Combustion Theory and Modelling
Vol. 11, No. 5, October 2007, 697–714

Flame propagation in a small-scale parallel flow

J. DAOU* and P. SPARKS

University of Manchester, School of Mathematics, Manchester M60 1QD, UK

Received 27 July 2006; in final form 12 December 2006

We consider the propagation of laminar premixed flames in the presence of a parallel flow whose scale is smaller than the laminar flame thickness. The study addresses fundamental aspects with relevance to flame propagation in narrow channels, to the emerging micro-combustion technology, and to the understanding of the effect of small scales in a (turbulent) flow on the flame structure. In part, the study extends the results of a previous analytical study carried out in the *thick flame asymptotic limit* which has in particular addressed the validity of Damköhler's second hypothesis in the context of laminar *steady* parallel flows. Several new contributions are made here.

Analytical contributions include the derivation of an explicit formula for the effective speed of a premixed flame U_T in the presence of an *oscillatory* parallel flow whose scale ℓ (measured with the laminar flame thickness δ_L) is small and amplitude A (measured with the laminar flame speed U_L) is $\mathcal{O}(1)$. The formula shows a quadratic dependence on both the amplitude and the scale of the flow. The validity of the formula is established analytically in two distinguished limits corresponding to $\mathcal{O}(1)$ frequencies of oscillations (measured with the natural frequency of the flame U_L/δ_L), and to higher frequencies of $\mathcal{O}(A/\ell)$ (the natural frequency of the flow). The analytical study yields partial support of Damköhler's second hypothesis in that it shows that the flame behaves as a planar flame (to leading order) with an increased propagation speed which depends on both the scale and amplitude of the velocity fluctuation. However our formula for U_T contradicts the formula given by Damköhler in his original paper where U_T has a square root dependence on the scale and amplitude.

Numerical contributions include a significant set of two-dimensional calculations which determine the range of validity of the asymptotic findings. In particular, these account for volumetric heat loss and differential diffusion effects. Good agreement between the numerics and asymptotics is found in all cases, both for steady and oscillatory flows, at least in the expected range of validity of the asymptotics. The effect of the frequency of oscillation is also discussed. Additional related aspects such as the difference in the response of thin and thick flames to the combined effect of heat loss and fluid flow are also addressed. It is found for example that the sensitivity of thick flames to volumetric heat loss is negligibly affected by the parallel flow intensity, in marked contrast to the sensitivity of thin flames. Interestingly, and somewhat surprisingly, thin flames are found to be more resistant to heat loss when a flow is present, even for unit Lewis number; this ceases to be the case, however, when the Lewis number is large enough.

Keywords: Damköhler's hypothesis; Laminar flame speed Premixed flames; Thick flame asymptotics; Turbulent combustion

1. Introduction

The propagation of premixed flames in the presence of a flow whose scale is comparable or smaller than the laminar flame thickness is a fundamental problem relevant to many areas.

*Corresponding author. E-mail: joel.daou@manchester.ac.uk

Combustion Theory and Modelling
ISSN: 1364-7830 (print), 1741-3559 (online) © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/13647830601173174

Such areas include flame propagation in channels and tubes [1], combustion in potentially promising micro-devices currently under development [2], and turbulent combustion where the small scales of the flow-field can have significant effects on the flame, see, e.g. [3, 4].

The present investigation is carried out in the context of *laminar parallel flows*. In this important and rather extensively studied context, we have been involved in various recent contributions [5–8] where several aspects of flame propagation have been addressed. These include the derivation of analytical formulae for the effective propagation speed in terms of the scale and amplitude of the flow, the effect of heat losses and non-unit Lewis numbers, flashback phenomena and the dependence of the quenching distance on the flow. However, our main goal in this paper is to extend and complement the analytical results of [5] entitled *the thick flame asymptotic limit and Damköhler's hypothesis*. Thus, although the results can be useful in other fields of study, such as micro-combustion in MEMS devices [2], we shall only invoke Damköhler's two hypotheses of turbulent combustion [9] in order to provide motivation for our work.

The first hypothesis postulates that the large scales in the flow increase the effective flame speed U_T by wrinkling the flame, thus increasing its area, without a significant change in its structure. This is relevant in the so-called *flamelet regime* [10] and has received significant attention and confirmation in the literature. An example of analytical contributions relevant to this regime is the often cited Clavin–Williams formula derived for flames propagating in large-scale, low-intensity turbulence [11]. For a detailed recent account on turbulent combustion, see [12] and references therein.

The second hypothesis postulates that the small scales in the flow do not cause any significant flame wrinkling but do change the flame structure by enhancing the diffusive processes; the change in the effective flame speed and thickness relative to those of a laminar planar flame could be described using effective diffusion coefficients to account for the flow. As discussed in [5] and pointed out in [1] however, this second hypothesis seems to have received little attention or support. This lack of support is striking even in the context of simple prescribed flows, in particular as far as analytical work is concerned. An analytical contribution aimed at testing this hypothesis was carried in [5] in the framework of prescribed *steady parallel flows*. Its main result is the formula

$$\frac{U_T}{U_0} \sim 1 + \frac{\ell^2}{2} \int_0^1 \left[\int_0^{\eta} u(\eta_1)\,\mathrm{d}\eta_1 \right]^2 \mathrm{d}\eta\,, \tag{1}$$

valid for small values of the scale ℓ of the flow u. Here ℓ and u are measured against the thickness δ_L and speed U_L of the *adiabatic* planar flame. U_T and U_0 represent the effective flame speed and the *planar* laminar flame speed, also measured with U_L. In the absence of heat losses $U_0 = 1$, but more generally U_0 is the larger root of $U_0^2 \ln U_0 = -\kappa$, where κ represent the intensity of heat loss (see equation (3) below). In equation (1) the argument of u must lie in [0, 1] and its spatial mean must be equal to zero (this is always possible by an appropriate choice of the origin and scale on the transverse axis and of the reference frame). The formula describes the increase in the effective flame speed U_T which is seen to depend quadratically on both the scale and intensity of the flow while being independent of the Lewis number.

It is useful to extend this result to more realistic situations, e.g. by accounting for flow unsteadiness and for more complex flows. In the present paper, we have two main objectives.

- First, to extend the analytical formula (1) to *time-periodic* parallel flows; a formula will be thus derived in two distinguished limits corresponding to $\mathcal{O}(1)$ frequencies of oscillations (measured with the natural frequency of the flame U_L/δ_L), and to higher frequencies of oscillations of $\mathcal{O}(A/\ell)$ (the natural frequency of the flow). The latter choice of the frequency is intended to mimic the fact that the characteristic frequency of eddies in turbulent combustion is commonly identified with the inverse of their turnover time which is proportional to their scale divided by their velocity.

- Second, to complement the analytical results with two-dimensional computations. These are aimed at: (a) assessing the range of validity of the analytical findings, both for steady and unsteady flows, and accounting for heat loss and preferential diffusion effects; (b) examining the influence of the frequency of oscillation; (c) pointing out certain differences between thin and thick flames.

The paper is structured as follows. We begin by presenting the thermodiffusive model used in the study. An asymptotic analysis is then carried out for time-periodic flows in two distinguished limits. This is followed by a presentation of the numerical calculations which are compared against the analytical predictions, and by a discussion of the main findings.

2. Formulation

We consider a two-dimensional flame propagating in the x-direction against a parallel flow $u(y, t)$ as represented in figure 1. A simple non-dimensional thermodiffusive model characterizing the problem as in [5] consists of the equations

$$Y_t + u(t, y)Y_x = \mathrm{Le}^{-1}(Y_{xx} + Y_{yy}) - \omega, \tag{2}$$

$$T_t + u(t, y)T_x = T_{xx} + T_{yy} + \omega - \frac{\kappa}{\beta}T, \tag{3}$$

and the boundary conditions

$$Y = 1, \quad T = 0 \quad \text{as} \quad x \to -\infty, \tag{4}$$

$$Y_x = T_x = 0 \quad \text{as} \quad x \to +\infty, \tag{5}$$

$$Y_y = T_y = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = \ell, \tag{6}$$

along with suitable initial conditions. Here T and Y are the (scaled) temperature and mass fraction of the fuel, which is assumed to limit the reaction, and Le is the Lewis number. The reaction rate, ω, is taken to follow an Arrhenius form

$$\omega = \frac{\beta^2}{2\mathrm{Le}} Y \exp\left\{\frac{\beta(T-1)}{1+\alpha(T-1)}\right\},$$

where β is the Zeldovich number and α an exothermicity parameter. Volumetric heat losses are modelled by a linear sink term of strength κ/β. The non-dimensionalization corresponds to the units for speed, length, and time being based on U_L and δ_L, the propagation speed and the thickness of the adiabatic unstretched planar flame respectively (more precisely to the asymptotic values of these as $\beta \to \infty$). The boundary conditions (4) and (5) correspond to a

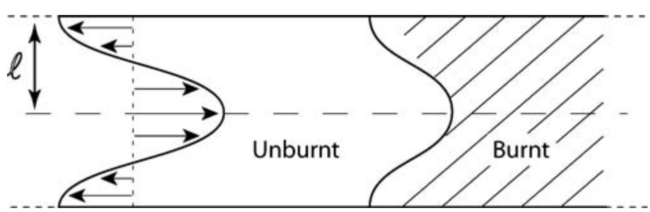

Figure 1. Schematic of a flame propagating against a parallel flow.

frozen mixture with prescribed temperature and composition upstream and uniform properties far downstream. The boundary conditions (6) assume that all profiles have zero slope at $y = 0$ and $y = \ell$, where ℓ represents a characteristic transverse scale of the flow (measured with δ_L); for example, for a flow which is periodic in the y-direction, ℓ may be taken equal to the spatial period, and the origin of the y axis is to be chosen so that the flame is vertical at $y = 0$ and $y = \ell$.

An important quantity to be determined by the solution is the total burning rate per unit transverse area

$$\Omega \equiv \frac{1}{\ell} \int_{-\infty}^{\infty} \int_{0}^{\ell} \omega \, \mathrm{d}y \, \mathrm{d}x, \tag{7}$$

which can be used to define an *effective* non-dimensional propagation speed U_T as conventionally done in turbulent combustion (possible divergence of the integral in (7) associated with the cold-boundary difficulty is easily and routinely avoided, e.g. by introducing a cut-off temperature below which w is set to zero). For time independent flows, $u = u(y)$, $U_T = \Omega$. For time-periodic flows sustaining an oscillatory flame propagation, $U_T = \bar{\Omega}$; here and below bars indicate time averages.

3. Asymptotic analysis for time-periodic parallel flows

In this section our aim is to extend the analytical formula (1) to oscillatory situations corresponding to time-periodic parallel flows $u(y, t)$. The analysis will be carried out using the asymptotic limit of small flow scale $\ell \to 0$ and large β (with $\beta^{-1} \ll \ell$).

We shall assume that $u(y, t)$ has a zero spatial mean (over the transverse length ℓ) in the frame of reference under consideration; thus we take u to be of the form $u = \bar{u}(y) + v(t, y)$, where $\bar{u}(y)$ has a zero spatial mean and $v(t, y)$ is a fluctuation with zero spatial and temporal means. This is the case, e.g. for the time and space harmonic flows to be considered in the numerics. This form also covers the time-independent parallel flow situations (for which v can be set equal to zero and the zero-spatial mean of u is a consequence of a suitable choice of the frame of reference).

For simplicity, we shall focus on the equidiffusional, adiabatic case (Le $= 1, \kappa = 0$) which can be described by a single dependent variable, since $Y + T = 1$ then follows from the governing equations and boundary conditions (assuming that this is compatible with the initial conditions). In the limit $\beta \to \infty$, the reaction is confined to a thin sheet given by $x_{\text{flame}} = -U_T t + f(t, y)$, say. Using the transverse scale $\eta = y/\ell$ and the longitudinal coordinate $\zeta = x - x_{\text{flame}} = x + U_T t - f(t, y)$, and writing $f(t, y) = \ell^2 F(t, \eta)$, the problem takes the form $T \equiv 1$ in the burnt gas ($\zeta > 0$) and

$$T_t + [U_T + u(t, \bar{\eta}) + F_{\eta\eta} - \ell^2 F_t] \, T_\zeta = \left(1 + \ell^2 F_\eta^2\right) T_{\zeta\zeta} + \ell^{-2} T_{\eta\eta} - 2F_\eta T_{\zeta\eta} \tag{8}$$

in the unburnt gas ($\zeta < 0$). Our task is to solve equation (8) subject to the upstream condition

$$T = 0 \quad \text{as} \quad \zeta \to -\infty, \tag{9}$$

the standard jump conditions (see, e.g. [13, 14])

$$T = 1, \quad T_\zeta = \left(1 + \ell^2 F_\eta^2\right)^{-1/2} \quad \text{at} \quad \zeta = 0^-, \tag{10}$$

and the zero-slope conditions

$$T_\eta = F_\eta = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1. \tag{11}$$

In the problem thus formulated, now free from β, we take the limit $\ell \to 0$ with the flow amplitude assumed to be an arbitrary $\mathcal{O}(1)$ quantity as in [5]. Furthermore, we assume that the temporal period is $\mathcal{O}(1)$, that is flow oscillations occur over (dimensional) time scales of the order of the flame time δ_L/U_L. However, the results that we shall derive can be extended to higher frequencies, as discussed in the following section.

We begin by writing expansions in the form

$$T = T_0 + \ell T_1 + \ell^2 T_2 + \cdots, \quad U_T = U_0 + \ell U_1 + \ell^2 U_2 + \cdots, \quad F = F_0 + \ell F_1 + \cdots,$$

which we substitute into equations (8)–(11).

To $\mathcal{O}(\ell^{-2})$ we find $T_{0\eta\eta} = 0$ which, when used with (11), implies that T_0 must be independent of η, $T_0 = T_0(t, \zeta)$.

To $\mathcal{O}(\ell^{-1})$ we have similarly $T_{1\eta\eta} = 0$ and $T_1 = T_1(t, \zeta)$.

To $\mathcal{O}(1)$ we obtain

$$T_{0t} + [U_0 + u(t, \eta) + F_{0\eta\eta}(t, \eta)]T_{0\zeta} - T_{0\zeta\zeta} = T_{2\eta\eta}\,, \tag{12}$$

which we integrate with respect to η over the range (0, 1), using (11) and the fact that u has a zero spatial mean. This yields the equation

$$T_{0t} + U_0 T_{0\zeta} - T_{0\zeta\zeta} = 0\,,$$

subject to $T_0 \to 0$ as $\zeta \to -\infty$, and $T_0 = T_{0\zeta} = 1$ at $\zeta = 0$, which is clearly solved by taking

$$U_0 = 1\,, \quad T_0 = \exp(\zeta) \quad (\zeta \le 0)\,, \tag{13}$$

at least outside an initial transient regime. Thus, in first approximation, the solution is independent of the flow and corresponds to the laminar planar flame. We now integrate equation (12) twice with respect to η from 0 to η using (13). This yields

$$T_2 = [S(t, \eta) + F_0(t, \eta)] \exp\zeta + \tilde{T}_2(t, \zeta)\,,$$

where $\tilde{T}_2$ is an arbitrary function of integration and S is a function which satisfies $S_{\eta\eta} = u$. The continuity of temperature at the reaction sheet, $T_2 = 0$ at $\zeta = 0$, then implies that $F_0 = -S$, i.e.

$$F_0(t, \eta) = -\int_0^{\eta} \mathrm{d}\eta_2 \int_0^{\eta_2} u(t, \eta_1)\,\mathrm{d}\eta_1\,, \tag{14}$$

within an additive function of time which can be set to zero by modifying the function $\tilde{T}_2(t, \zeta)$, if needed. Thus T_2 must be independent of η, namely $T_2 = T_2(t, \zeta)$ with $T_2 = 0$ at $\zeta = 0$. Incidentally, we note that (14) describes the leading order effect of the flow field on the flame shape.

To $\mathcal{O}(\ell)$ we obtain

$$T_{1t} + T_{1\zeta} - T_{1\zeta\zeta} = -(U_1 + F_{1\eta\eta})e^{\zeta} + T_{3\eta\eta}\,,$$

which integrated with respect to η over the range (0, 1) yields $T_{1t} + T_{1\zeta} - T_{1\zeta\zeta} = -U_1 e^{\zeta}$, whose solution, subject to $T_1 = 0$ as $\zeta \to -\infty$ and $T_1 = T_{1\zeta} = 0$ at $\zeta = 0^-$, is clearly

$$U_1 = 0\,, \quad T_1 = 0\,, \tag{15}$$

at least after an initial transient time.

To $\mathcal{O}(\ell^2)$ we find

$$T_{2t} + T_{2\zeta} - T_{2\zeta\zeta} = -\big(U_2 + F_{2\eta\eta} - F_{0t} - F_{0\eta}^2\big)e^{\zeta} + T_{4\eta\eta}\,,$$

which we again integrate from $\eta = 0$ to 1 to get

$$T_{2t} + T_{2\zeta} - T_{2\zeta\zeta} = \left[\int_0^1 F_{0t}\,\mathrm{d}\eta + \int_0^1 F_{0\eta}^2\,\mathrm{d}\eta - U_2\right] e^{\zeta}.$$

Upon time averaging, this yields the ordinary differential equation

$$\bar{T}_{2\zeta} - \bar{T}_{2\zeta\zeta} = \left[\int_0^1 \overline{F_{0\eta}^2}\,\mathrm{d}\eta - U_2\right] e^{\zeta} \tag{16}$$

whose solution subject to $\bar{T}_2(-\infty) = 0$ and $\bar{T}_2(0) = 0$ is

$$\bar{T}_2 = C\zeta e^{\zeta} \quad \text{where} \quad C \equiv U_2 - \int_0^1 \overline{F_{0\eta}^2}\,\mathrm{d}\eta. \tag{17}$$

Now, the jump conditions imply that $\bar{T}_{2\zeta}(0^-) = -\overline{F_{0\eta}^2}/2$, which should determine the constant C; clearly, this is impossible since the right-hand side of this equation is a function of η while the left-hand side is not. This suggests the need to reconsider the problem in an inner layer near the flame. To this end, we denote the straightforward expansion above by

$$T^{\text{outer}} \sim e^{\zeta} + \ell^2 T_2(t,\zeta) + \ell^3 T_3(t,\zeta,\eta) + \cdots, \tag{18}$$

where T_2 satisfies (17), and write an inner expansion in the form

$$T^{\text{inner}} \sim 1 + \ell\xi + \ell^2\frac{\xi^2}{2} + \ell^3\Theta_3(t,\xi,\eta) + \cdots \quad (\zeta = \ell\xi). \tag{19}$$

Note that the first three terms of (19) have been given explicitly, using a Taylor expansion of the uniformly valid leading order solution e^{ζ} as $\zeta \to 0$ and the condition $T_2 = 0$ at $\zeta = 0$. Substitution of (19) into equations (8) to (11) shows that Θ_3 is governed by

$$\Theta_{3\xi\xi} + \Theta_{3\eta\eta} = \xi + F_{1\eta\eta}(t,\eta), \tag{20}$$

so that

$$\Theta_3 = \xi^3/6 + F_1(t,\eta) + \theta(t,\xi,\eta), \tag{21}$$

where θ satisfy

$$\theta_{\xi\xi} + \theta_{\eta\eta} = 0, \tag{22}$$

and

$$\theta_\eta(t,\xi,0) = 0, \quad \theta_\eta(t,\xi,1) = 0, \quad \theta_\xi(t,0,\eta) = -\frac{F_{0\eta}^2}{2}. \tag{23}$$

It follows on applying the divergence theorem to the integral of equation (22) over the rectangular domain $[\xi, 0] \times [0, 1]$ that

$$\int_0^1 \theta_\xi(t,\xi,\eta)\,\mathrm{d}\eta = -\frac{1}{2}\int_0^1 F_{0\eta}^2(t,\eta)\,\mathrm{d}\eta, \tag{24}$$

which must hold, in particular, as $\xi \to -\infty$. Now the matching of the outer and inner expansions (18) and (19) to $\mathcal{O}(\ell^3)$ requires

$$\Theta_3(t,\xi,\eta) \sim \frac{\xi^3}{6} + T_{2\zeta}(t,0^-)\xi + T_3(t,0,\eta) \quad \text{as } \xi \to -\infty, \tag{25}$$

which, together with (21) and (24), implies that

$$T_{2\zeta} = -\frac{1}{2}\int_0^1 F_{0\eta}^2\,\mathrm{d}\eta \quad \text{at } \zeta = 0^-,$$

and upon time averaging

$$\bar{T}_{2\zeta}(0^-) = -\frac{1}{2}\int_0^1 \overline{F_{0\eta}^2}\,\mathrm{d}\eta.$$

This relation, together with (17), yields

$$U_2 = \frac{1}{2}\int_0^1 \overline{F_{0\eta}^2}\,\mathrm{d}\eta. \tag{26}$$

To summarize, using (13), (15), and (26), we have

$$\frac{U_T}{U_0} \sim 1 + \frac{\ell^2}{2}\int_0^1 \overline{\left[\int_0^\eta u(t,\eta_1)\,\mathrm{d}\eta_1\right]^2}\,\mathrm{d}\eta, \tag{27}$$

with $U_0 = 1$ in the present adiabatic equidiffusional case under consideration. The result can be extended to account for non-zero heat losses and non-unit Lewis numbers, provided U_0 is taken again as the larger root of $U_0^2 \ln U_0 = -\kappa$. The proof of this generalization, which is not given here (but which will be tested numerically below), follows a similar generalization carried out in the steady flow cases based on a near-equidiffusion-flame approximation ($\mathrm{Le} = 1 + le/\beta$ with $le = \mathcal{O}(1)$ as $\beta \to \infty$) detailed in [5]. At this point, we remind the reader that the derivation of this section has been based on the distinguished limit $\ell \to 0$, $A = \mathcal{O}(1)$, and $\tau = \mathcal{O}(1)$, where ℓ, A, and τ represent the scale, the amplitude, and the period of the oscillatory flow, respectively.

In the next section, we shall show that formula (27) can be derived for higher frequencies of the oscillating flow.

4. Asypmtotics with higher frequencies of the oscillating flow

In order to extend the derivation to oscillations with frequencies higher than the ones considered in the previous section, we let the period of the oscillatory flow τ tend to zero as $\ell \to 0$. More precisely, we consider the distinguished limit $\ell \to 0$, $A = \mathcal{O}(1)$, and $\tau = \mathcal{O}(\ell)$. As argued in the introduction, the choice of the scaling for τ may be seen as intended to mimic the fact that the natural time scale for eddies in turbulent combustion is their turnover time which is proportional to their size divided by their speed, i.e. $\tau \sim \ell/A$ in our non-dimensional notations. Of course, other distinguished limits could be examined by the reader, although some of these may turn out to be analytically untractable.

In the sequel, we show that formula (27) can be derived in the limit $\ell \to 0$, $A = \mathcal{O}(1)$, and $\tau = \mathcal{O}(\ell)$. Only a quick description of the main steps, especially those differing from those of last section, needs to be given.

We begin by rescaling time in equation (8), using $t' = t/\ell$, in order to examine periods of oscillations of $\mathcal{O}(\ell)$. Then dropping primes we obtain

$$\ell^{-1}T_t + [U_T + u(t,\eta) + F_{\eta\eta} - \ell F_t]\,T_\zeta = \left(1 + \ell^2 F_\eta^2\right)T_{\zeta\zeta} + \ell^{-2}T_{\eta\eta} - 2F_\eta T_{\zeta\eta}\,, \tag{28}$$

subject to the same auxiliary conditions as in the previous section.

To $\mathcal{O}(\ell^{-2})$ we find $T_0 = T_0(t,\zeta)$ exactly as before.

To $\mathcal{O}(\ell^{-1})$ we have now however $T_{0t} = T_{1\eta\eta}$ which may be integrated with respect to η to yield $T_{1\eta} = T_{0t}(t,\zeta)\eta + \tilde{T}_1(t,\zeta)$; the integration function $\tilde{T}_1$ is in fact identically zero and so is T_{0t}, on using the zero-slope conditions (11); therefore $T_{1\eta}$ is also identically zero. We conclude that $T_0 = T_0(\zeta)$ and $T_1 = T_1(t,\zeta)$.

To $\mathcal{O}(1)$ we obtain

$$T_{1t} + [U_0 + u(t,\eta) + F_{0\eta\eta}(t,\eta)]T_{0\zeta} - T_{0\zeta\zeta} = T_{2\eta\eta}\,, \tag{29}$$

instead of (12), which we integrate with respect to η over the range (0, 1) to obtain

$$T_{0\zeta\zeta} - U_0 T_{0\zeta} = T_{1t}\,.$$

Now since the left-hand side of this equation is a function independent of t, say $C = C(\zeta)$, we conclude that T_1 will grow linearly in time at each location ζ and thus becomes unbounded unless C is set to zero, whence $T_1 = T_1(\zeta)$ and

$$U_0 T_{0\zeta} - T_{0\zeta\zeta} = 0\,.$$

The solution of this equation, which is subject to $T_0 \to 0$ as $\zeta \to -\infty$, and $T_0 = T_{0\zeta} = 1$ at $\zeta = 0$, corresponds to the planar flame solution given by (13). At this stage it becomes clear that equation (29) is in fact identical to (12), and as result equation (14) is still valid, and $T_2 = T_2(t,\zeta)$ with $T_2 = 0$ at $\zeta = 0$, as before.

To $\mathcal{O}(\ell)$ we obtain

$$T_{2t} + T_{1\zeta} - T_{1\zeta\zeta} = -(U_1 + F_{1\eta\eta} - F_{0t})e^{\zeta} + T_{3\eta\eta}\,,$$

which integrated over η from zero to one and then time-averaged yields $T_{1\zeta} - T_{1\zeta\zeta} = -U_1 e^{\zeta}$; the latter equation, subject to the boundary conditions $T_1 = 0$ as $\zeta \to -\infty$ and $T_1 = T_{1\zeta} = 0$ at $\zeta = 0^-$, is clearly solved by $U_1 = 0$ and $T_1 = 0$.

To $\mathcal{O}(\ell^2)$ we find

$$T_{3t} + T_{2\zeta} - T_{2\zeta\zeta} = -\big(U_2 + F_{2\eta\eta} - F_{1t} - F_{0\eta}^2\big)e^{\zeta} + T_{4\eta\eta}\,,$$

which when integrated from $\eta = 0$ to 1 then time-averaged leads back to equation (16) whose solution is given by (17). The inner analysis is almost identical to that given above except that an additional term appears in (20) which now reads

$$\Theta_{3\xi\xi} + \Theta_{3\eta\eta} = \xi + F_{1\eta\eta}(t,\eta) - F_{0t}(t,\eta)\,, \tag{30}$$

and thus

$$\Theta_3 = \xi^3/6 + F_1(t,\eta) - G(t,\eta) + \theta(t,\xi,\eta)\,, \tag{31}$$

where $G(t,\eta)$ is a function obtained by integrating $F_{0t}(t,\eta)$ twice with respect to η, i.e. $G\eta\eta = F_{0t}$, and θ satisfies Laplace's equation (22) and the boundary conditions (23). The rest of the analysis leading to (26) proceeds exactly as before.

5. Numerical calculations and comparison with the asymptotic results

In this section, we present an extensive set of numerical calculations, mainly in order to assess the validity of the asymptotic findings. Several additional aspects such as the influence of the frequency of the oscillatory flow are also discussed.

Equations (2)–(6) are solved numerically using a finite volume discretization combined with an algebraic multigrid solver. The transverse dimension of the computational domain corresponds to the transverse scale of the flow, and its longitudinal extent is taken to be several hundred times the planar flame thickness. A non-uniform grid with typically 80,000 points is used. The grid is translated during the iterations so that the flame remains around the origin. The independence of the numerical results of the spatio-temporal grids used (within one percent accuracy) has been satisfactorily tested. In the computations we take $\beta = 8$, $\alpha = 0.85$, and mainly vary the flow scale ℓ and its amplitude A. The influence of other parameters such as the *reduced* Lewis number $le \equiv \beta(\mathrm{Le} - 1)$, the heat loss coefficient κ, and the period of the oscillatory flow τ are also considered for sake of completeness.

For time-independent flows, the non-dimensional form

$$u = A\cos\frac{\pi y}{\ell} \tag{32}$$

is adopted where A is the flow amplitude (measured with U_L) and ℓ the flow scale (measured with δ_L). When used in (1) along with $\eta \equiv y/\ell$, this yields the asymptotic result

$$\frac{U_T}{U_0} = 1 + \frac{A^2\ell^2}{4\pi^2}, \tag{33}$$

against which the numerics will be compared; here $U_0 = 1$ in the absence of heat loss, but more generally U_0 is the larger root of $U_0^2 \ln U_0 = -\kappa$.

For time-dependent situations, the harmonic form

$$u = A\cos\frac{2\pi t}{\tau}\cos\frac{\pi y}{\ell} \tag{34}$$

is adopted, where τ is the time-period measured against δ_L/U_L. When used in (27), this yields

$$\frac{U_T}{U_0} = 1 + \frac{A^2\ell^2}{8\pi^2}. \tag{35}$$

We remind the reader again that formula (27) of which (35) is a particular case has been derived in the asymptotic limit $\ell \to 0$, $A = \mathcal{O}(1)$, and with either (a) $\tau = \mathcal{O}(1)$ (slow oscillations), or (b) $\tau = \mathcal{O}(\ell)$ (faster oscillations). In the computations, case (b) will be mainly adopted (by taking $\tau = 2\ell/A$), when testing the validity of the asymptotics for oscillatory flows. However, we shall also examine numerically, albeit briefly, the influence of τ on the results. Before embarking on this programme however, we begin with the time-independent situations.

5.1 *Time-independent flows*

In this subsection we address the time-independent cases pertinent to (32). Shown in figure 2 is the effective flame speed U_T versus the flow amplitude A for selected values of ℓ decreasing from top to bottom in the adiabatic, equidiffusional case, $\kappa = 0$ and $le = 0$. The dashed lines correspond to the numerical results* and the solid lines are based on the analytical formula (33) (the circles indicate where values are computed based on the analytical formula or on the numerics). It is seen that there is good agreement between the numerics and asymptotics, provided A is not too large. The range of agreement extends to higher values of A as ℓ is decreased, up to $A \approx 50$ when $\ell = 0.1$, for example. Now, formula (33) suggests that it is the combination $A\ell$, representing the Peclet number of the flow, which is significant in determining U_T. This is confirmed in figure 3 where the same results are plotted versus $A\ell$. It is seen that there is a good agreement provided $A\ell \leq 5$, approximately.

We now examine the effect of non-zero heat loss and non-unit Lewis number ($\mathrm{Le} \neq 1$ or $le \neq 0$). The results are summarized in figures 4 and 5, both similar to figure 2. The figures again show a good agreement between the asymptotics and numerics and allow us to draw similar conclusions.

* Here and below, we have normalized the numerical values of U_T by the corresponding *numerical* value of the *planar*, unstretched flame speed. This allows a fair comparison between the asymptotics (based on an infinite β assumption) and the numerics, without having to use excessively large values of β.

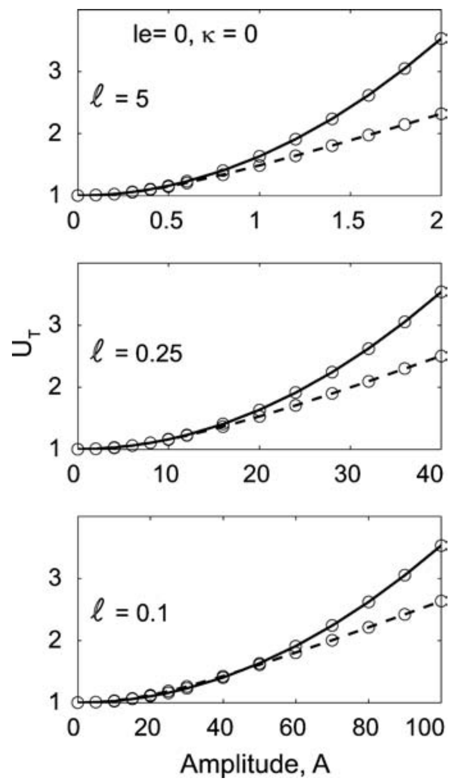

Figure 2. U_T versus A predicted by asymptotics (solid line) and numerics (dashed line) for Le $= 0$ and $\kappa = 0$.

5.1.1 Effect of the flow on the response of thin and thick flames to heat loss. Before leaving the time-independent cases, it is instructive to highlight the difference in the response of thin and thick flames to the combined effect of heat loss and flow. This is illustrated in figures 6 and 7, where U_T is plotted versus κ for fixed values of A, and selected values of Le increasing from top to bottom. The individual curves terminate at the extinction points determined numerically. Figure 6, corresponding to a thick flame ($\ell = 0.1$), shows that U_T versus κ is practically unaffected by the flow. This observation is in agreement with the

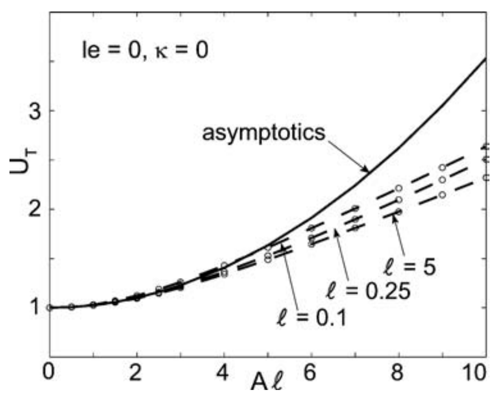

Figure 3. U_T versus $A\ell$ predicted by asymptotics (solid line) and numerics (dashed lines) for Le $= 0$ and $\kappa = 0$.

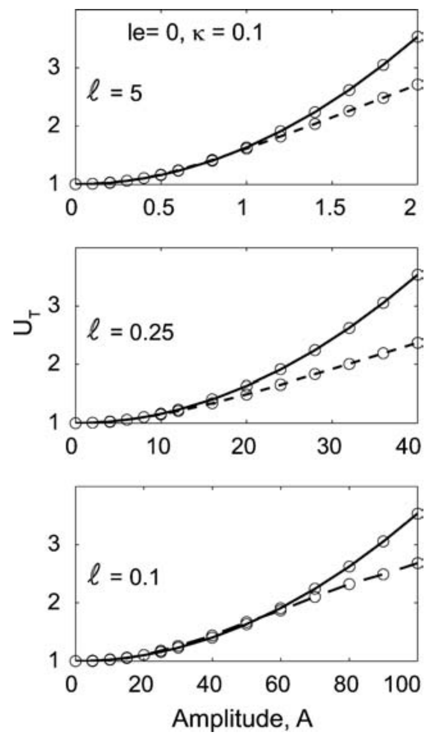

Figure 4. U_T versus A predicted by asymptotics (solid line) and numerics (dashed line) for $\mathrm{Le} = 0$ and $\kappa = 0.1$.

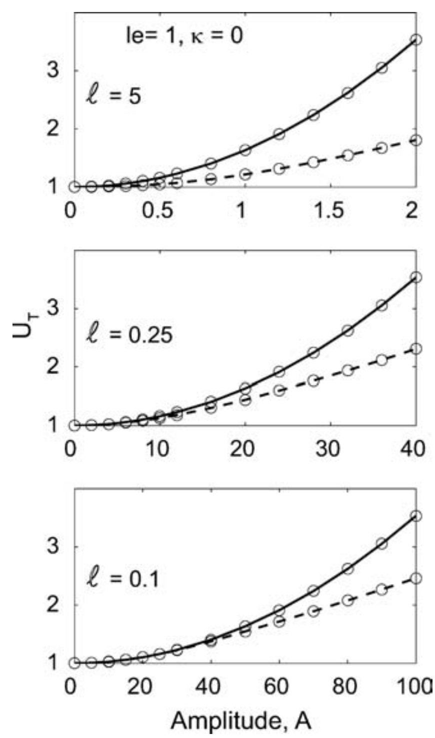

Figure 5. U_T versus A predicted by asymptotics (solid line) and numerics (dashed line) for $\mathrm{Le} = 1$ and $\kappa = 0$.

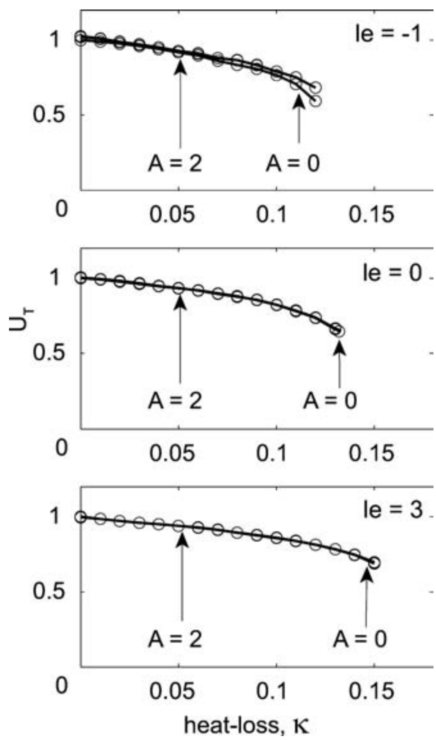

Figure 6. U_T versus κ for $\ell = 0.1$ (thick flame) and selected values of A and Le.

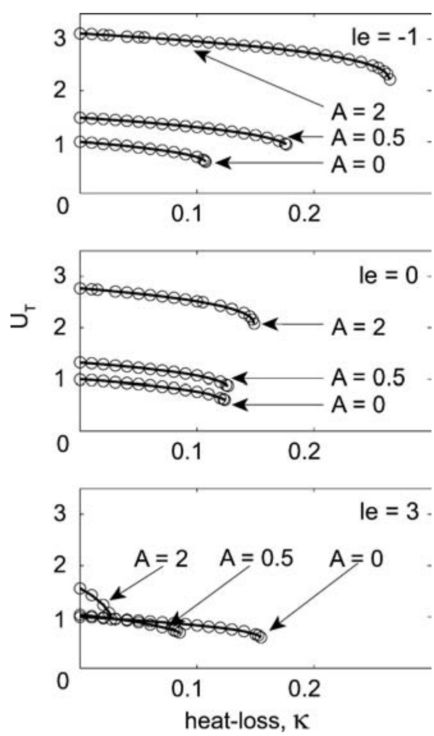

Figure 7. U_T versus κ for $\ell = 10$ (thin flame) and selected values of A and Le.

asymptotic formula (33) which predicts that U_T must deviate from its planar value U_0 by an amount of $\mathcal{O}(\ell^2)$ which is very small (for $\mathcal{O}(1)$ values of A).

In contrast, figure 7, corresponding to a thin flame ($\ell = 10$), shows that U_T versus κ *is* significantly affected by the flow. Interestingly, and may be somewhat surprisingly, *the thin flame is found to be more resistant to heat loss in the presence of a flow*, even for unit Lewis number (Le $= 0$). When the Lewis number is large, however, the flame becomes less resistant to heat loss in the presence of a flow, as seen on the bottom of figure 7. For near unit values of the Lewis number at least, a plausible explanation for the increased resistance of the thin-flame to heat losses is the increased surface area of the flame (due to the flow); this results in an increased rate of heat generation per unit area perpendicular to direction of propagation; thus stronger volumetric heat losses are required to cause flame extinction. Of course, flame stretch and its sign along the flame front has also an important contribution which strongly depends on the flow adopted and on the deviation of the Lewis number from unity, and which seems difficult to predict without recourse to numerics. Although this is an important topic in its own right, we shall not pursue it any further herein; see however [15] for a recent related study in the framework of a vortical flow.

5.2 *Oscillatory flows*

We now turn our attention to the time-dependent cases, with the flow being of the form (34). We begin with the adiabatic, equidiffusional case shown in figure 8, corresponding to $\ell = 1$, $A = 2$ and $\tau = 1$. Plotted are the instantaneous amplitude, $\hat{A} \equiv A \cos 2\pi t/\tau$, and total burning rate Ω (defined in equation (7)) versus time t, after an initial transient which is not shown. In each period τ of the flow, two maxima of Ω can be observed. These are due to the flow amplitude reaching its maximum positive and negative values, leading to maxima in the flame area and thus in Ω, which are seen to occur after a time lag however. From the plot, $U_T = \bar{\Omega}$, the time average of Ω, can be extracted and is found to be approximately equal to 0.92 in this case.

5.2.1 Testing the validity of the asymptotics for oscillatory flows.

Repeating the calculation for several values of ℓ and extracting $\bar{\Omega}$ as just described generates figure 9, where

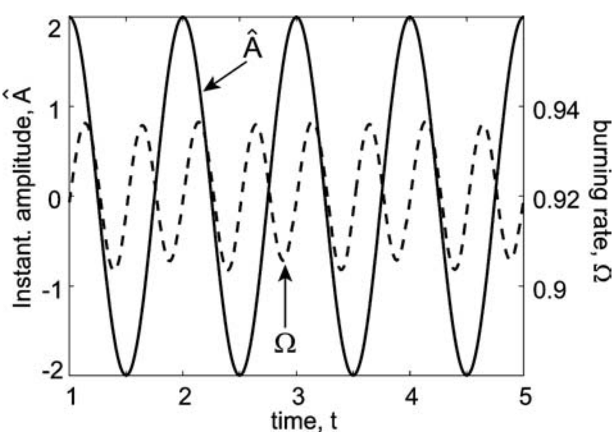

Figure 8. Instantaneous amplitude $\hat{A}$ (solid line) and total burning rate Ω (dashed line) versus time t for Le $= 0$ and $\kappa = 0$ (with $\ell = 1$, $A = 2$ and $\tau = 1$).

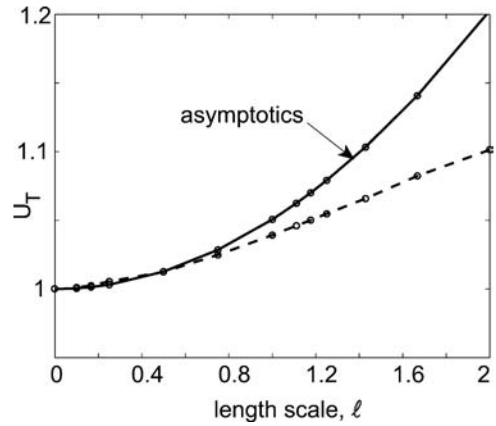

Figure 9. U_T versus ℓ predicted by asymptotics (solid line) and numerics (dashed line) for $\mathrm{Le} = 0$ and $\kappa = 0$ (with $A = 2$ and $\tau = 2\ell/A$).

U_T (normalized as before) is plotted versus ℓ along with the asymptotic curve based on formula (35). In this figure, A is kept fixed, equal to 2, while taking $\tau = 2\ell/A$. A good agreement between the numerics and asymptotics can be observed, provided $\ell < 0.8$, approximately. Similar calculations were performed in the presence of heat loss which are summarized in figures 10 and 11 and lead to similar conclusions. Finally, we have also tested the influence of non-unit Lewis number for two cases with $\mathrm{Le} = -1$ and 3; the corresponding results appear in figure 12. In summary, the numerical results confirm the validity of the time-dependent asymptotics provided ℓ is sufficiently small, although not excessively so, say $\ell \approx 0.8$ for $A = 2$.

5.2.2 Influence of the period of the oscillatory flow. Since the analytical formula (27) was derived both for $\tau = \mathcal{O}(1)$ and $\tau = \mathcal{O}(\ell)$ in the asymptotic limit $\ell \to 0$ with $A = \mathcal{O}(1)$, we briefly examine the influence of the period of oscillations τ restricting ourselves to the adiabatic equidiffusional case. A first set of computations is reported in figure 13 where U_T

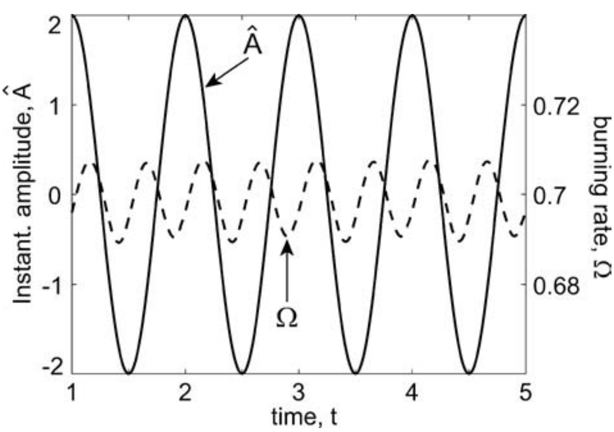

Figure 10. Instantaneous amplitude $\hat{A}$ (solid line) and total burning rate Ω (dashed line) versus time t for $\mathrm{Le} = 0$ and $\kappa = 0.1$ (with $\ell = 1$, $A = 2$ and $\tau = 1$).

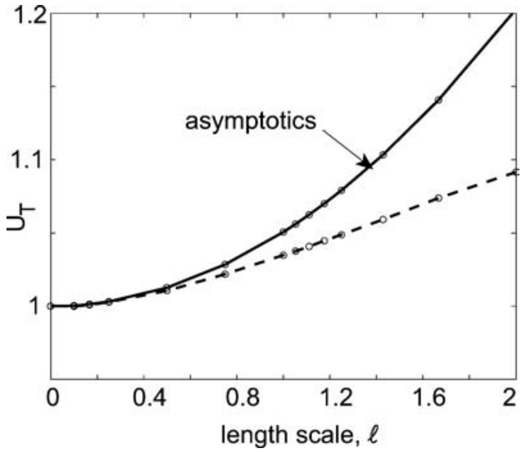

Figure 11. U_T versus ℓ predicted by asymptotics (solid line) and numerics (dashed line) for Le $= 0$ and $\kappa = 0.1$ (with $A = 2$ and $\tau = 2\ell/A$).

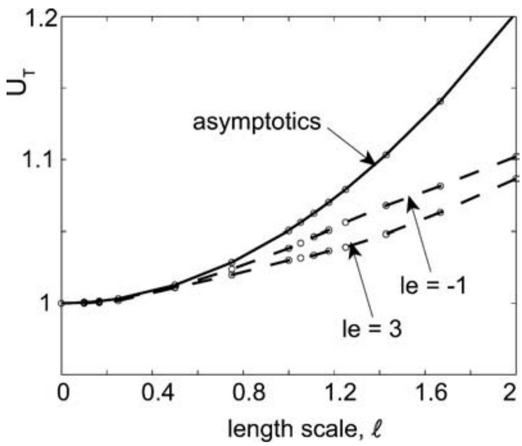

Figure 12. U_T versus ℓ predicted by asymptotics (solid line) and numerics (dashed lines) for $\kappa = 0$ and two cases with Le $= -1$ and $le = 3$.

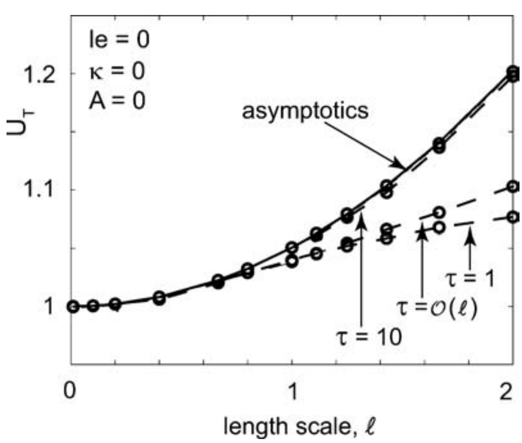

Figure 13. U_T versus ℓ predicted by asymptotics (solid line) and numerics (dashed line) for Le $= 0$, $\kappa = 0$, $A = 2$ and three cases with $\tau = 1$, $\tau = \ell$ and $\tau = 10$.

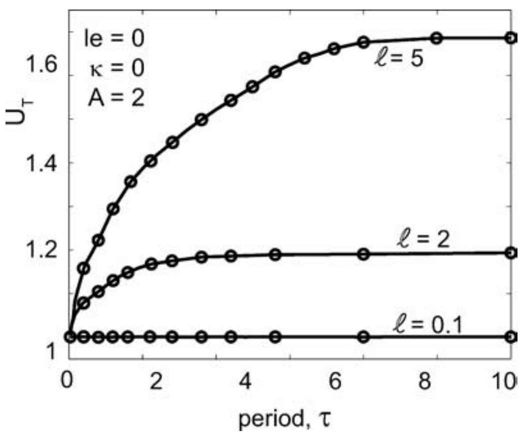

Figure 14. U_T versus τ determined numerically for Le $= 0$, $\kappa = 0$, $A = 2$ and three cases with $\ell = 0.1$, $\ell = 2$ and $\ell = 5$.

is plotted versus ℓ for $\tau = \ell$ (as in figure 9) as well as for $\tau = 1$ and $\tau = 10$. This figure confirms the validity of the asymptotics for both $\tau = \mathcal{O}(1)$ and $\tau = \mathcal{O}(\ell)$ cases, and shows furthermore that the applicability of the asymptotics extends to higher values of ℓ as τ is increased. Another way of examining the influence of the period of the oscillatory flow τ is to plot U_T versus τ for selected fixed values of ℓ. This is done in figure 14 where three values of ℓ are considered with the smallest value corresponding to a relatively thick flame and the larger values to thinner flames. The figure illustrates that whereas thin flames are strongly affected by the frequency of the oscillatory flows, thick flames are very weakly affected. More relevant to this study, the figure illustrates that irrespective of the flow scale ℓ, U_T tends to unity (the laminar flame speed) as $\tau \to 0$. This observation is in agreement with the results of [16] which shows, within a G-equation model, that U_T must tend to unity when the frequency of the flow tends to infinity. Their result can in fact be derived from the model we use in this study by considering the distinguished limit $\tau \to 0$ with A and ℓ fixed. We conclude that *formula (27) must ultimately break down for τ sufficiently small*. It does hold, however, for $\tau = \mathcal{O}(\ell)$ as shown analytically and confirmed numerically.

6. Conclusion

In this investigation, several new contributions have been made.

Firstly, we have derived an analytical formula for the effective flame speed U_T in the presence of a prescribed, oscillating, parallel flow, whose scale is small compared with the laminar flame thickness and whose amplitude is of the order of the laminar flame speed. The derivation was carried out in two distinguished limits in which the frequency of the oscillating flow is of the order of the natural frequency of the flame U_L/δ_L or of the order of the natural frequency of the flow (equal to the flow amplitude divided by its scale). The formula derived in both limits, given by equation (27), shows that there is an increase in U_T attributed to the flow which depends quadratically on both the scale and intensity of the flow. We note that our study yields partial support to Damköhler's second hypothesis in that it shows that the flame indeed behaves as a planar flame (to leading order) with and increased propagation speed which depends on both the scale and amplitude of the velocity fluctuation. However our formula for U_T contradicts the formula given by Damköhler in his original paper [9] (pages 3 and 35 of the English translation) which shows that U_T has a square root dependence on the scale and amplitude of the flow (i.e. $U_T \sim (A\ell)^{1/2}$ in our non-dimensional notations), instead of the quadratic dependence found in this paper (see also [12], page 123).

Secondly, we have carried out a large number of numerical calculations as a systematic test of the asymptotic findings for both stationary and time-dependent flows. These accounted for the effects of volumetric heat loss as well as differential diffusion. A good agreement was found between the numerics and asymptotics for all cases, at least in the expected asymptotic range of validity of the analytical results; of course the numerics do provide a concrete and useful estimate of the actual range of validity of the asymptotics.

Thirdly the effect of the frequency of the oscillating flow has also been examined along with additional aspects such as the differences in the response of thin and thick flames to heat loss in the presence of a flow.

We close the paper by stressing the limitations of the present model and indicating a number of possibilities to extend the range of validity of the findings. In doing so, our remarks will focus on aspects relevant to turbulent combustion, although, as pointed out in the introduction, the study is relevant to many other fields such as the emerging micro-combustion technology. The first obvious limitation is the adoption of the constant density assumption for sake of analytical tractability, which precludes to account for the effects of gas-expansion and the Darrieus–Landau instability on flame interaction with small-scale turbulence (see, e.g. [17, 18]). The second limitation is that the flow considered is unidirectional and mono-scale. The third limitation is related to the distinguished limits investigated (based here on $\ell \to 0$ with A constant) under which we have examined Damköhler's second hypothesis. A full understanding of the problem requires naturally a knowledge of flame behaviour under a variety of distinguished limits such as $A \to \infty$ with ℓ fixed, or $A \to \infty$ with the Peclet (or Reynolds) number $A\ell$ fixed, etc. The latter two limits are important to understand the behaviour of the effective flame speed U_T for large values of the amplitude, say A, of the turbulent flow, and the corresponding *bending effect* of U_T versus A, which is frequently, but somewhat inconclusively, discussed in the literature for parallel, vortical, and more general flows (see, e.g. [19–24]). The interested reader may consult these references for an instructive discussion of the bending effect and other aspects; of particular relevance to our study are references [22] and [23] since they consider time-dependent flows without addressing the limit $\ell \to 0$ however. Here we simply point out that the bending effect is believed by Ronney and Yakhot [3] to be associated with the increasingly felt presence of the small scales in the (multi-scale) flow by the flame, as the turbulent intensity is increased; these authors seem however to have in mind a highly disrupted flame fronts similar to those investigated by Kagan and Sivashinsky [15] for vortical flows, which cannot be obtained within the model of the present study. The model, despite its simplicity and its extensive use in the literature however, can still be used to provide significant analytical insight into flame behaviour in a flow field, under a variety of distinguished limits which we are currently investigating. Similar investigations in the context of multi-scale and more complex flows can also help get a clearer picture of turbulent premixed flames in future studies.

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