hat{v} = 0$ as $y \rightarrow \pm\infty$. The solution to $\hat{p}_{yy} - k^2 \hat{p} = 0$ vanishing as $y \rightarrow \pm\infty$ is given by

$$\hat{p} = \begin{cases} Ae^{-ky} & \text{for } y > 0, \\ Be^{ky} & \text{for } y < 0, \end{cases}$$

where A and B are constants and the wave number k is assumed positive. The solutions $\hat{u}(y)$ and $\hat{v}(y)$ can then be determined, and these take the form

$$\hat{u} = \frac{i}{k} \frac{d\hat{v}}{dy} \quad \text{and} \quad \hat{v} = \begin{cases} A\gamma_2 k e^{-ky} + D e^{-ky-y/\gamma_2 \rho_1 S_L^0} & \text{for } y > 0, \\ -B\gamma_1 k e^{ky} & \text{for } y < 0, \end{cases} \quad (\text{A6})$$

after enforcing the boundary conditions as $y \rightarrow \pm\infty$, and assuming $\text{Re}(s) > 0$, which is sufficient to the ultimate goal of capturing the presence of unstable modes. Here, for brevity, we have introduced $\gamma_1 = (\rho_1 s + \mu_1/\kappa_1 + \rho_1 S_L^0 k)^{-1}$ and $\gamma_2 = (\rho_2 s + \mu_2/\kappa_2 - \rho_1 S_L^0 k)^{-1}$. Using now the interfacial conditions (A5), a linear homogeneous system of four equations for the unknown constants A , B , C , and D is obtained, namely,

$$\begin{bmatrix} \gamma_2 k & \gamma_1 k & (1 - \rho_1/\rho_2) S_L^0 \mathcal{L} k^2 & 1 \\ 1 & -1 & -\alpha + 2(\rho_1/\rho_2 - 1) \rho_1 S_L^{0^2} \mathcal{L} k^2 & 0 \\ \gamma_2 k & -\gamma_1 k & (1 - \rho_1/\rho_2) S_L^0 k & 1 + 1/\rho_1 S_L^0 \gamma_2 k \\ 0 & \gamma_1 k & s + S_L^0 \mathcal{L} k^2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \mathbf{0}. \quad (\text{A7})$$

The solvability condition for this system, corresponding to its determinant being set to zero, leads to dispersion relation (29), whose derivation is the main aim of this Appendix.

To close the Appendix, we note that the derivation of the dispersion relation (37), based on the improved propagation speed model (18), can be obtained in a similar fashion. Without repeating the details, the same procedure can be carried out in this case, leading ultimately to the linear stability

system,

$$\begin{bmatrix} \gamma_2 k & \gamma_1 k [1 + (\rho_1/\rho_2 - 1) \mathcal{L}_s k] & (1 - \rho_1/\rho_2) S_L^0 \mathcal{L}_c k^2 & 1 \\ 1 & -1 - 2\gamma_1 (\rho_1/\rho_2 - 1) \rho_1 S_L^0 \mathcal{L}_s k^2 & -\alpha + 2(\rho_1/\rho_2 - 1) \rho_1 S_L^0 \mathcal{L}_c k^2 & 0 \\ \gamma_2 k & -\gamma_1 k & (1 - \rho_1/\rho_2) S_L^0 k & 1 + 1/\rho_1 S_L^0 \gamma_2 k \\ 0 & \gamma_1 k (1 - \mathcal{L}_s k) & s + S_L^0 \mathcal{L}_c k^2 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \mathbf{0}, \quad (\text{A8})$$

whose solvability condition coincides with dispersion relation (37).

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- [1] G. Darrieus, Propagation d'un front de flamme, *Tech. Mod.* **30** (1938).
- [2] L. D. Landau, On the theory of slow combustion, *Zh. Eksp. Teor. Fiz.* **14**, 240 (1944).
- [3] L. Rayleigh, Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, *Proc. London Math. Soc.* **s1-14**, 170 (1882).
- [4] G. I. Taylor, The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I, *Proc. R. Soc. Lond. A* **201**, 192 (1950).
- [5] P. G. Saffman and G. I. Taylor, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid, *Proc. R. Soc. Lond. A* **245**, 312 (1958).
- [6] G. Joulin and G. I. Sivashinsky, Influence of momentum and heat losses on the large-scale stability of quasi-2D premixed flames, *Combust. Sci. Technol.* **98**, 11 (1994).
- [7] S. H. Kang, H. G. Im, and S. W. Baek, A computational study of Saffman–Taylor instability in premixed flames, *Combust. Theory Model.* **7**, 343 (2003).
- [8] R. C. Aldredge, Saffman–Taylor influence on flame propagation in thermoacoustically excited flow, *Combust. Sci. Technol.* **177**, 53 (2004).
- [9] T. Miroshnichenko, V. Gubernov, and S. Minaev, Hydrodynamic instability of premixed flame propagating in narrow planar channel in the presence of gas flow, *Combust. Theory Model.* **24**, 362 (2020).
- [10] P. Gondret and M. Rabaud, Shear instability of two-fluid parallel flow in a Hele-Shaw cell, *Phys. Fluids* **9**, 3267 (1997).
- [11] C. Ruyer-Quil, Inertial corrections to the Darcy law in a Hele-Shaw cell, *C. R. Acad. Sci., Ser. IIB: Mech.* **329**, 337 (2001).
- [12] F. Plouraboué and E. J. Hinch, Kelvin–Helmholtz instability in a Hele-Shaw cell, *Phys. Fluids* **14**, 922 (2002).
- [13] D. Fernández-Galisteo, V. N. Kurdyumov, and P. D. Ronney, Analysis of premixed flame propagation between two closely-spaced parallel plates, *Combust. Flame* **190**, 133 (2018).
- [14] D. Martínez-Ruiz, F. Veiga-López, D. Fernández-Galisteo, V. N. Kurdyumov, and M. Sánchez-Sanz, The role of conductive heat losses on the formation of isolated flame cells in Hele-Shaw chambers, *Combust. Flame* **209**, 187 (2019).
- [15] P. Rajamanickam and J. Daou, Effect of a shear flow on the Darrieus–Landau instability in a Hele-Shaw channel, *Proc. Combust. Inst.* **40**, 105671 (2024).
- [16] D. Fernández-Galisteo and V. N. Kurdyumov, Impact of the gravity field on stability of premixed flames propagating between two closely spaced parallel plates, *Proc. Combust. Inst.* **37**, 1937 (2019).
- [17] J. Daou and M. Matalon, Flame propagation in Poiseuille flow under adiabatic conditions, *Combust. Flame* **124**, 337 (2001).
- [18] D. A. Vasquez, J. W. Wilder, and B. F. Edwards, Chemical wave propagation in Hele-Shaw cells and porous media, *J. Chem. Phys.* **104**, 9926 (1996).
- [19] A. De Wit, Fingering of chemical fronts in porous media, *Phys. Rev. Lett.* **87**, 054502 (2001).
- [20] R. Demuth and E. Meiburg, Chemical fronts in Hele-Shaw cells: Linear stability analysis based on the three-dimensional Stokes equations, *Phys. Fluids* **15**, 597 (2003).
- [21] J. D'Hernoncourt, A. Zebib, and A. De Wit, On the classification of buoyancy-driven chemo-hydrodynamic instabilities of chemical fronts, *Chaos* **17**, 013109 (2007).

- [22] E. Al Sarraf, C. Almarcha, J. Quinard, B. Radisson, B. Denet, and P. Garcia-Ybarra, Darrieus–Landau instability and Markstein numbers of premixed flames in a Hele-Shaw cell, *Proc. Combust. Inst.* **37**, 1783 (2019).
- [23] F. Veiga-López, D. Martínez-Ruiz, E. Fernández-Tarrazo, and M. Sánchez-Sanz, Experimental analysis of oscillatory premixed flames in a Hele-Shaw cell propagating towards a closed end, *Combust. Flame* **201**, 1 (2019).
- [24] F. Veiga-López, M. Kuznetsov, D. Martínez-Ruiz, E. Fernández-Tarrazo, J. Grune, and M. Sánchez-Sanz, Unexpected propagation of ultra-lean hydrogen flames in narrow gaps, *Phys. Rev. Lett.* **124**, 174501 (2020).
- [25] A. Dejoan, Z. Zhou, D. Fernández-Galisteo, P. D. Ronney, and V. N. Kurdyumov, Effect of confinement on the propagation patterns of lean hydrogen–air flames, *Proc. Combust. Inst.* **40**, 105431 (2024).
- [26] J. Chang, W. Zhu, and X. Kang, An experimental study on premixed methane-air flame propagation with intrinsic instabilities in a Hele-Shaw chamber at various inclined angles, *Combust. Sci. Technol.* **1** (2024).
- [27] G. H. Markstein, Experimental and theoretical studies of flame-front stability, in *Dynamics of Curved Fronts* (Elsevier, New York, 1988), pp. 413–423.
- [28] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, England, 2000).
- [29] B. Karlovitz, D. W. Denniston, Jr., D. H. Knapschafer, and F. E. Wells, Studies on turbulent flames: A. flame propagation across velocity gradients, B. Turbulence measurement in flames, *Symp. (Int.) Combust.* **4**, 613 (1953).
- [30] W. Eckhaus, Theory of flame-front stability, *J. Fluid Mech.* **10**, 80 (1961).
- [31] G. H. Markstein, Theory of flame propagation, in *AGARDograph* (Elsevier, New York, 1964), Vol. 75, pp. 5–14.
- [32] P. Clavin and F. A. Williams, Effects of molecular diffusion and of thermal expansion on the structure and dynamics of premixed flames in turbulent flows of large scale and low intensity, *J. Fluid Mech.* **116**, 251 (1982).
- [33] M. Matalon and B. J. Matkowsky, Flames as gasdynamic discontinuities, *J. Fluid Mech.* **124**, 239 (1982).
- [34] F. A. Williams, A review of some theoretical considerations of turbulent flame structure, in *AGARD Conference Proceedings, 1975* (1975).
- [35] J. D. Buckmaster and G. S. S. Ludford, *Lectures on Mathematical Combustion* (Society for Industrial and Applied Mathematics, Philadelphia, PA, 1983).
- [36] P. Clavin and G. Searby, *Combustion Waves and Fronts in Flows: Flames, Shocks, Detonations, Ablation Fronts and Explosion of Stars* (Cambridge University Press, Cambridge, England, 2016).
- [37] P. Clavin and J. C. Graña-Otero, Curved and stretched flames: The two Markstein numbers, *J. Fluid Mech.* **686**, 187 (2011).
- [38] P. Pelce and P. Clavin, Influence of hydrodynamics and diffusion upon the stability limits of laminar premixed flames, in *Dynamics of Curved Fronts* (Elsevier, New York, 1988), pp. 425–443.
- [39] P. Clavin and G. Joulin, Premixed flames in large scale and high intensity turbulent flow, *J. Phys. Lett. (Paris)* **44**, 1 (1983).
- [40] D. Keller and N. Peters, Transient pressure effects in the evolution equation for premixed flame fronts, *Theor. Comput. Fluid Dyn.* **6**, 141 (1994).
- [41] P. Rajamanickam and J. Daou, Hydrodynamic theory of premixed flames under Darcy’s law, *Phys. Fluids* **36**, 123620 (2024).
- [42] M. Matalon, The Darrieus–Landau instability of premixed flames, *Fluid Dyn. Res.* **50**, 051412 (2018).
- [43] P. Clavin and P. Garcia, The influence of the temperature dependence of diffusivities on the dynamics, *J. Mech. Theor. Appl.* **2**, 245 (1983).
- [44] In particular, for premixed flames in wide channels we may replace (5), motivated by the study in [17], by the approximate formula $S_L/S_L^0 = (1 + \mathcal{V})(1 + \mathcal{L}\nabla \cdot \mathbf{n})$ for $\mathcal{V} > 0$ and $S_L/S_L^0 = (1 - \mathcal{V}/2)(1 + \mathcal{L}\nabla \cdot \mathbf{n})$ for $\mathcal{V} < 0$. For small channels, relevant to our study, as shown in [15,56,57] the appropriate formula is given by $S_L/S_L^0 = (1 + \mathcal{L}\nabla \cdot \mathbf{n})(1 + \gamma\epsilon^2 m^2 \mathcal{V}^2/r^2)/\sqrt{1 + \gamma\epsilon^2 m^2 \mathcal{V}^2 \text{Le}^2/r^2}$, where Le is the Lewis number, and γ is the Taylor dispersion coefficient, which takes the value $\gamma = 1/210$ for a Poiseuille flow. The corrections introduced here are equivalent to simply redefining S_L^0 in (5) and elsewhere by multiplying by

- a constant. With the redefined S_L^0 , the dispersion relation (30) is still applicable, without the need for a separate stability analysis.
- [45] F. Creta and M. Matalon, Strain rate effects on the nonlinear development of hydrodynamically unstable flames, *Proc. Combust. Inst.* **33**, 1087 (2011).
 - [46] F. A. Williams, *Combustion Theory* (CRC Press, Boca Raton, FL, 1985).
 - [47] J. Daou and P. Rajamanickam, Diffusive-thermal instabilities of a planar premixed flame aligned with a shear flow, *Combust. Theory Model.* **28**, 20 (2024).
 - [48] P. Rajamanickam, A. Kelly, and J. Daou, Stability of diffusion flames under shear flow: Taylor dispersion and the formation of flame streets, *Combust. Flame* **257**, 113003 (2023).
 - [49] A. Kelly, P. Rajamanickam, J. Daou, and J. Landel, Influence of heat loss on the stability of diffusion flames in a narrow-channel shear flow, *Combust. Sci. Technol.* **1** (2023).
 - [50] A. Kelly, P. Rajamanickam, J. Daou, and J. R. Landel, Three-dimensional diffusive-thermal instability of flames propagating in a plane Poiseuille flow, *Proc. Combust. Inst.* **40**, 105258 (2024).
 - [51] J. Daou, A. Kelly, and J. Landel, Flame stability under flow-induced anisotropic diffusion and heat loss, *Combust. Flame* **248**, 112588 (2023).
 - [52] J. Daou and P. Rajamanickam, Premixed flame stability under shear-enhanced diffusion: Effect of the flow direction, *Phys. Rev. Fluids* **8**, 123202 (2023).
 - [53] O. Thual, U. Frisch, and M. Henon, Application of pole decomposition to an equation governing the dynamics of wrinkled flame fronts, in *Dynamics of Curved Fronts* (Elsevier, New York, 1988), pp. 489–498.
 - [54] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevFluids.10.013201> for temperature fields and streamlines for three values of \mathcal{V} and $G = 0$, as seen from a frame moving with a speed $S_T(t)$.
 - [55] M. Battikh, E. Al Sarraf, B. Radisson, C. Almarcha, and B. Denet, Nonlinear dynamics of upward propagating flames, *Phys. Rev. E* **107**, 065110 (2023).
 - [56] J. Daou, P. Pearce, and F. Al-Malki, Taylor dispersion in premixed combustion: Questions from turbulent combustion answered for laminar flames, *Phys. Rev. Fluids* **3**, 023201 (2018).
 - [57] P. Rajamanickam and J. Daou, A thick reaction zone model for premixed flames in two-dimensional channels, *Combust. Theory Model.* **27**, 487 (2023).