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# A Lyapunov-Lurye functional parametrization of discrete-time Zames-Falb multipliers

Jingfan Zhang and Joaquin Carrasco

Abstract—We consider the absolute stability of discrete-time Lurye systems with SISO/MIMO (non-repeated SISO) nonlinearities that are sector bounded and slope restricted. For this class of systems, we present a parametrization of Lyapunov-Lurye functional (LLF) that is the time-domain equivalence to finite impulse response (FIR) Zames-Falb multipliers. As searches over FIR Zames-Falb multipliers provide the best-known results in the literature, the parametrization here provides the best-known Lyapunov function for absolute stability. A motivation of this alternative is making it easy to analyze the system in the time domain, especially when the frequency domain expression of the system is not straightforward. In this letter, we show the equivalence between the proposed LLF and FIR multipliers theoretically and numerically.

Index Terms—Nonlinear systems, Absolute stability, Lyapunov-Lurye functional, Zames-Falb multipliers

## I. Introduction

This letter studies discrete-time Lurye[1] systems (see Fig. 1), where $G$ is a linear time-invariant (LTI) stable plant, and $\phi$ is a memoryless nonlinear operator, which is sector-bounded in the range $[0, \Psi]$ and slope-restricted in the range $[0, \Gamma]$. The system is said to be absolutely stable if the origin is globally uniformly asymptotically stable for all nonlinearities in the class [2]. The problem to show absolute stability is called Lurye problem [3], and it has been being an open question since the early 1940s due to the lack of computational necessary and sufficient conditions.

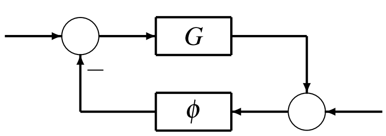


Fig. 1: The Lurye system

Recently, major breakthroughs of the stability problem were achieved by the input-output analysis in frequency domain, in particular, with the use of Zames–Falb multipliers. On one hand, tighter slope bounds for which systems are absolutely stable were obtained by searches over FIR Zames-Falb multipliers for both SISO case [4], [5] and MIMO case [6]. On the other hand, in [7], counterexamples with periodic solutions in the SISO case were developed by using the corresponding (conservative) dual bound of LTI Zames-Falb multipliers [8]. As the gap between these two slope bounds are often very small (see [8, Table II, III]), it was conjectured in [8], [9] that the existence of LTI Zames-Falb multipliers is a sufficient and necessary condition for absolute stability.

Meanwhile, the least conservative stability result in Lyapunov approach was provided in [10] by the search over a class of LLFs. The conservativeness is reduced because the Lurye term is bounded tightly [11]. Furthermore, as the bound is in quadratic form, the positivity condition of the whole LLF takes the combination of the coefficient matrix in the quadratic term and the coefficients in the Lurye terms (as we will show in Section III-A), instead of making them positive definite respectively. See similar technique to relax the positivity condition in continuous-time [12], [13]. Nevertheless, as shown in [5], the LLF in [10] is the time-domain equivalence to a restricted subclass of second order FIR Zames-Falb multipliers (also see the equivalence by numerical examples in Sec.IV); as shown in [14], the extensions of the parametrization in [10] are equivalent to restricted subclasses of higher order FIR Zames-Falb multipliers.

In this letter, we present a novel parametrization of LLFs, which is the time-domain equivalence to the full class of FIR Zames-Falb multipliers for the case with SISO/non-repeated MIMO nonlinearities. In all existing literature, the least conservative stability condition is obtained by searches over FIR Zames-Falb multipliers, so the proposed parametrization of LLFs would be the best Lyapunov function currently. Although the Lyapunov criterion in this letter does not provide less-conservative results than FIR Zames-Falb multipliers, it may be significant to analyze the system purely in the time domain, especially when there is no direct frequency domain description of the system [15]. Moreover, time-domain approaches provide convenience to analyze local stability [16], while an input-output local stability in the frequency domain using Zames–Falb multipliers is given in [17].

It is also possible to convert a particular Zames-Falb multiplier to a Lyapunov function by using J-spectral factorization [15]. Once we have obtained the multiplier, then a Lyapunov function can be derived, however no parametrization of the Lyapunov function is provided. In addition, the state-space technique is used to search unstructured noncausal Zames-Falb multipliers in [18], which shows the possibility to construct the Lyapunov function. It is worth highlighting that noncausal FIR

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[1]Also written as Lur'e or Lurie due to different translations [1].

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multipliers do not have a state-space representation [5]. Our objective is to derive a prior parametrization of LLFs.

The outline of the rest parts of the letter is as follows. In Section II, some preliminary notations and theorems are introduced, especially the Zames-Falb theorem and FIR Zames-Falb multipliers. In Section III, the main result of the parametrization of the LLF is provided, following by its frequency domain expression with FIR Zames-Falb multipliers. In Section IV, examples are used to show the equivalence between the proposed LLF and FIR multipliers. The results are the least conservative in all existing literature. In Section V, the letter is concluded with future studies.

## II. Preliminaries

### A. Notations

Consider the feedback interconnection in Fig. 1,

$$x_{i+1} = Ax_i + Bu_i, \tag{1a}$$
$$y_i = Cx_i, \tag{1b}$$
$$u_i = -\phi(y_i), \tag{1c}$$

where $x_i \in \mathbb{R}^{n_p}$, $u_i \in \mathbb{R}^{m_p}$ and $y_i \in \mathbb{R}^{m_p}$ are the state, input and output of the system $G$ at time instant $i$ respectively; the matrices $A \in \mathbb{R}^{n_p \times n_p}$, $B \in \mathbb{R}^{n_p \times m_p}$, $C \in \mathbb{R}^{m_p \times n_P}$, and $A$ is Schur. The memoryless nonlinearity operator $\phi$ is defined by a static MIMO function $\phi : \mathbb{R}^{m_p} \mapsto \mathbb{R}^{m_p}$ as

$$\phi(y_i) \equiv \begin{bmatrix} \phi_1(y_i^1) & \phi_2(y_i^2) & \cdots & \phi_{m_p}({y_i}^{m_p}) \end{bmatrix}^\top, \tag{2}$$

which consists of non-repeated SISO function $\phi_j$ satisfying sector bounded and slope restricted conditions,

$$0 \le \frac{\phi_j(\sigma)}{\sigma} \le \psi_j, \quad \forall \sigma \neq 0, \tag{3a}$$
$$0 \le \frac{\phi_j(\sigma_2) - \phi_j(\sigma_1)}{\sigma_2 - \sigma_1} \le \gamma_j, \quad \forall \sigma_1 \neq \sigma_2, \tag{3b}$$

where $y_i^j$ $(j = 1, 2, \cdots, m_p)$ is the $j^{th}$ element of the output $y_i$; $\psi_j$ and $\gamma_j$ are the $j^{th}$ elements of the positive diagonal matrices $\Psi$ and $\Gamma$ respectively.

The expression $G^*(z)$ denotes the complex conjugate transpose of $G(z)$ on $|z| = 1$, i.e. $G^*(z) = G^\top\left(\frac{1}{z}\right)$, where the superscript $\top$ indicates the transpose. For a matrix $M$, $\mathrm{He}\{M\} = M + M^\top$; $M > (\ge) 0$ means $M$ is positive (semi)definite. The expression $\mathrm{diag}(\cdots)$ denotes a diagonal matrix; $\star$ represents a term of a symmetric matrix that can be inferred by symmetry.

### B. Bounds of the Lurye term

In order to convert LLF inequalities to LMIs, the Lurye term is bounded in quadratic form below.

Lemma 1 ([11]): For the nonlinearity $\phi$ in (2) and satisfying (3), the lower and upper bounds of the Lurye term of each element $\phi_j$ are $L \le \int_{\sigma_1}^{\sigma_2} \phi_j(\sigma) d\sigma \le U$, where

$$L = \phi_j(\sigma_1)(\sigma_2 - \sigma_1) + \frac{1}{2\gamma_j}\left\{\phi_j(\sigma_2) - \phi_j(\sigma_1)\right\}^2, \tag{4a}$$
$$U = \phi_j(\sigma_2)(\sigma_2 - \sigma_1) - \frac{1}{2\gamma_j}\left\{\phi_j(\sigma_2) - \phi_j(\sigma_1)\right\}^2. \tag{4b}$$

### C. Zames-Falb theorem and FIR multipliers

Zames-Falb theorem is a based on passivity. By appropriate class of Zames-Falb multipliers and loop-transformation, slope-restricted nonlinearites are preserved to be passive. Then, with the same multiplier, if the linear part can be shown strictly passive, then the closed-loop system is stable.

For the nonlinearities $\phi$ defined in (2) (3), the class of Zames-Falb multipliers are defined as follows.

Definition 1 (Discrete-time LTI Zames-Falb multipliers for non-repeated SISO nonlinearities [19], [20]): The convolution operator $M : \ell_2^{m_p} \mapsto \ell_2^{m_p}$ is a discrete-time LTI Zames-Falb multiplier if its impulse response $m$ satisfies the $\ell_1$-norm condition $\sum_{i=-\infty}^{\infty} |m_i| \le 2m_0$, where $m_i \in \mathbb{R}^{m_p \times m_p}$ are diagonal, and $m_i \le 0$ for all $i \neq 0$.

Theorem 1 (Zames-Falb theorem [19]): Consider the feedback interconnection (1) with $\phi$ satisfying (2) and (3). The system is absolutely stable if

$$\mathrm{He}\left\{M(z)\left(G(z) + \Gamma^{-1}\right)\right\} > 0 \quad \forall |z| = 1, \tag{5}$$

where $M$ is a discrete-time LTI Zames-Falb multiplier.

Particularly, a subclass of discrete-time Zames-Falb multipliers with finite impulse response $(i < \infty)$ is considered.

Definition 2 (FIR Zames-Falb multipliers [5], [6]): The convolution operator is an FIR Zames-Falb multiplier if

$$M(z) = m_{n_b} z^{-n_b} \cdots + m_1 z^{-1} + m_0 + m_{-1} z \cdots + m_{-n_f} z^{n_f}, \tag{6}$$

where the causal part is with the backward-shift operator $z^{-i_b}$ $(i_b = 1, 2, \cdots, n_b)$, and the anticausal part is with the forward-shift operator $z^{i_f}$ $(i_f = 1, 2, \cdots, n_f)$. In addition, the coefficients $m_{i_b}, m_{-i_f} \in \mathbb{R}^{m_p \times m_p}$ are non-positive diagonal matrices and satisfy the $\ell_1$-norm condition $\sum_{i_b=1}^{n_b} |m_{i_b}| + \sum_{i_f=1}^{n_f} |m_{-i_f}| \le m_0$.

### D. Kalman-Yakubovich-Popov (KYP) lemma

The time domain stability condition in this letter will be converted into frequency domain by the KYP lemma.

Lemma 2: Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $\Pi = \Pi^\top \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(zI - A) \neq 0$ for all $|z| = 1$, the following statements are equivalent:

1) There is a symmetric matrix $P \in \mathbb{R}^{n \times n}$ and

$$\begin{bmatrix} A^\top PA - P & A^\top PB \\ B^\top PA & B^\top PB \end{bmatrix} + \Pi < 0. \tag{7}$$

2) For all $|z| = 1$,

$$\begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix} < 0. \tag{8}$$

Remark 1: The above equivalence holds with non-strict inequalities only if the pair $(A, B)$ are controllable. However, strict inequalities are used in the next section, and the pair $(A, B)$ are often uncontrollable.

## III. Main results

### A. Main theorem

Theorem 2: Given a free parameter $n \ge 2$. Consider the feedback interconnection (1) with $\phi$ satisfying (2) and (3). The system is absolutely stable if there exist

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a symmetric matrix $P \in \mathbb{R}^{(n_p n+m_p n)\times(n_p n+m_p n)}$ and positive diagonal matrices $L \in \mathbb{R}^{m_p\times m_p}$, $M_r \in \mathbb{R}^{m_p\times m_p}$, $N_r \in \mathbb{R}^{m_p\times m_p}$ ($r = 1,2,\cdots,n$), such that

$$\widehat{P} = P + \Xi > 0, \tag{9}$$

$$\widehat{\Omega} = \begin{bmatrix} \widehat{A}^T \widehat{P} \widehat{A} - \widehat{P} & \widehat{A}^T \widehat{P} \widehat{B} \\ \widehat{B}^T \widehat{P} \widehat{A} & \widehat{B}^T \widehat{P} \widehat{B} \end{bmatrix} + \Omega < 0, \tag{10}$$

where $Q_r = M_r + N_r$, and $(\widehat{A}, \widehat{B})$, $\Xi$, $\Omega$ are defined in (11), (12), (13) respectively on the next page.

Proof: The proof is based on the standard Lyapunov theorem for asymptotic stability (e.g. [21, Theorem 13.2]). See the proof in Appendix A. ■

Remark 2: As shown in (23), some terms are added and subtracted at the same time. It is not mandatory here, but it makes the LMI (10) consistent with (7). This relation provides convenience to convert Theorem 2 into frequency domain by the KYP lemma in the next part.

The proof of Theorem 2 is valid for any $n \geq 1$. However, due to the slight difference in the expression of $\Omega$, the case for $n = 1$ is claimed as a corollary below.

Corollary 1: Consider the feedback interconnection (1) with $\phi$ satisfying (2) and (3). The system is absolutely stable if there exist a symmetric matrix $P \in \mathbb{R}^{(n_p+m_p)\times(n_p+m_p)}$ and positive diagonal matrices $L \in \mathbb{R}^{m_p\times m_p}$, $M_1 \in \mathbb{R}^{m_p\times m_p}$, $N_1 \in \mathbb{R}^{m_p\times m_p}$, such that (9), (10) are satisfied, where

$$\widehat{A} = \left[\begin{array}{c|c} A & -B \\ \hline 0 & 0 \end{array}\right], \quad \widehat{B} = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right],$$

$$\Xi = \left[\begin{array}{c|c} C^\top N_1 \Psi C & \star \\ \hline -N_1 C & Q_1 \Gamma^{-1} \end{array}\right],$$

$$\Omega = \left[\begin{array}{c|c:c} 0 & \star & \star \\ \hline -N_1 CA + LC & \mathrm{He}\left\{N_1 CB\right\} - 2L\Psi^{-1} & \star \\ \hdashline -M_1 C + Q_1 CA & Q_1 \Gamma^{-1} - Q_1 CB & -2Q_1 \Gamma^{-1} \end{array}\right].$$

Proof: The proof follows from the proof of Theorem 2 by setting $n = 1$, so it is omitted. ■

B. Frequency domain interpretation

Similar to the technique in [5], [14], the main theorem is converted into frequency domain below.

Theorem 3: Consider the feedback interconnection in (1). If Theorem 2 or Corollary 1 is satisfied with some $n$, $\Psi$, $\Gamma$ by solutions $L$, $M_r$, $N_r$, then the frequency domain condition

$$\mathrm{He}\left\{L(G(z) + \Psi^{-1}) + M(z)(G(z) + \Gamma^{-1})\right\} > 0 \tag{14}$$

holds for all $|z| = 1$ with the FIR Zames-Falb multiplier

$$M(z) = -M_n z^{-n} \cdots - M_1 z^{-1} + Q - N_1 z \cdots - N_n z^n, \tag{15}$$

where $Q = \sum_{r=1}^n M_r + \sum_{r=1}^n N_r$.

Proof: As the LMI (10) is in the same structure of (7), it is clear by the KYP lemma that (10) is equivalent to the frequency domain condition

$$\begin{bmatrix} (zI - \widehat{A})^{-1} \widehat{B} \\ I \end{bmatrix}^* \Omega \begin{bmatrix} (zI - \widehat{A})^{-1} \widehat{B} \\ I \end{bmatrix} < 0,$$

which can be simplified as (14) with (15). Note that $zG(z) = CA(zI - A)^{-1}B + CB$. The detailed procedures are omitted, and refer to [5] for the similar proof. ■

Remark 3: Theorem 1 and 2 are sufficient for asymptotic stability. Furthermore, in recent studies [22], [23], a subset of FIR Zames-Falb multipliers are invoked to explore the exponential decay rate of Lurye systems for which asymptotic stability is guaranteed. Similarly, if the Lyapunov function is further bounded by $\alpha\|x_i\|^p \leq V(x_i) \leq \beta\|x_i\|^p$ and certifies $\rho V(x_{i+1}) - V(x_i) \leq 0$ with some $\alpha, \beta > 0$, $p \geq 1$ and $\rho > 1$, then the system is proved to be exponentially stable in the way $\|x_i\| \leq (\beta/\alpha)^{1/p} \|x_0\| (\rho^{1/p})^{-i}$ [21, Theorem 13.2]. Then, it is also possible to show the equivalence between the Lyapunov criterion and the Zames-Falb theorem in [22], [23] for exponential stability, but this problem is still open.

In (15), the steps of causal and anticausal terms are equal ($n_b = n_f = n$). It is also possible to make them different. Assume that $n_b \neq n_f$ and $n = \max(n_b, n_f)$. In (17), by removing the terms with $n_r$ if $n_f < r \leq n$, or removing the terms with $m_r$ if $n_b < r \leq n$, it leads to FIR Zames-Falb multipliers with causal steps $n_b$ and anticausal steps $n_f$. Alternatively, it is also feasible by vanishing the corresponding coefficients $M_r$ or $N_r$ in the LMIs. Nevertheless, we fix $n_b = n_f = n$ here.

## IV. Numerical examples

### A. Examples

1) $G_1(z) = \frac{2z+0.92}{z^2-0.5z}$
2) $G_2(z) = \frac{0.2343z^2+0.1224z+0.04805}{z^3+1.611z^2+1.065z+0.08843}$
3) $G_3(z) = \frac{z^4-1.5z^3+0.5z^2-0.5z+0.5}{4.4z^5-8.957z^4+9.893z^3-5.671z^2+2.207z-0.5}$
4) $G_4(z) = \begin{bmatrix} \frac{0.2}{z-0.98} & \frac{-0.2}{z-0.92} \\ \frac{0.3}{z-0.97} & \frac{0.1}{z-0.91} \end{bmatrix}$
5) $G_5(z) = \begin{bmatrix} \frac{-0.551z+0.02933}{z^2+0.5z+0.1327} & \frac{-0.852z-0.3544}{z^2+0.5z+0.1327} & \frac{-0.255z-0.2061}{z^2+0.5z+0.1327} \\ \frac{-2.386z-0.1756}{z^2+0.5z+0.1327} & \frac{-2.054z-0.5572}{z^2+0.5z+0.1327} & \frac{2.347z+1.259}{z^2+0.5z+0.1327} \\ \frac{0.2442}{z^2+0.5z+0.1327} & \frac{0.5568z-0.02436}{z^2+0.5z+0.1327} & \frac{0.3456z-0.6324}{z^2+0.5z+0.1327} \end{bmatrix}$
6) $G_6(z) = \begin{bmatrix} \frac{-1.375z^2-2.312z-0.9785}{z^3+2.29z^2+1.774z+0.4605} & \frac{-0.8586z^2-0.5926z-0.06916}{z^3+0.75z^2+0.0973z-0.002785} \\ \frac{1.345z^2+1.879z+0.6163}{z^3+2.38z^2+1.856z+0.4714} & \frac{1.497z^2-1.582z+0.3669}{z^3-1.78z^2+0.9876z-0.1586} \end{bmatrix}$

Six plants are included to compare the results, where $G_1$, $G_3$, $G_4$ are taken from literature [5], [10]; the rest of plants are new in this letter to address that less-conservative results are possible by adding more terms in the LLF. We present the results in Table I by Theorem 2/Corollary 1 with $n = 1,2,3$ and 10 respectively. For the MIMO examples, we set the maximum sector bound $\psi_j$ as a constant for all $j$. Meanwhile, we set $\gamma_j = \psi_j$ for all $j$. All LMIs are solved in Matlab using semidefinite program (SDP) toolbox Yalmip [24] with solver sdpt3 [25].

### B. Numerical Results

The results are compared with the corresponding multipliers results, which are obtained by searching FIR multipliers where $n_b = n_f = n$ with factorization by lifting in [5], [6]. The results are also compared with [10], which is least-conservative in existing Lyapunov literature (see numerical examples therein).

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$$\widehat{A}=\left[\begin{array}{ccccc|ccccc} 0 & I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A & 0 & 0 & 0 & \cdots & -B \\ \hline 0 & 0 & 0 & \cdots & 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & I & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right]_{(n_pn+m_pn)\times(n_pn+m_pn)} \qquad \widehat{B}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ I \end{bmatrix}_{(n_pn+m_pn)\times m_p} \tag{11}$$

$$\Xi=\left[\begin{array}{ccccc|ccccc} C^T\sum_{r=1}^n N_r\Psi C & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & C^T\sum_{r=2}^n N_r\Psi C & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & C^T\sum_{r=3}^n N_r\Psi C & \star & \star & \star & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \cdots & C^T\sum_{r=n}^n N_r\Psi C & \star & \star & \star & \star & \star \\ \hline -\sum_{r=1}^n N_rC & 0 & 0 & \cdots & 0 & \sum_{r=1}^n Q_r\Gamma^{-1} & \star & \star & \star & \star \\ 0 & -\sum_{r=2}^n N_rC & 0 & \cdots & 0 & 0 & \sum_{r=2}^n Q_r\Gamma^{-1} & \star & \star & \star \\ 0 & 0 & -\sum_{r=3}^n N_rC & \cdots & 0 & 0 & 0 & \sum_{r=3}^n Q_r\Gamma^{-1} & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \star \\ 0 & 0 & 0 & \cdots & -\sum_{r=n}^n N_rC & 0 & 0 & 0 & \cdots & \sum_{r=n}^n Q_r\Gamma^{-1} \end{array}\right] \tag{12}$$

$$\Omega=\left[\begin{array}{ccccc|cccccc} 0 & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star & \star & \star & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \star & \star & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & 0 & 0 & \star & \star & \star & \star & \star & \star \\ \hline LC & -N_1C & -N_2C & \cdots & -N_{n-1}C-N_nCA & -2L\Psi^{-1} & \star & \star & \star & \star & \star \\ -M_1C & Q_1C & 0 & \cdots & 0 & Q_1\Gamma^{-1} & -2Q_1\Gamma^{-1} & \star & \star & \star & \star \\ -M_2C & 0 & Q_2C & \cdots & 0 & Q_2\Gamma^{-1} & 0 & -2Q_2\Gamma^{-1} & \star & \star & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \star & \star \\ -M_{n-1}C & 0 & 0 & \cdots & Q_{n-1}C & \begin{array}{c} Q_{n-1}\Gamma^{-1} \\ +B^TC^TN_n \end{array} & 0 & 0 & \cdots & -2Q_{n-1}\Gamma^{-1} & \star \\ -M_nC & 0 & 0 & \cdots & Q_nCA & Q_n\Gamma^{-1} & 0 & 0 & \cdots & -Q_nCB & -2Q_n\Gamma^{-1} \end{array}\right] \tag{13}$$

Particularly, the less-conservative SISO results are compared with the slope bounds when periodic solutions exist. As these two slope bounds are very close, the existence of FIR multiplier/LLF is almost a necessary and sufficient condition for absolute stability in SISO case, while the techniques in [7] [8] need further development for the MIMO case.

As shown in Table I, the maximum slope bounds are less-conservative by increasing the order $n$ in LLFs and the order $n_f$, $n_b$ in FIR multipliers. Further less-conservative results are possible by using larger $n$ and $n_b$, $n_f$ at the cost of increasing the computational complexity, e.g. numerous decision variables and large size of LMIs. In consequence, numerical results may deteriorate when the order of FIR multipliers or LLF is overlarge (see [5, Fig.3]).

Because the criterion in [10] is equivalent to a restricted second order FIR multiplier [5], the results in [10] are close to first order FIR multipliers in all examples besides Example 6. In Example 6, the result is between first order and second order FIR multipliers. Moreover, according to Theorem 3, the main theorem with a given $n$ is the time-domain equivalence of FIR multipliers with $n_f = n_b = n$. Hence, the results in [10] are also between the main theorem with $n = 1$ and $n = 2$, except for Example 4.

### C. Discussion

The search of LLFs by Theorem 2 is slightly conservative than the search of FIR multipliers numerically, especially for Example 4, 5, 6, when $n = 10$. One reason may be that two LMIs $\widehat{P} > 0$ and $\widehat{\Omega} < 0$ are solved in the main theorem, while only one LMI in the form of (7) is solved in the multiplier approach. Note that $M_r, N_r \geq 0$, $m_{i_b}, m_{-i_f} \leq 0$ and the $\ell_1$-norm condition can be converted as scalar inequalities because they are diagonal. Another reason may be the large size of $\widehat{P}$ and $\widehat{\Omega}$. For instance, for Example 6, the sizes of $\widehat{P}$ and $\widehat{\Omega}$ are $140\times 140$ and $142\times 142$ respectively, while the size of the LMI by multiplier approach is $54\times 54$ by lifting factorization [6, Sec.5]. As a result, LLF takes about 100 s per iteration in the bisection

TABLE I: Maximum slope bounds by FIR multipliers and LLF

| | Ex (1) | Ex (2) | Ex (3) | Ex (4) | Ex (5) | Ex (6) |
|---|---|---|---|---|---|---|
| Park et al. (2019) [10] | 0.9108076 | 0.665928 | 2.590419 | 3.809132 | 0.288438 | 0.620123 |
| FIR multiplier [5], [6] ($n_f = n_b = 1$) | 0.9108076 | 0.665932 | 2.590419 | 3.809133 | 0.288439 | 0.572377 |
| FIR multiplier [5], [6] ($n_f = n_b = 2$) | 0.9114577 | 1.366665 | 2.590428 | 3.810275 | 0.348746 | 0.690220 |
| FIR multiplier [5], [6] ($n_f = n_b = 3$) | 0.9114579 | 1.366665 | 3.225440 | 3.812292 | 0.534200 | 0.696217 |
| FIR multiplier [5], [6] ($n_f = n_b = 10$) | 0.9114579 | 1.661334 | 3.824034 | 3.822113 | 0.553116 | 0.737294 |
| Main theorem ($n = 1$) | 0.9108076 | 0.665931 | 2.590423 | 3.807457 | 0.288439 | 0.572376 |
| Main theorem ($n = 2$) | 0.9114580 | 1.366680 | 2.590423 | 3.808584 | 0.348746 | 0.689670 |
| Main theorem ($n = 3$) | 0.9114580 | 1.366680 | 3.225431 | 3.809960 | 0.533969 | 0.691339 |
| Main theorem ($n = 10$) | 0.9114580 | 1.661325 | 3.824028 | 3.809960 | 0.545365 | 0.723420 |
| Slope bound when periodic solution exists [7], [8] | 0.9114583 | 1.661340 | 3.824040 | N/A | N/A | N/A |
| Nyquist Value | 1.086956 | 2.285526 | 7.907000 | 3.850132 | 0.556590 | 0.878303 |

search on $\Gamma$, while FIR multiplier only takes about 4 s per iteration.

The computational burden of Theorem 2 might be reduced by expressing $y_{i+l}$ $(l = 1, 2, \cdots, r)$ in (18) and (23) by $y_{i+l} = CA^l x_i - C\sum_{\kappa=1}^{l} A^{l-\kappa} B\phi(y_{i+(\kappa-1)})$. Hence, $\eta_i = [x_i^\top\ \phi(y_i)^\top \cdots \phi\left(y_{i+(n-1)}\right)^\top]^\top$, and $\zeta_i = [x_i^\top\ \phi(y_i)^\top\ \phi(y_{i+1})^\top \cdots \phi(y_{i+n})^\top]^\top$, which indicates that the sizes of $\widehat{P}$ and $\widehat{\Omega}$ of Example 6 can be reduced to $32 \times 32$ and $34 \times 34$ respectively. Nevertheless, it is complicated to provide the general form of LMIs in a clear pattern as (12) and (13), so this possible improvement for the numerical calculation is left as a future study.

In short, Theorem 2 is the least conservative in Lyapunov approach for Lurye systems with sector-bounded, slope-restricted nonlinearities. Moreover, it is theoretically equivalent to the search of FIR multipliers in frequency domain, although the FIR parametrization is more numerically efficient in particular for large values of $n$.

## V. Conclusions

We have presented a time-domain stability criterion for Lurye systems with sector-bounded, slope-restricted nonlinearities. The stability criterion is based on LLFs, which is equivalent to the frequency domain condition with FIR Zames-Falb multipliers. The numerical results are least conservative in all existing Lyapunov literature, and equivalent to the search of FIR Zames-Falb multipliers although minor computational issue may occur when the size of LMIs becomes large.

Nevertheless, for Lurye systems where nonlinearities are also odd and/or repeated, less-conservative stability results can be obtained by searching over wider classes of FIR Zames-Falb multipliers [5], [6], [20]. The equivalent Lyapunov parametrization remains open.

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# Appendix

## A. Proof of Theorem 2

Proof: In brief, we prove a LLF candidate $V$ is positive if (9) holds, and its difference $\Delta V$ is negative if (10) holds. Hence, it is sufficient for absolute stability.

Here, we denote $M_r = \mathrm{diag}(m_r^1, m_r^2, \cdots, m_r^{m_p})$ and $N_r = \mathrm{diag}(n_r^1, n_r^2, \cdots, n_r^{m_p})$, where all elements are positive.

First, we consider the LLF candidate

$$\begin{aligned} &V\left(x_i, \cdots, x_{i+(n-1)}, \phi(y_i), \cdots, \phi\left(y_{i+(n-1)}\right)\right) = \\ &\quad V_0\left(x_i, \cdots, x_{i+(n-1)}, \phi(y_i), \cdots, \phi\left(y_{i+(n-1)}\right)\right) \\ &\quad + V_1(x_i, \phi(y_i)) + V_2(x_i, x_{i+1}, \phi(y_i), \phi(y_{i+1})) \\ &+ \cdots + V_n\left(x_i, \cdots, x_{i+(n-1)}, \phi(y_i), \cdots, \phi\left(y_{i+(n-1)}\right)\right). \end{aligned} \tag{16}$$

Henceforth, the variables of $V$ are omitted for simplicity. In the LLF, $V^0 = \eta_i^\top P \eta_i$, and the augmented state is

$$\eta_i = [x_i^\top \ \cdots \ x_{i+(n-1)}^\top \ \phi(y_i)^\top \ \cdots \ \phi\left(y_{i+(n-1)}\right)^\top]^\top.$$

In addition, for $r = 1, 2, \cdots, n$,

$$\begin{aligned} V_r = &\, 2\sum_{j=1}^{m_p} m_r^j \int_0^{y_i^j} \phi^j(\sigma)^\top d\sigma + 2\sum_{j=1}^{m_p} m_r^j \int_0^{y_{i+1}^j} \phi^j(\sigma)^\top d\sigma \\ &+ \cdots + 2\sum_{j=1}^{m_p} m_r^j \int_0^{y_{i+(r-1)}^j} \phi^j(\sigma)^\top d\sigma \\ &+ 2\sum_{j=1}^{m_p} n_r^j \int_0^{y_i^j} \left\{\psi^j \sigma - \phi^j(\sigma)\right\}^\top d\sigma \\ &+ 2\sum_{j=1}^{m_p} n_r^j \int_0^{y_{i+1}^j} \left\{\psi^j \sigma - \phi^j(\sigma)\right\}^\top d\sigma \\ &+ \cdots + 2\sum_{j=1}^{m_p} n_r^j \int_0^{y_{i+(r-1)}^j} \left\{\psi^j \sigma - \phi^j(\sigma)\right\}^\top d\sigma. \end{aligned} \tag{17}$$

By Lemma 1, the lower bound of $V_r$ is given by

$$\begin{aligned} V_r \geq &\, y_i^\top N_r \Psi y_i + y_{i+1}^\top N_r \Psi y_{i+1} + \cdots + y_{i+(r-1)}^\top N_r \Psi y_{i+(r-1)} \\ &- \mathrm{He}\left\{\phi(y_i)^\top N_r y_i\right\} - \mathrm{He}\left\{\phi(y_{i+1})^\top N_r y_{i+1}\right\} \\ &- \cdots - \mathrm{He}\left\{\phi(y_{i+(r-1)})^\top N_r y_{i+(r-1)}\right\} \\ &+ \phi(y_i)^\top Q_r \Gamma^{-1} \phi(y_i) + \phi(y_{i+1})^\top Q_r \Gamma^{-1} \phi(y_{i+1}) \\ &+ \cdots + \phi(y_{i+(r-1)})^\top Q_r \Gamma^{-1} \phi(y_{i+(r-1)}). \end{aligned} \tag{18}$$

After some calculation, the lower bounds sum up as

$$V_1 + V_2 + \cdots + V_n \geq \eta_i^\top \Xi \eta_i, \tag{19}$$

where $\Xi$ is in (12). Hence, together with the quadratic term, the LLF (16) is bounded quadratically as $V \geq \eta_i^\top (P + \Xi)\eta_i$, and it is positive for any nonzero $\eta_i$ if and only if (9) holds.

Next, we consider the difference of $V$,

$$\Delta V = \Delta V_0 + \Delta V_1 + \Delta V_2 + \cdots + \Delta V_n, \tag{20}$$

where $\Delta V_0 = \eta_{i+1}^\top P \eta_{i+1} - \eta_i^\top P \eta_i$, and for $r = 1, 2, \cdots, n$,

$$\begin{aligned} \Delta V_r = &\, 2\sum_{j=1}^{m_p} m_r^j \int_{y_i^j}^{y_{i+r}^j} \phi^j(\sigma)^\top d\sigma \\ &+ 2\sum_{j=1}^{m_p} n_r^j \int_{y_i^j}^{y_{i+r}^j} \left\{\psi^j \sigma - \phi^j(\sigma)\right\}^\top d\sigma. \end{aligned} \tag{21}$$

In $\Delta V_0$, the augmented state $\eta_i$ satisfies the state-space representation $\eta_{i+1} = \widehat{A}\eta_i + \widehat{B}\phi(y_{i+n})$, where the pair $(\widehat{A}, \widehat{B})$ are in (11), so we have

$$\eta_{i+1}^\top P \eta_{i+1} - \eta_i^\top P \eta_i = \zeta_i^\top \begin{bmatrix} \widehat{A}^\top P \widehat{A} - P & \widehat{A}^\top P \widehat{B} \\ \widehat{B}^\top P \widehat{A} & \widehat{B}^\top P \widehat{B} \end{bmatrix} \zeta_i,$$

where $\zeta_i = [x_i^\top \ x_{i+1}^\top \ \cdots \ x_{i+(n-1)}^\top \ \phi(y_i)^\top \ \phi(y_{i+1})^\top \ \cdots \ \phi(y_{i+n})^\top]^\top$, and $x_{i+n}$ is expressed by

$$x_{i+n} = CAx_{i+(n-1)} - CB\phi(y_{i+(n-1)}). \tag{22}$$

The integral terms $\Delta V_1$ to $\Delta V_n$ above are bounded quadratically by Lemma 1, and some terms are added and subtracted at the same time in each term as discussed in Remark 2. Then, we have the general form

$$\begin{aligned} \Delta V_r \leq &\, y_{i+r}^\top N_r \Psi y_{i+r} - y_i^\top N_n \Psi y_i \\ &+ 2\phi(y_{i+r})^\top M_r (y_{i+r} - y_i) - 2\phi(y_i)^\top N_r (y_{i+r} - y_i) \\ &- \left[\phi(y_{i+r}) - \phi(y_i)\right]^\top Q_r \Gamma^{-1} \left[\phi(y_{i+r}) - \phi(y_i)\right] \\ &+ y_{i+1}^\top (N_r - N_r)\Psi y_{i+1} + y_{i+2}^\top (N_r - N_r)\Psi y_{i+2} \\ &+ \cdots + y_{i+(r-1)}^\top (N_r - N_r)\Psi y_{i+(r-1)} \\ &+ \phi(y_{i+1})^\top (N_r - N_r) y_{i+1} + \phi(y_{i+2})^\top (N_r - N_r) y_{i+2} \\ &+ \cdots + \phi(y_{i+r})^\top (N_r - N_r) y_{i+r} \\ &+ \phi(y_{i+1})^\top (Q_r - Q_r)\phi(y_{i+1}) + \phi(y_{i+2})^\top (Q_r - Q_r)\phi(y_{i+2}) \\ &+ \cdots + \phi(y_{i+r})^\top (Q_r - Q_r)\phi(y_{i+r}). \end{aligned} \tag{23}$$

Meanwhile, the sector condition (3a) implies that

$$2\phi^\top(y_i) L \left[y_i - \Psi^{-1}\phi(y_i)\right] \geq 0 \tag{24}$$

holds with a positive diagonal matrix $L$.

Involving the upper bounds of $\Delta V_0$ and $\Delta V_r$ for $r = 1, 2, \cdots, n$, and involving the sector condition (24), by complicated but straightforward calculation, we have

$$\Delta V \leq \zeta_i^\top \left( \begin{bmatrix} \widehat{A}^\top (P+\Xi)\widehat{A} - (P+\Xi) & \widehat{A}^\top (P+\Xi)\widehat{B} \\ \widehat{B}^\top (P+\Xi)\widehat{A} & \widehat{B}^\top (P+\Xi)\widehat{B} \end{bmatrix} + \Omega \right) \zeta_i, \tag{25}$$

where $x_{i+n}$ is expressed by (22), and $\Omega$ is given in (13). Hence, $\Delta V < 0$ if (10) holds.

In summary, (9) and (10) is sufficient for asymptotic stability [21, Theorem 13.2]. ■