Absolute stability of systems with integrator and/or time delay via off-axis circle criterion

Jingfan Zhang, Harun Tugal, Joaquin Carrasco, and William P. Heath

Abstract—Graphical methods are a key tool to analyse Lur'e systems with time delay. In this paper we revisit clockwise properties of the Nyquist plot and extend results in the literature to critically stable systems and time-delayed systems. It is known that rational transfer functions with no resonant poles and no zeros satisfy the Kalman conjecture. We show that the same class of transfer functions in series with a time delay also satisfies the Kalman conjecture. Furthermore the same class of transfer functions in series with an integrator and delay (which may be zero) satisfies a suitably relaxed form of the Kalman conjecture. Useful results are also obtained where the delay is constant but unknown. Results in this paper can be used as benchmarks to test sufficient stability conditions for the Lur'e problem with time-delay systems.

Index Terms—Stability of nonlinear systems; Delay systems.

I. INTRODUCTION

THE Lur'e problem [1] consists of analysing the stability of the feedback interconnection between an LTI system and any nonlinearity within a class of systems. To highlight that stability is guaranteed for a class of system, the term *absolute stabilty* was coined; see [2] for an early review of the topic. Although originally the Lur'e problem was established using asymptotic stability of the origin, the definition of the Lur'e problem has evolved to accommodate different stability definitions as they have been proposed [3]. In this paper, we focus on input-output stability as defined in [4]–[6].

The exploration of sufficient stability conditions has become a core topic in control theory. The Lur'e system is shown in Fig. 1, which is the negative feedback interconnection between an LTI plant and a nonlinearity that belongs to a class of nonlinearities, and where the injected signal f can be treated as a disturbance or a signal that generates the initial condition of the plant [7]. One of the classical problems is to find frequency domain conditions on the LTI stable plant G such that the feedback interconnection between G and any nonlinearity ψ within a slope-restricted sector $[0,k]$ is stable, e.g. [8].

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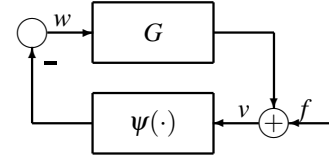
Fig. 1: Input-output version for the Lur'e system to consider crically stable systems [7]. When G stable, it is straightforward to show that this is the most general input-output configuration.

An early solution to the Lur'e problem for some plants was given by Popov [9]. Although the Popov criterion was developed for finite dimensional systems, Răsvan extended Popov's method to time-delay and critical plants, e.g. [10], [11] and references therein. Recently, there has been some attention to the case when G is a plant with an unknown but constant time delay. Several different techniques can be used to determine stability such as Lyapunov-Krasovskii functionals [12], quadratic separation, and integral quadratic constraints (IQC) [13], [14]. In addition, IQC stability conditions for Lur'e systems with critically stable plants have been explored in [7], [15]–[18]. These are of practical significance because such systems are common in applications such as servo systems with saturating PID control and phase-locked loops with nonlinear phase detectors [19]. However, the performance of these stability techniques are unclear, due to the lack of benchmark examples where necessary and sufficient conditions for absolute stability can be determined.

In this paper, we are concerned with the development of a class of systems where such necessary and sufficient conditions can be obtained. In particular, we focus on Lur'e systems where the linear plant takes the form $G(s) = G_n(s)e^{-sT_d}$ and $G(s) = \frac{G_n(s)}{s}e^{-sT_d}$ (Fig. 2), and we will propose conditions on G_n that allow us to ensure that G satisfies either the Kalman conjecture or a suitably relaxed version of the Kalman conjecture in the case where G is critically stable. We propose such plants as benchmarks to judge the performance of sufficient conditions developed by other stability methods.
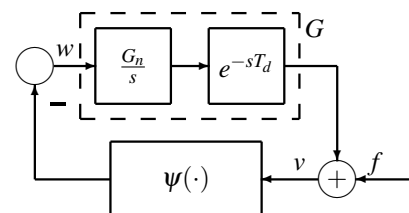

Fig. 2: Lur'e system with integrator and time delay.

For time delayed systems, analytic methods have been used by [20] and [21], but their feedback structure is different from this paper. Some analytic tools are also available for some low order unstable systems [22]. Graphical methods provide a significant advantage for the structure in Fig. 2 as it is not possible to obtain a closed-form for the Nyquist value in general. Therefore, we will deliberate the clockwise properties of the Nyquist plot, which have been studied in [23]–[28]. Particularly, in [26], a class of rational plants are proved that satisfy the Kalman conjecture. The main contribution of this paper is to show that this set of plants together with an integrator and/or time delay also satisfies the Kalman conjecture (or relaxed version).

There are several conventional methods to design compensators for linear time-delayed plants such as the Smith predictor [29] and the Dahlin controller [30]. These conventional methods cannot guarantee stability and robustness with respect to actuator saturation. The method presented in this paper can guarantee stability and robustness for plants with time-delays and saturation, and does not require a perfect knowledge of the delay as in the Smith predictor. Initial design results are presented in [31].

Similar properties have been used to develop classes of plants satisfying the Kalman conjecture in [23], [32] and references therein. In [23], critically stable systems were considered, but the second order derivative of the phase was missed in the proof, which led to incorrect results. In [32] time-delayed plants were introduced; here we provide results for a wider class of plants.

II. NOTATION AND PRELIMINARY RESULTS

A. Lur'e system and off-axis circle criterion (OACC)

Let $\mathbf{RH}_\infty$ be the space consisting of proper real rational transfer functions with no pole in the closed right-half complex plane. A minimal state space realisation of the transfer function is $G(s) = C(sI - A)^{-1}B + D$. We use the notation $\Re\{s\}$ and $\Im\{s\}$ represent the real part and imaginary part of the complex number s, respectively.

Let $\mathbb{R}^+$ be the set of non-negative real numbers, and $\mathscr{L}_2(\mathbb{R}^+)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$. A truncation of the function f at $T \in \mathbb{R}$ is given by $f_T(t) = f(t)$, $\forall t \leq T$; $f_T(t) = 0$, $\forall t > T$. The function f belongs to the extended space $\mathscr{L}_{2e}(\mathbb{R}^+)$ if $f_T \in \mathscr{L}_2(\mathbb{R}^+)$ for all $T > 0$.

A nonlinearity $\psi : \mathscr{L}_{2e}(\mathbb{R}^+) \mapsto \mathscr{L}_{2e}(\mathbb{R}^+)$ is said to be memoryless if there exists a map $N : \mathbb{R} \to \mathbb{R}$ such that $(\psi(\upsilon))(t) = N(\upsilon(t))$, $\forall t \in \mathbb{R}$. The nonlinearity ψ is said to be slope restricted, denoted by $\psi \in S[0,k]$, if

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq k, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2. \tag{1}$$

A slope restricted nonlinearity is also sector bounded, but the reverse is not necessarily true.

The Lur'e system in Fig. 1 is given by

$$\begin{aligned} v &= f + Gw, \\ w &= -\psi(v) \end{aligned}$$

where the signals are in the extended space $\mathscr{L}_{2e}(\mathbb{R}^+)$. The feedback interconnection is said to be well-posed if the map $(v, w) \mapsto (0, f)$ has a causal inverse on $\mathscr{L}_{2e}(\mathbb{R}^+)$. Additionally, the interconnection is said to be $\mathscr{L}_2$-stable if the inverse operation is bounded, i.e. $||w_T|| + ||v_T|| < \gamma ||f_T||$ with a positive γ. For the system with an integrator, $\mathscr{L}_2$-stability only implies that the output from the critically stable plant converges to any feasible equilibrium point of the nonlinearity when $t \to \infty$ (see [7]). Henceforth, we use the term weakly $\mathscr{L}_2$-stable for this case.

The Kalman conjecture and the off-axis circle criterion are given as follows.

Definition 1: (Nyquist value, k_N) The Nyquist value of a stable transfer function $G(s)$ is

$$k_N = \sup_k \{k > 0 : (1 + \tau k G(s))^{-1} \text{ is stable } \forall \tau \in [0,1]\}.$$

Conjecture 1 (Kalman conjecture [34]): Let ψ be a memoryless nonlinearity, and $\psi \in S[0,k]$. The feedback interconnection between G and ψ is asymptotically stable if $k < k_N$.

Theorem 1 (Off-axis circle criterion (OACC) [35]): Let $G \in \mathbf{RH}_\infty$ be a nominal system in a feedback interconnection with a slope restricted nonlinearity $\psi \in S[0,k]$. If the Nyquist curve of the nominal system $G(j\omega)$ lies entirely to the right of a straight line with non-zero slope passing through the point $(-\frac{1}{k} - \varepsilon, 0)$ with $\varepsilon > 0$, then the proposed feedback interconnection is $\mathscr{L}_2$-stable.

B. Clockwise properties

Following [36], let the magnitude and phase of a transfer function G at frequency ω be $M(\omega)$ and $\Phi(\omega)$ respectively. Let Γ be the Nyquist curve of G in the complex plane, defined by two parametric equations

$$X(\omega) = \Re\{G(j\omega)\} = \frac{\cos\phi(\omega)}{m(\omega)} \tag{2}$$

$$Y(\omega) = \Im\{G(j\omega)\} = \frac{-\sin\phi(\omega)}{m(\omega)}, \tag{3}$$

where $\phi(\omega) = -\arg[G(j\omega)] = -\Phi(\omega)$, $m(\omega) = |G(j\omega)|^{-1} = M(\omega)^{-1}$. The curvature of Γ is defined as

$$\begin{aligned} \mathscr{C}(\omega) &= \frac{X_\omega Y_{\omega\omega} - X_{\omega\omega} Y_\omega}{(X_\omega^2 + Y_\omega^2)^{\frac{3}{2}}} \\ &= \frac{-\left(\frac{\cos\phi}{m}\right)_\omega \left(\frac{\sin\phi}{m}\right)_{\omega\omega} + \left(\frac{\cos\phi}{m}\right)_{\omega\omega} \left(\frac{\sin\phi}{m}\right)_\omega}{\left[\left(\frac{\cos\phi}{m}\right)_\omega^2 + \left(\frac{\sin\phi}{m}\right)_\omega^2\right]^{\frac{3}{2}}}. \end{aligned} \tag{4}$$

It can be simplified as

$$\mathscr{C}(\omega) = -\frac{\phi_\omega^3 + \phi_\omega \tilde{m}_{\omega\omega} - \phi_{\omega\omega} \tilde{m}_\omega}{(\phi_\omega^2 + \tilde{m}_\omega^2)^{\frac{3}{2}}} m \tag{5}$$

where $\tilde{m}_\omega = \frac{m_\omega}{m}$, $\tilde{m}_{\omega\omega} = \frac{m_{\omega\omega}}{m}$.

The Nyquist plot at a frequency ω is clockwise if $\mathscr{C}(\omega) < 0$. A system is said to have a clockwise Nyquist plot if it is clockwise at all frequencies. The sign of the curvature can be derived by analysing the numerator in (5), i.e.

$$n(\omega) = \phi_\omega^3 + \phi_\omega \tilde{m}_{\omega\omega} - \phi_{\omega\omega} \tilde{m}_\omega. \tag{6}$$

Its terms can be expressed as the sums

$$\tilde{m}_\omega = \sum_i \tilde{m}_{i\omega}, \quad \tilde{m}_{\omega\omega} = \sum_i \tilde{m}_{i\omega\omega} + \sum_{i,j}^{i\neq j} \tilde{m}_{i\omega}\tilde{m}_{j\omega};$$
$$\phi_\omega = \sum_i \phi_{i\omega}, \quad\quad \phi_{\omega\omega} = \sum_i \phi_{i\omega\omega}, \quad\quad (7)$$

where i, j indicates the real or complex link in G, see [26] for definition of link. Assume the plant G is comprised by n_r real links and n_c complex links as

$$G_i(s) = \frac{1}{s+r_i} \quad (i=1,\cdots,n_r),$$

$$G_i(s) = \frac{1}{s^2 + 2\zeta_{i-n_r}\omega_{i-n_r}\omega + \omega_{i-n_r}^2} \quad (i=n_r+1,\cdots,n_r+n_c).$$

The related functions are listed in Table 1 in [26].

Definition 2: Let G be a rational transfer function. G is said to belong to the subset $\mathscr{G}_c$ if

1) the poles of G are either real or complex conjugate with damping factor greater than $\frac{1}{\sqrt{2}}$;
2) G has no zero.

Lemma 1 ([26]): If $G \in \mathscr{G}_c$, then the Nyquist plot of G is clockwise at all frequencies, and G satisfies the Kalman conjecture.

C. Stability for systems with an integrator

In Theorem 1, G is required to be bounded, but if G has a pole at 0 then $G \notin \mathbf{RH}_\infty$. In the following part, we revisit stability results in [7], [16] for Lur'e systems with critically stable plants.

First, the integrator should be encapsulated in the nonlinearity by the loop transformation from Fig. 1 to Fig. 3 [7]. In Fig. 3, G_{n0} and κ are obtained from the partial fractional expansion $\frac{-G_n}{s} = \kappa(G_{n0} - \frac{1}{s})$, where $\kappa = G_n(0) > 0$.

Then, following [7], the operator Δ_ψ is bounded on $\mathscr{L}_2(\mathbb{R}^+)$ with the maximum gain $k\kappa$, and a class of modified Zames-Falb multiplier is given to preserve its positivity. For a proper definition of the class of Zames-Falb multipliers $\mathscr{M}$, see [6], [37]. Following [7], the stability condition can be expressed in terms of the slope of the nonlinearity as follows:

Theorem 2 ([7]): For the Lur'e system with the plant $\frac{G_n}{s}$ where $G_n \in \mathbf{RH}_\infty$ and the nonlinearity $\psi \in S[0,k]$, if

1) $G_n(0) > 0$;
2) there exist $\varepsilon > 0$ and a Zames-Falb multiplier $M(j\omega) \in \mathscr{M}$, such that

$$\Re\left\{ \left(\frac{G_n(j\omega)}{j\omega} + \frac{1}{k} \right) M(j\omega) \right\} \geq \varepsilon \quad \forall \omega \neq 0,$$

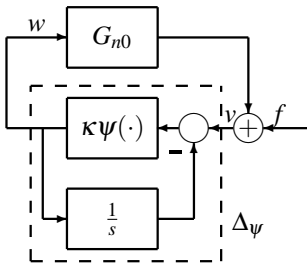then the system is weakly $\mathscr{L}_2$-stable.

Fig. 3: Encapsulation of the integrator

The Nyquist value can be trivially modified so stability is not required for $k = 0$.

Definition 3 (Modified Nyquist value, $\tilde{k}_N$): The Nyquist value of a critically stable transfer function $G(s)$ is $\tilde{k}_N = \sup_k\{k > 0 : (1 + \tau k G(s))^{-1}$ is stable $\forall \tau \in (0,1]\}$.

Although it is not explicitly mentioned in [7], the condition $G_n(0) > 0$ can be taken without loss of generality for SISO systems; see Theorem 2 in [38]. In short, if $G_n(0) < 0$, then $\tilde{k}_N = 0$.

Corollary 1: For a plant $\frac{G_n}{s}$ where $G_n \in \mathbf{RH}_\infty$ and $G_n(0) > 0$, the negative feedback interconnection between G and any nonlinearity $\psi \in S[0,k]$ is weakly $\mathscr{L}_2$-stable if the Nyquist plot of $G(j\omega)$ is clockwise for all $\omega > 0$.

Proof: The proof follows the same arguments as the original proof in [35], but the multiplier may require some further positive phase as $\omega \to 0$. Nonetheless as $G_n \in \mathbf{RH}_\infty$ then it is always possible to choose the first zero of the RL multiplier low enough to compensate the phase of the slowest poles of the system as the phase of $1 + kG$ will approach $-\pi/2 - r\omega$ for some finite value of r as $\omega \to 0$. ∎

Remark 1: This simple result provides a significant insight to the class of multipliers required for absolute stability of critically stable systems. The properties as $\omega \to 0$ restrict us to RL multipliers (with positive phase) in the construction of the off-axis circle criterion.

This analysis suggests a suitable modification of the Kalman conjecture appropriate for critically stable systems:

Conjecture 2 (Relaxed Kalman conjecture): Let ψ be a memoryless nonlinearity, and $\psi \in S[0,k]$ with $\psi(x) \neq 0$ if $x \neq 0$. The feedback interconnection between $G = \frac{G_n}{s}$ and ψ is stable if $k < \tilde{k}_N$.

Remark 2: The classical literature [39], [40] considers other relaxations where strict inequalities in (1) are used. Such a relaxation may accommodate the critical case, however they may lead to pathological cases (see [40] for a discussion of the Aizerman conjecture).

III. ABSOLUTE STABILITY OF SYSTEMS WITH TIME DELAY

In this section, the clockwise properties are extended to systems with constant time delays. Next, the absolute stability of time-delayed systems is discussed.

A. Clockwise properties of systems with time delay

Lemma 2: The set of time-delayed systems $G(s) = G_n(s)e^{-sT_d}$ ($T_d \in \mathbb{R}^+$) have clockwise Nyquist plots when $G_n \in \mathscr{G}_c$.

Proof: For any given $T_d \geq 0$, the real and complex parts of $G(j\omega)$ are given by

$$X(\omega) = \frac{\cos(\phi + \omega T_d)}{m}, \quad Y(\omega) = \frac{-\sin(\phi + \omega T_d)}{m}.$$

where $\phi(\omega) = -\arg[G_n(j\omega)]$ and $m(\omega) = |G(j\omega)|^{-1} = |G_n(j\omega)|^{-1}$.

By straightforward calculation using (4), the curvature of a time delayed system is

$$\mathscr{C}(\omega) = -\frac{\left(\phi_\omega^3 + \phi_\omega \tilde{m}_{\omega\omega} - \phi_{\omega\omega}\tilde{m}_\omega\right)}{\left(\phi_\omega^2 + \tilde{m}_\omega^2 + T_d^2 + 2T_d\phi_\omega\right)^{\frac{3}{2}}}m$$
$$- \frac{\left(T_d^3 + T_d\tilde{m}_{\omega\omega}\right) + \left(3T_d\phi_\omega^2 + 3T_d^2\phi_\omega\right)}{\left(\phi_\omega^2 + \tilde{m}_\omega^2 + T_d^2 + 2T_d\phi_\omega\right)^{\frac{3}{2}}}m. \tag{8}$$

The plant G has a clockwise Nyquist curve if its curvature is less than zero, or equivalently the numerator in (8) is positive, i.e.

$$n(\omega) = \left(\phi_\omega^3 + \phi_\omega\tilde{m}_{\omega\omega} - \phi_{\omega\omega}\tilde{m}_\omega\right)$$
$$+ \left(T_d^3 + T_d\tilde{m}_{\omega\omega}\right) + \left(3T_d\phi_\omega^2 + 3T_d^2\phi_\omega\right) > 0 \quad \forall \omega \geq 0. \tag{9}$$

It is clear the term in the first bracket corresponds to the curvature of the nominal system, which is positive. Meanwhile, according to (7) and Table 1 in [26], ϕ_ω and $\tilde{m}_{\omega\omega}$ are both positive for $G \in \mathscr{G}_c$. Hence, the Nyquist plot of G is clockwise. ∎

B. OACC of systems with time delay

With the geometrical properties of time-delayed systems where $G_n \in \mathscr{G}_c$, the absolute stability problem is extended to this set of plants with a range of time delays using the OACC in Theorem 1.

Theorem 3: Let $G = G_n e^{-sT_d}$ with $G_n \in \mathscr{G}_c$ and $T_d \in \mathbb{R}^+$; then G satisfies the Kalman conjecture.

Proof: The theorem follows the same arguments that were used in Theorem 2 in [26]. ∎

A few numerical examples are provided in Appendix, where the feedback interconnections with the strictly stable plant $G = \frac{e^{-3s}}{(s+1)(s^2+1.5s+1)^2}$ with four nonlinearities $\psi \in S[0, k_N - 0.0001]$ are shown to be $\mathscr{L}_2$-stable respectively.

Moreover, we can show that the same multiplier which is used for a delay T_d is also valid for any delay τT_d ($0 < \tau \leq 1$).

Lemma 3: Consider a feedback interconnection with a time-delayed plant $G_1(j\omega) = G_n(j\omega)e^{-j\omega T_d}$ ($G_n \in \mathscr{G}_c$) and a slope-restricted nonlinearity $\psi \in S[0, k]$. Assume the Nyquist plot of $G_1(j\omega)$ is on the right of a straight line, L, with non-zero slope passing through the point $(1/k\varepsilon, 0)$ with $\varepsilon > 0$. Then, any $G_2(j\omega) = G_n(j\omega)e^{-j\omega\tau T_d}$ ($\tau \in [0, 1)$) is also on the right of L.
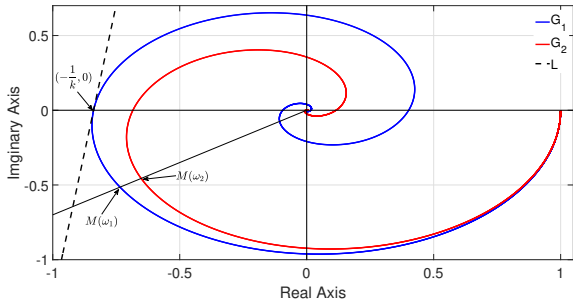
Proof: As stated, the Nyquist plot of G_1, denoted by Γ_1, is entirely on the right side of the non-horizontal line, say L, which goes through $(-\frac{1}{k} - \varepsilon, 0)$, where $\varepsilon > 0$. Hence, the Nyquist plot of G_2, denoted by Γ_2, should not cross L. Alternatively, at the same phase, if the magnitude of G_2 is smaller than that of G_1, Γ_2 will be inside Γ_1, and will not cross L (see an example in Fig. 4).

Firstly, let ω_1 and ω_2 be frequencies such that the phase of G_1 at ω_1 is equal to the phase of G_2 at ω_2, i.e. $\bar{\phi} = -\arg[G_1(j\omega_1)] = -\arg[G_2(j\omega_2)]$. Equivalently,

$$\bar{\phi} = \phi(\omega_1) + T_d\omega_1 = \phi(\omega_2) + \tau T_d\omega_2. \tag{10}$$

The special case is at $\omega_1 = \omega_2 = 0$, where the magnitude of both systems are the same. For general cases, $\omega_1 \neq \omega_2$. Subtracting $\tau T_d\omega_1$ from both sides of (10), and rearranging the terms,

$$\phi(\omega_2) - \phi(\omega_1) + \tau T_d(\omega_2 - \omega_1) = (T_d - \tau T_d)\omega_1. \tag{11}$$

Dividing $(\omega_2 - \omega_1)$ on both sides,

$$\frac{\phi(\omega_2) - \phi(\omega_1)}{\omega_2 - \omega_1} + \tau T_d = \frac{T_d - \tau T_d}{\omega_2 - \omega_1}\omega_1. \tag{12}$$

According to Table 1 in [26] and (7), $\phi(\omega)$ is monotonically increasing with respect to ω, then $\frac{\phi(\omega_2) - \phi(\omega_1)}{\omega_2 - \omega_1} > 0$. Hence, $\frac{T_d - \tau T_d}{\omega_2 - \omega_1} > 0$, then $\omega_2 > \omega_1$. Finally, because $M(\omega)$ is monotonically decreasing with respect to ω, $M(\omega_1) > M(\omega_2)$, and Γ_2 is 'inside' Γ_1. Therefore, the interconnection with G_2 and $\psi \in S[0, k]$ is stable by the OACC. ∎

With the above lemma, now we can establish a straightforward condition for absolute stability:

Corollary 2: Let $G = G_n e^{-sT_d}$ with $G_n \in \mathscr{G}_c$ and $T_d \in \mathbb{R}^+$. Let k_N be the Nyquist value of G. Then for any $\tau \in [0, 1]$, the feedback interconnection $G_\tau = G_n e^{-s\tau T_d}$ and $\psi \in S[0, k]$ is absolutely stable if and only if $k < k_N$.

IV. ABSOLUTE STABILITY OF SYSTEMS WITH INTEGRATOR AND TIME DELAY

As discussed in Section II Part B, the OACC is also available for critically stable plants. Hence, the clockwise properties are also important for this class of plants. Similar to Section III, the clockwise properties are studied first, and then the absolute stability is discussed.

A. Clockwise properties of systems with integrator and time delay

In this section, the clockwise properties of critically stable systems are proved first, and then the delay is added.

Lemma 4: The set of critically stable systems $G(s) = \frac{G_n(s)}{s}$ have clockwise Nyquist plots when $G_n \in \mathscr{G}_c$.

Proof: By the same method and notations, the real and complex parts of $G(j\omega)$ are

$$X(\omega) = \frac{\cos(\phi + \frac{\pi}{2})}{m\omega} = \frac{-\sin(\phi)}{m\omega}, \tag{13}$$

$$Y(\omega) = \frac{-\sin(\phi + \frac{\pi}{2})}{m\omega} = \frac{-\cos(\phi)}{m\omega}. \tag{14}$$
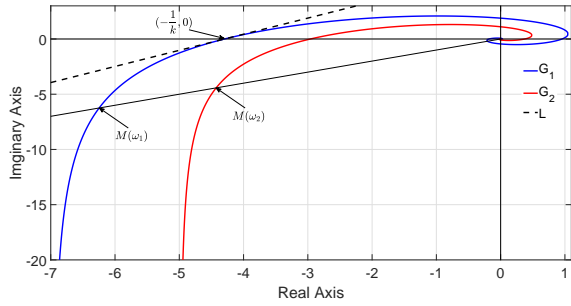

Fig. 4: Nyquist plots of $G_1 = \frac{e^{-3s}}{(s+1)(s^2+1.5s+1)^2}$ and $G_2 = \frac{e^{-s}}{(s+1)(s^2+1.5s+1)^2}$.

The curvature of $\frac{G_n(s)}{s}$ is

$$\mathscr{C}(\omega) = -\frac{\omega\left(\phi_\omega^3 + \phi_\omega\tilde{m}_{\omega\omega} - \phi_{\omega\omega}\tilde{m}_\omega\right) + 2\tilde{m}_\omega\phi_\omega - \phi_{\omega\omega}}{\left(\phi_\omega^2 + m_\omega^2 + 2\frac{\tilde{m}_\omega}{\omega} + \frac{1}{\omega^2}\right)^{\frac{3}{2}}}m. \quad (15)$$

Then, the curvature $\mathscr{C}(\omega)$ is negative for all frequencies if the numerator in (15) is positive, i.e.

$$n(\omega) = \omega\left(\phi_\omega^3 + \phi_\omega\tilde{m}_{\omega\omega} - \phi_{\omega\omega}\tilde{m}_\omega\right) + 2\tilde{m}_\omega\phi_\omega - \phi_{\omega\omega} > 0. \quad (16)$$

for all $\omega > 0$. The first term corresponds to the curvature of the nominal system G_n, which is positive. However, the remaining parts are positive when $\zeta > \sqrt{5/8}$ [26], which is larger than $\frac{1}{\sqrt{2}}$. Hence, $n(\omega)$ should be expressed in detail as

$$n(\omega) = \omega\left[\left(\sum_j \phi_{j\omega}\right)^3 + \left(\sum_i \phi_{j\omega}\right)\left(\sum_j \tilde{m}_{j\omega\omega} + \sum_{j,k}^{j\neq k} \tilde{m}_{j\omega}\tilde{m}_{k\omega}\right) \right.$$
$$\left. -\left(\sum_i \tilde{m}_{i\omega}\right)\left(\sum_j \phi_{j\omega\omega}\right)\right] + 2\left(\sum_i \tilde{m}_{i\omega}\right)\left(\sum_j \phi_{j\omega}\right) - \left(\sum_j \phi_{j\omega\omega}\right),$$

which can be simplified as

$$n(\omega) = \sum_i n_i(\omega) + \sum_{i,j}^{i<j}\Lambda_{ij}(\omega)$$
$$+ \omega\sum_{i,j,k}^{i\neq j\neq k}\left(\tilde{m}_{i\omega}\tilde{m}_{j\omega}\phi_{k\omega} + \phi_{i\omega}\phi_{j\omega}\phi_{k\omega}\right), \quad (17)$$

where

$$n_i(\omega) = \omega\left(\phi_{i\omega}^3 + \phi_{i\omega}\tilde{m}_{i\omega\omega} - \phi_{i\omega\omega}\tilde{m}_{i\omega}\right) + 2\tilde{m}_{i\omega}\phi_{i\omega} - \phi_{i\omega\omega}, \quad (18)$$

$$\Lambda_{ij}(\omega) = 3\omega\phi_{i\omega}\phi_{j\omega}\left(\phi_{i\omega} + \phi_{j\omega}\right) + \omega\tilde{m}_{i\omega}\left(2\phi_{j\omega}\tilde{m}_{j\omega} - \phi_{j\omega\omega}\right)$$
$$+ \omega\tilde{m}_{j\omega}\left(2\phi_{i\omega}\tilde{m}_{i\omega} - \phi_{i\omega\omega}\right) + \omega\tilde{m}_{i\omega\omega}\phi_{j\omega}$$
$$+ \omega\tilde{m}_{j\omega\omega}\phi_{i\omega} + 2\tilde{m}_{i\omega}\phi_{j\omega} + 2\tilde{m}_{j\omega}\phi_{i\omega}. \quad (19)$$

Firstly, we are concerned with the term $n_i(\omega)$ in (18). According to Table 1 in [26], assume G_i is a real link,

$$n_i(\omega) = 6\omega r_i m_i^{-4},$$

which is positive for any r_i; assume G_i is a complex link,

$$n_i(\omega) = 8\zeta_i\omega_i\omega\left[3\omega_i^4\left(2\zeta_i^2 - 1\right) + 2\omega_i^2\omega^2\left(\zeta_i^2 + 2\right) + 3\omega^4\right]m_i^{-4},$$

which is positive for $\zeta_i > \frac{1}{\sqrt{2}}$.

Secondly, for the term $\Lambda_{ij}(\omega)$ in (19), it rewrites as

$$\Lambda_{ij}(\omega) = \omega\Lambda_{ij}^*(\omega) + 2\tilde{m}_{j\omega}\phi_{i\omega} + 2\tilde{m}_{i\omega}\phi_{j\omega}, \quad (20)$$

where the properties of $\Lambda_{ij}^*(\omega)$ are studied in [26] for $G_n \in \mathscr{G}_c$. Here, the possible terms to break the clockwise properties are those with $\phi_{i(j)\omega\omega}$ that is negative when $\zeta_i > \frac{\sqrt{3}}{2}$.

Assume G_i and G_j are both real links; it is straightforward that $\Lambda_{ij}(\omega)$ is positive, because $\phi_{i(j)\omega\omega}$ is negative as shown in Table 1 in [26].

Assume G_i and G_j are both complex links with $\zeta_i > \frac{1}{\sqrt{2}}$ and $\zeta_j > \frac{1}{\sqrt{2}}$. Via Lemma 3 in [26], the term $\Lambda_{ij}^*(\omega) > 0$. The remaining terms are also positive as shown in Table 1 in [26]. Hence, $\Lambda_{ij}(\omega)$ is positive for two complex links.

Assume G_i is a complex link with $\zeta_i > \frac{1}{\sqrt{2}}$ and G_j is a real link. According to Lemma 4 in [26],

$$\Lambda_{ij}^*(\omega) + 2\tilde{m}_{j\omega}^2\phi_{i\omega} > 0. \quad (21)$$

Multiplying $\omega > 0$ on both sides, and rearranging,

$$\omega\Lambda_{ij}^*(\omega) + 2\tilde{m}_{j\omega}\phi_{i\omega}(\tilde{m}_{j\omega}\omega) > 0, \quad (22)$$

where $\tilde{m}_{j\omega}\omega = \frac{\omega^2}{m_j^2} = \frac{\omega^2}{r_i^2 + \omega^2} < 1$. Therefore,

$$2\tilde{m}_{j\omega}\phi_{i\omega} > 2\tilde{m}_{j\omega}\phi_{i\omega}(\tilde{m}_{j\omega}\omega), \text{ or } \omega\Lambda_{ij}^*(\omega) + 2\tilde{m}_{j\omega}\phi_{i\omega} > 0. \quad (23)$$

As a result, since the remaining terms in (19) are positive for $G_n \in \mathscr{G}_c$, $\Lambda_{ij}(\omega)$ is positive for all positive frequencies.

Finally, in (17), all the three terms are positive when $\omega > 0$ for $G_n \in \mathscr{G}_c$. The Nyquist plot of $G = \frac{G_n}{s}$ is clockwise for positive frequencies. $\blacksquare$

With the curvature of the delay-free plant, the time delay can be added by the same method as in Section III.

Lemma 5: The set of time delayed systems $G(s) = \frac{G_n(s)}{s}e^{-sT_d}$ $(T_d \in \mathbb{R}^+)$ have clockwise Nyquist plots if $G_n \in \mathscr{G}_c$.

Proof: Similarly, the real part and imaginary part of $G = \frac{G_n}{s}e^{-sT_d}$ $(T_d \in \mathbb{R}^+)$ are

$$X(\omega) = \frac{\cos(\phi + \frac{\pi}{2} + T_d\omega)}{m\omega} = \frac{-\sin(\phi + T_d\omega)}{m\omega},$$
$$Y(\omega) = \frac{-\sin(\phi + \frac{\pi}{2} + T_d\omega)}{m\omega} = \frac{-\cos(\phi + T_d\omega)}{m\omega}.$$

The curvature of $\frac{G_n(s)}{s}e^{-sT_d}$ is

$$\mathscr{C}(\omega) = -\frac{\omega\left(\phi_\omega^3 + \phi_\omega\tilde{m}_{\omega\omega} - \phi_{\omega\omega}\tilde{m}_\omega\right) + 2\tilde{m}_\omega\phi_\omega - \phi_{\omega\omega}}{\left((\phi_\omega + T_d)^2 + m_\omega^2 + 2\frac{\tilde{m}_\omega}{\omega} + \frac{1}{\omega^2}\right)^{\frac{3}{2}}}m$$
$$- \frac{\omega\left(T_d^3 + T_d\tilde{m}_{\omega\omega} + 3T_d\phi_\omega^2 + 3T_d^2\phi_\omega\right) + 2\tilde{m}_\omega T_d}{\left((\phi_\omega + T_d)^2 + m_\omega^2 + 2\frac{\tilde{m}_\omega}{\omega} + \frac{1}{\omega^2}\right)^{\frac{3}{2}}}m. \quad (24)$$

Similar to Section III, the first term in (24) is negative according to Lemma 4; the second term is negative by (7) and Table 1 in [26]. As a result, G has a clockwise Nyquist plot when $\omega > 0$ with $G_n \in \mathscr{G}_c$. $\blacksquare$

B. OACC of systems with integrator and time delay

Similarly, the OACC can be extended to time delay case for critical stable plants.

Theorem 4: Let $G = \frac{G_n}{s}e^{-sT_d}$ with $G_n \in \mathscr{G}_c$; then G satisfies the relaxed Kalman conjecture.

Remark 3: The case $T_d = 0$ establishes the counterpart of the result of Theorem 2 in [26] for critically stable systems.

Proof: By Lemma 4, G has a clockwise Nyquist plot when $G_n \in \mathscr{G}_c$. Meanwhile, both the magnitudes of G_n and $\frac{1}{s}$ are decreasing with respect to frequency. Hence, the geometrical properties satisfy the condition in [26]. On the other hand, $G_n(s) > 0$, which satisfies the condition in Theorem 2, so the multiplier theory is valid for this class of plants. Finally, by the same argument in [26], G satisfies the Kalman conjecture. $\blacksquare$

Similarly, a few numerical examples are provided in Appendix with the critically stable plant $G = \frac{e^{-3s}}{s(s+1)(s^2+1.5s+1)^2}$.

The simulation results indicates the feedback interconnections are weakly $\mathscr{L}_2$-stable.

Lemma 6: Consider a feedback interconnection with a time delayed plant $G_1(j\omega) = \frac{G_n(j\omega)}{j\omega}e^{-j\omega T_d}$ ($G_n \in \mathscr{G}_c$) and a slope restricted nonlinearity $\psi \in S[0,k]$. Assume the Nyquist plot of $G_1(j\omega)$ is on the right of a straight line, L, with non-zero slope passing through the point $(1/k\varepsilon, 0)$ with $\varepsilon > 0$. Then, any $G_2(j\omega) = \frac{G_n(j\omega)}{j\omega}e^{-j\omega\tau T_d}$ ($\tau \in [0,1)$) is also on the right of L (see an example in Fig. 5).

Fig. 5: Nyquist plots of $G_1 = \frac{e^{-3s}}{s(s+1)(s^2+1.5s+1)^2}$ and $G_2 = \frac{e^{-s}}{s(s+1)(s^2+1.5s+1)^2}$.

Similarly, we can establish the straightforward condition for absolute stability:

Corollary 3: Let $G = \frac{G_n}{s}e^{-sT_d}$ with $G_n \in \mathscr{G}_c$ and $T_d \in \mathbb{R}^+$. Let $\tilde{k}_N$ be the Nyquist value of G. Then for any $\tau \in [0,1]$, the feedback interconnection $G_\tau = \frac{G_n}{s}e^{-s\tau T_d}$ and $\psi \in S[0,k]$ is absolutely stable if and only if $k < \tilde{k}_N$.

V. CONCLUSION

This paper develops a class of time-delayed stable transfer functions and a class of time-delayed critically stable transfer functions that satisfy the Kalman conjecture. Firstly, the clockwise properties have been extended to the time delayed case. Then, the stability issues of the time-delayed plants have been analysed based on the clockwise Nyquist plots, and the Kalman conjecture is proved to be satisfied via the OACC. In addition, the result is further extended to the system with bounded and constant time delay. The same process is applied to the critical stable plants, and the conclusions are consistent. Following [7], we have shown that the OACC can be applied to critically stable systems.

The particular geometrical properties of the forward path systems can be used to design the feedback system where saturation and time delay are considered. Corollaries 2 and 3 provide a wide class of plants which can be used as benchmarks to judge the performance of stability conditions in the literature; for example they correspond directly to the LMI conditions given in [14] and [18] respectively.

It remains open whether similar results could be obtained using the Popov criterion. Hence it also remains open whether the same class of time-delayed stable transfer functions and/or time-delayed critically stable transfer functions also satisfy the Aizerman conjecture.

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APPENDIX

A. Selected nonlinearities and input

We have selected 4 different nonlinearities:

- Saturation with a gain:

$$\psi_1(u)(t) = \text{sat}(ku(t)), \qquad (25)$$

where

$$\text{sat}(x) = \begin{cases} 0.5 & \text{if } 0.5 < x, \\ x & \text{if } -0.5 \le x \le 0.5, \\ -0.5 & \text{if } x < -0.5. \end{cases}$$

- Deadzone with a gain:

$$\psi_2(u)(t) = \text{dz}(ku(t)), \qquad (26)$$

where

$$\text{dz}(x) = \begin{cases} x - 0.5 & \text{if } 0.5 < x, \\ 0 & \text{if } -0.5 \le x \le 0.5, \\ x + 0.5 & \text{if } x < -0.5. \end{cases}$$

- Deadzone combined with saturation:

$$\psi_3(u)(t) = \text{sat}(k\ \text{dz}(u(t))). \qquad (27)$$

- Arctangent with gain:

$$\psi_4(u)(t) = \arctan(ku(t)). \qquad (28)$$

As a result, all the above nonlinearities belong to $S[0,k]$. The input $f \in \mathscr{L}_2$ is defined as

$$f(t) = \begin{cases} 10 & \text{if } t < 10, \\ 0 & \text{if } t \ge 10. \end{cases} \qquad (29)$$

B. LTI plant

*1) Strictly stable:* Let us consider the time delayed plant given by

$$G_1(s) = \frac{e^{-3s}}{(s+1)(s^2+1.5s+1)^2}. \qquad (30)$$

Its Nyquist gain is $k_N < 1.1916$. Figure 6 shows four simulations of the feedback interconnection between G and the four nonlinear memoryless operators defined in Appendix A with $k = 1.1915$. As expected, all simulations have stable behaviour.

2) Marginally stable: Let us consider the time delayed plant with an integrator given by

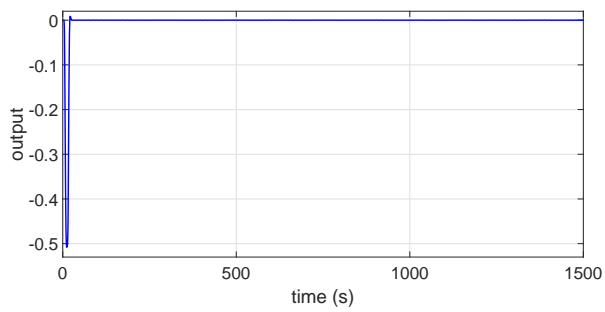$$G_2(s) = \frac{e^{-3s}}{s(s+1)(s^2+1.5s+1)^2}. \qquad (31)$$

Its Nyquist gain is $k_N < 0.2328$. Figure 7 shows four simulations of the feedback interconnection between G and the four nonlinear memoryless operators defined in Appendix A with $k = 0.2327$. As expected, all simulations have stable behaviour using the weaker $\mathscr{L}_2$-stability definition.
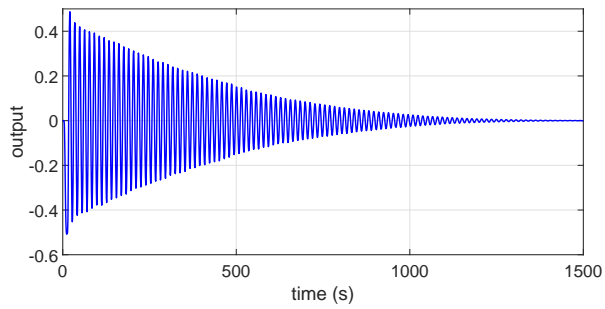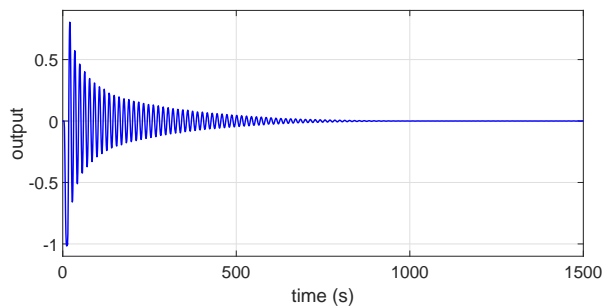
(a) Feedback interconnection between G_1 and ψ_1
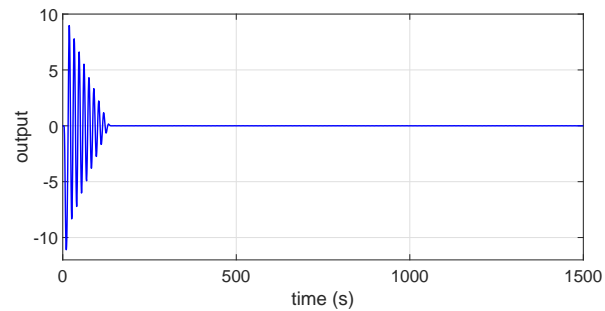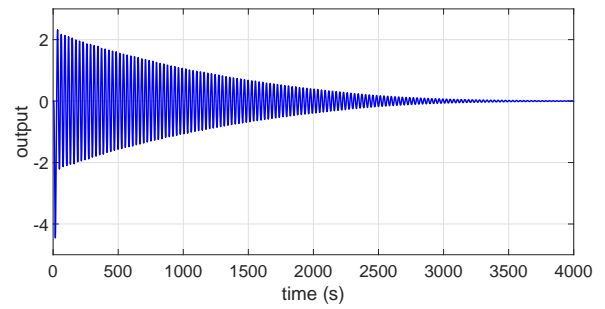

(b) Feedback interconnection between G_1 and ψ_2

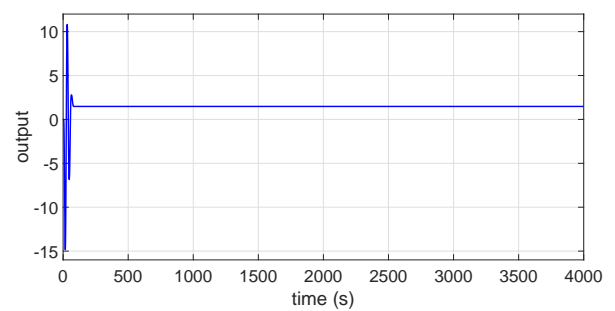(c) Feedback interconnection between G_1 and ψ_3

(d) Feedback interconnection between G_1 and ψ_4

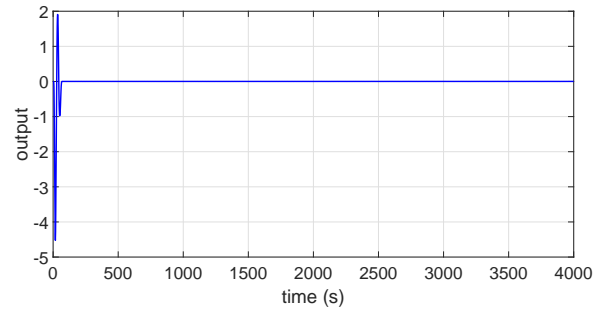Fig. 6: Examples of simulations of feedback interconnections between G_1 and slope-restricted nonlinearities.
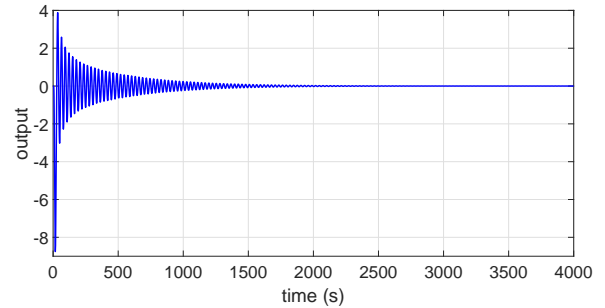

(a) Feedback interconnection between G_2 and ψ_1

(b) Feedback interconnection between G_2 and ψ_2

(c) Feedback interconnection between G_2 and ψ_3

(d) Feedback interconnection between G_2 and ψ_4

Fig. 7: Examples of simulations of feedback interconnections between G_2 and slope-restricted nonlinearities.