# Phase limitations of Zames-Falb multipliers

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***Abstract*—Phase limitations of both continuous-time and discrete-time Zames-Falb multipliers and their relation with the Kalman conjecture are analysed. A phase limitation for continuous-time multipliers given by Megretski is generalised and its applicability is clarified; its relation to the Kalman conjecture is illustrated with a classical example from the literature. It is demonstrated that there exist fourth-order plants where the existence of a suitable Zames-Falb multiplier can be discarded and for which simulations show unstable behavior. A novel phase-limitation for discrete-time Zames-Falb multipliers is developed. Its application is demonstrated with a second-order counterexample to the Kalman conjecture. Finally, the discrete-time limitation is used to show that there can be no direct counterpart of the off-axis circle criterion in the discrete-time domain.**

## I. Introduction

The absolute stability of a negative feedback interconnection between an LTI system $G$ and a nonlinearity $\phi$ with a slope restriction $k$ has aroused the interests of many researchers. The stability tests include the circle criterion, Popov criterion [1], [2], and off-axis circle criterion [3], [4] in continuous time and the circle criterion [5], Tsypkin criterion [6] and Jury-Lee criterion [7], [8] in discrete time. For a recent discussion, see [9] and [10]. Apart from these, loop transformation and multiplier theory are both important tools to establish the stability of feedback interconnections. The Zames-Falb multipliers are a class of multipliers with the property of preserving the positivity of monotone and bounded nonlinearities, and hence of slope-restricted nonlinearities after loop transformation. The class of Zames-Falb multipliers can be defined in either continuous time [11], [12] or discrete time [13], [14]. Specifically, after loop transformation, the stability of the negative interconnection between an LTI system $G$ and a nonlinearity $\phi$ with a slope restriction $k$ is guaranteed if there exists a Zames-Falb multiplier $M$ such that

$$\mathrm{Re}\{M(1+kG)\} > 0, \tag{1}$$

with $M$ and $G$ evaluated over all frequencies. That is to say, at $j\omega$, $\omega \in \mathbb{R}$ for continuous-time systems and at $e^{j\omega}$, $\omega \in [0, 2\pi]$ for discrete-time systems.

The Zames-Falb multipliers may be considered a classical tool [15]. Nevertheless, there has been considerable recent interest, largely sparked by the availability of numerical searches ([16], [17], [18], [19], [20], [21], [22], for continuous time; [23], [24] for discrete time) and their encapsulation within an IQC (integral quadratic constraint) framework [25], [26], [27], [28]. There has also been interest in generalising the class, both to MIMO (multi-input, multi-output) nonlinearities [29], [30], [31], [32], [33] and to nonlinearities outside the original classes considered by Zames and Falb [34], [35], [28], [36]. In addition to determining stability conditions, they can be used to analyse performance [37], [38]; further, they can be used to obtain tighter versions of the Popov criterion [39]. Applications of Zames-Falb multipliers range from input-constrained model predictive control [40] to first order numerical optimisation algorithms [41].

Although both continuous-time and discrete-time Zames-Falb multipliers are defined with similar conditions, there are clear distinctions between their properties. In discrete time the Zames-Falb multipliers are the full set of multipliers preserving the positivity of monotone and bounded nonlinearities, besides direct phase substitutions [14], [43]. In continuous time matters are more nuanced, but the class of Zames-Falb multipliers remains the widest known class

TABLE I
Various slope restrictions discussed in the text.

| | |
|---|---|
| $\hat{k}_{ZF}$ | Maximum slope for which a Zames-Falb multiplier is known |
| $k_{ZF}$ | Maximum slope for which there exists a Zames-Falb multiplier |
| $k_S$ | Maximum slope for which the Lur'e system is absolutely stable |
| $k_{PL}$ | Minimum slope for which phase limitation implies there is no Zames-Falb multiplier |
| $\hat{k}_C$ | Minimum slope for which a counterexample to absolute stability is known |
| $k_O$ | Slope for direct discrete-time counterpart off-axis circle criterion (which is false) |
| $k_{RO}$ | Slope for Reduced Off-axis circle criterion in [42] |
| $k_N$ | Nyquist value |

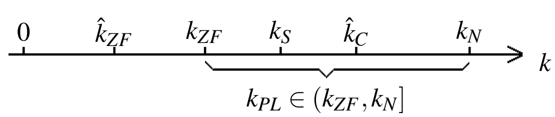


Fig. 1. Relations between slope restrictions discussed in the text. Conjecture I.2 is that $k_{ZF} = k_S$ and hence $k_S < k_{PL}$.

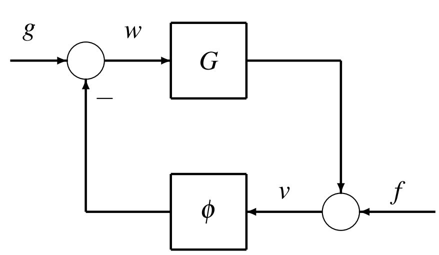


Fig. 2. Lur'e problem

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DOI: 10.1109/TAC.2017.2729162

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of multipliers preserving the positivitiy of monotone and bounded nonlinearities, up to phase equivalence [39], [44]. For a tutorial introduction to the phase properties of continuous-time Zames-Falb multipliers, phase-equivalence results and the issues associated with causality, see [45]. Phase properties are essential to our understanding of Zames-Falb multipliers.

For example, if $k < k_N$ (see Table I for various slope restrictions discussed in this paper), then the phase of $(1+kG)$ lies between $-180^\circ$ and $180^\circ$. Meanwhile multipliers must be positive so are restricted to lie between $-90^\circ$ and $90^\circ$. But as the Kalman conjecture is false, any set of suitable multipliers must be restricted by some further fundamental limitations. This follows from the obvious but important fact:

**Fact I.1** *If the system is not absolutely stable, there can be no appropriate Zames-Falb multiplier.*

However, only a few papers discuss such limitations. Megretski [46] shows that there exists a phase limitation for continuous-time Zames-Falb multipliers. Another phase limitation of Zames-Falb multipliers is given by Jönsson and Laiou [47], [48]. Such limitations are often ignored when new searches for multipliers are presented (see for example [45] and references therein). Often only $k_N$ is provided as an upper limit for the slope restriction.

We discuss Megretski's phase restriction [46] with respect to a fourth-order continuous-time plant whose phase drops from $+180^\circ$ to $-180^\circ$; similarly with $k$ sufficiently big the phase of $1+kG$ drops from above $+90^\circ$ degrees to below $-90^\circ$. The limitation cannot be applied to first, second or third-order plants whose phase is in the range $(-180,90)$ degrees or $(-90,180)$ degrees; this agrees with the well-known result that the Kalman conjecture is true for such plants [49].

To the best of authors' knowledge, no similar limitation has been developed in the discrete-time domain. Since there exist second-order discrete-time counterexamples to the Kalman conjecture whose phase is in the range $(-180,0)$ degrees [50], [51] one might expect a *simpler* limitation for discrete-time multipliers; this turns out to be indeed the case.

The contribution of this paper is for both continuous-time and discrete-time multipliers. We generalise Megretski's limitation [46] for continuous-time multipliers to a wider choice of frequency intervals. Further, we show that Megrestki's limitation [46] only applies for the class of Zames-Falb multipliers which do not require the odd condition on the nonlinearity; we provide the corresponding result when the nonliearity is odd. We discuss the limitation's numerical calculation and demonstrate its application in the context of a classical example due to O'Shea [11], [45]. In particular we demonstrate a fourth-order counterexample of the Kalman conjecture for which the constraint is active. A further contribution of the paper is the development of a phase limitation for discrete-time Zames-Falb multipliers. This limitation is fundamentally different to Megrestki's limitation as it only requires the phase of $(1+kG)$ to be either in the interval $(90,180)$ degrees or in the interval $(-180,-90)$ degrees. The limitation is easy to compute, and is active for a second-order discrete-time counterexample of the Kalman conjecture. This close link between the preclusion of a Zames-Falb multiplier and unstable behaviour leads us to the following conjecture as the counterpart to Fact I.1; however no proof (or counterexample) is offered in this paper:

**Conjecture I.2** *If there is no appropriate Zames-Falb multiplier, the system is not absolutely stable.*

One direct application of the phase limitation is to show there can be no direct discrete-time counterpart of the off-axis circle criterion. The continuous-time off-axis circle criterion is a useful graphical stability test and is shown to be a less conservative criterion compared to the circle criterion [3], [4]. The derivation is based on the phase properties of RL/RC multipliers. The direct discrete-time counterpart of the off-axis circle criterion is sometimes assumed to be true in the literature (e.g. [52], [53]). However only a highly restrictive discrete-time version is proposed in [42], without discussion as to whether the direct discrete-time counterpart off-axis circle criterion is true or false. In this paper, we show that in some cases there are no Zames-Falb multipliers with the requisite phase properties for its derivation - i.e. the direct counterpart off-axis circle criterion cannot be derived using multiplier theory. The invalidation is completed by counterexample.

Some preliminary results related with Theorem IV.3 part (i) were presented in [54].

## II. Notation and Preliminary results

### A. Signal spaces

For continuous-time signals let $\mathscr{L}_2[0,\infty)$ be the Hilbert space of square integrable and Lebesgue measurable functions $f:[0,\infty)\to\mathbb{R}$ and let $\mathscr{L}_2$ be defined similarly for $f:\mathbb{R}\to\mathbb{R}$. Let $\mathscr{L}_{2e}[0,\infty)$ be the extended space of $\mathscr{L}_2[0,\infty)$ [43].

For discrete-time signals let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integer numbers and positive integer numbers including 0, respectively. Let $\ell$ be the space of all real-valued sequences, $h:\mathbb{Z}^+\to\mathbb{R}$ and let $\ell_2$ denote the Hilbert space of all square-summable real sequences $f:\mathbb{Z}^+\to\mathbb{R}$ ($\ell$ is the extended space of $\ell_2$). Similarly, we can define the Hilbert space $\ell_2(\mathbb{Z})$ by considering real sequences $f:\mathbb{Z}\to\mathbb{R}$.

### B. Lur'e problem and the Kalman conjecture

The feedback interconnection system is a Lur'e system represented in Fig. 2 with both $G$ and $\phi$ mapping $\mathscr{L}_{2e}[0,\infty)\to\mathscr{L}_{2e}[0,\infty)$ (continuous time) or $\ell\to\ell$ (discrete time). The object $G$ is assumed LTI stable and the object $\phi$ memoryless and slope-restricted (see below). The interconnection relationship is

$$\begin{cases} v = f + Gw, \\ w = -(\phi v) + g. \end{cases} \tag{2}$$

The system (2) is well-posed if the map $(v,w)\mapsto(g,f)$ has a causal inverse on $\ell\times\ell$, and this feedback interconnection is $\ell_2$-stable if for any $f,g\in\ell_2$, both $w,v\in\ell_2$.

**Definition II.1** *(Memoryless slope-restricted nonlinearity) The nonlinearity $\phi:\mathscr{L}_{2e}[0,\infty)\to\mathscr{L}_{2e}[0,\infty)$ or $\phi:\ell\to\ell$ is said to be memoryless and slope-restricted in $S[0,k]$, if there* *is a function $N : \mathbb{R} \to \mathbb{R}$ such that $(\phi u)(t) = N(u(t))$ or $(\phi u)(k) = N(u(k))$, $N(0) = 0$, and*

$$0 \leq \frac{N(x_1) - N(x_2)}{x_1 - x_2} \leq k, \quad \forall x_1, x_2 \in \mathbb{R}, x_1 \neq x_2. \tag{3}$$

*In addition, $\phi$ is said to be odd if $N$ is odd, i.e. $N(x) = -N(-x)$, for all $x \in \mathbb{R}$.*

We define the Nyquist value and state the Kalman conjecture for both continuous-time and discrete-time systems.

**Definition II.2 (Nyquist value)** *Given a stable LTI system $G$, the Nyquist value $k_N$ is the supremum of all the positive real numbers $k$ such that $\tau k G$ satisfies the Nyquist Criterion for all $\tau \in [0, 1]$. It can also be expressed as:*

$$k_N = \sup\{k > 0 : \inf_{\omega}\{|1 + \tau k G|\} > 0), \forall \tau \in [0, 1]\}, \tag{4}$$

*with $G$ evaluated over all frequencies (i.e. $\omega \in \mathbb{R}$ for continuous-time systems and $\omega \in [0, 2\pi]$ for discrete-time systems).*

**Conjecture II.3 (Kalman Conjecture, [55])** *Let $\phi$ be a memoryless slope-restricted nonlinearity such that there exists a continuously differentiable $N : \mathbb{R} \to \mathbb{R}$ and $S > 0$ such that $\phi(v)(t) = N(v(t))$ (or $\phi(v)(k) = N(v(k))$) and*

$$0 \leq \frac{dN(x)}{dx} \leq S, \quad \forall x \in \mathbb{R}. \tag{5}$$

*Then the negative feedback interconnection of the continuous-time (or discrete-time) LTI systems $G \sim [A, B, C, 0]$ and $\phi$ (Fig 2) is globally asymptotically stable if $A - BCk$ is Hurwitz (Schur) for all $k \in [0, S]$.*

There exist fourth-order continuous-time counterexamples to the Kalman conjecture [56], [49], [57] and second-order discrete-time counterexamples [50], [51].

### C. Zames-Falb multipliers

The characteristics of continuous-time Zames-Falb multipliers is given in the following theorem that defines two different classes of multipliers.

**Theorem II.4** ***(Continuous-time Zames-Falb multipliers, [12].)*** *Consider the continuous-time feedback system in Fig. 2 with $G$ a stable LTI system and $\phi$ memoryless and slope-restricted in $S[0, k]$. Suppose that there exists an LTI multiplier $M : \mathscr{L}_2 \to \mathscr{L}_2$ whose transfer function has the form*

$$M(s) = 1 - H(s) \tag{6}$$

*such that the impulse response $h$ of $H$ satisfies*

$$\int_{-\infty}^{\infty} |h(t)|\, dt < 1. \tag{7}$$

*Moreover, let us assume that either $\phi$ is odd or $h(t) > 0$. Suppose further there is some $\delta > 0$ such that*

$$Re\{M(j\omega)(1 + kG(j\omega))\} \geq \delta \text{ for all } \omega \in \mathbb{R}. \tag{8}$$

*Then the feedback interconnection (2) is $\mathscr{L}_2$-stable.* ■

**Remark II.5** *With some abuse of notation, we denote $h(t)$ as the addition of a real-valued function $h_a(t)$ and impulses at different instants, i.e.*

$$h(t) = h_a(t) + \sum_{i=1}^{\infty} h_i \delta(t_i). \tag{9}$$

**Definition II.6** *The class of continuous-time Zames-Falb multipliers $\mathscr{M}^c$ is defined as the LTI systems $M : \mathscr{L}_2 \to \mathscr{L}_2$ whose transfer function has the form*

$$M(s) = 1 - H(s) \tag{10}$$

*such that the impulse response $h$ of $H$ satisfies that $h(t) \geq 0$ for all $t$ and*

$$\int_{-\infty}^{\infty} h(t)\, dt < 1. \tag{11}$$

**Definition II.7** *The class of continuous-time "odd" Zames-Falb multipliers $\mathscr{M}^c_{odd}$ is defined as the LTI systems $M : \mathscr{L}_2 \to \mathscr{L}_2$ whose transfer function has the form*

$$M(s) = 1 - H(s) \tag{12}$$

*such that the impulse response $h$ of $H$ satisfies*

$$\int_{-\infty}^{\infty} |h(t)|\, dt < 1. \tag{13}$$

By definition, $\mathscr{M}^c \subset \mathscr{M}^c_{\text{odd}}$.

The counterpart result in discrete time is given in the following theorem and it also defines two different classes of multipliers:

**Theorem II.8** ***(Discrete-time Zames-Falb multipliers, [14], [43])*** *Consider the discrete-time feedback system in Fig. 2 with $G$ a stable LTI system and $\phi$ memoryless and slope-restricted in $S[0, k]$. Suppose that there exists an LTI multiplier $M : \ell_2 \to \ell_2$ whose transfer function has the form*

$$M(z) = 1 - H(z) \tag{14}$$

*such that the impulse response $h$ of $H$ satisfies that $h_0 = 0$ and*

$$\sum_{i=-\infty}^{\infty} |h_i| < 1. \tag{15}$$

*Moreover, let us assume that either the nonlinearity is odd or $h_i \geq 0$. Suppose further*

$$Re\{M(z)(1 + kG(z))\} > 0, \quad \forall |z| = 1. \tag{16}$$

*Then the feedback interconnection (2) is $\ell_2$-stable.* ■

**Remark II.9** *Inequality (8) is evaluated over $\omega \in \mathbb{R}$ whereas inequality (16) is evaluated over the frequency interval $\omega \in [0, 2\pi]$. Hence, by the Extreme Value Theorem [58], it is unnecessary to define any $\delta > 0$ for the discrete case corresponding to that used in the continuous case.*

Similarly to the previous definitions, we can define the classes of multipliers $\mathscr{M}^d$ and $\mathscr{M}^d_{\text{odd}}$.

### D. Off-axis circle criterion

The continuous-time off-axis circle criterion is given.

**Lemma II.10 (Off-axis circle criterion for continuous-time systems, [3])** *Consider the feedback system in Fig. 2 with $G$ LTI stable and $\phi$ is slope-restricted in $S[0,k]$. Suppose that the Nyquist plot of the linear part of the system $G(j\omega)$ lies entirely to the right of a straight line passing through the point $(-\frac{1}{k}+\delta,0)$ where $\delta>0$ and $\phi$ is monotonically increasing. Then the feedback interconnection (2) is $\mathscr{L}_2$-stable.*

For discrete time, only a highly restrictive version is proposed.

**Lemma II.11 (Reduced off-axis circle criterion for discrete-time systems, [42])** *Let the Nyquist plot of $G(e^{j\omega})$ for all $0\leq\omega\leq\pi$ lie entirely to the right of a straight line, whose slope $k$ is nonnegative passing through $(-\frac{1}{K_2},0)$. Let $\omega_0$ be such that Re $G(e^{j\omega_0})=-\frac{1}{K_2}$ and Re $G(e^{j\omega})\geq-\frac{1}{K_2}$ for $\omega\geq\omega_0$ and Im $G(e^{j\omega})\leq 0$ for $\omega_0\geq\omega\geq 0$. Then the system is asymptotically stable for all monotone $\phi$ with slope restriction $K_2$ in the feedback path if*

$$\theta\leq-\frac{1}{2}\omega_0+\frac{\pi}{2}, \tag{17}$$

*where $\theta$ is the angle made by the straight line and the imaginary axis, i.e., $\theta=\cot^{-1}k$. If Im $G(e^{j\omega})\geq 0$ for $\omega_0\geq\omega\geq 0$, the same argument can be used to prove the asymptotic stability of the system with nonpositive $k$ and*

$$\theta\geq\frac{1}{2}\omega_0-\frac{\pi}{2}. \tag{18}$$

### E. Further mathematical notation

For the convenience of solving potential numerical issues, the notation of $O(\cdot)$ is given.

**Definition II.12** *The condition*

$$f(t)=g(t)+O(t^n),\ \text{as}\ t\to 0. \tag{19}$$

*means that there exist $M$ and $t_0$ such that*

$$|f(t)-g(t)|\leq Mt^n\ \text{on}\ [0,t_0]. \tag{20}$$

The floor function, denoted by $\lfloor\nu\rfloor$, is defined by

$$\lfloor\nu\rfloor=\max\{m\in\mathbb{Z}\,|\,m\leq\nu\}. \tag{21}$$

## III. Continuous phase limitations and the Kalman conjecture

Megretski presents in [46] a phase limitation for continuous-time Zames-Falb multipliers. In this section we generalise the result to a wider set of frequency intervals, and derive separate results for both $\mathscr{M}^c$ and $\mathscr{M}^c_{odd}$. Although it is stated in [46] that the result there is valid for $\mathscr{M}^c_{odd}$ (in the terminology of this paper) we show by counterexample that it is in fact valid for $\mathscr{M}^c$ only. Finally, we bridge the limitation of [46] with the Kalman conjecture; this is the key motivation to develop a different set of phase limitations for the discrete-time Zames-Falb multipliers.

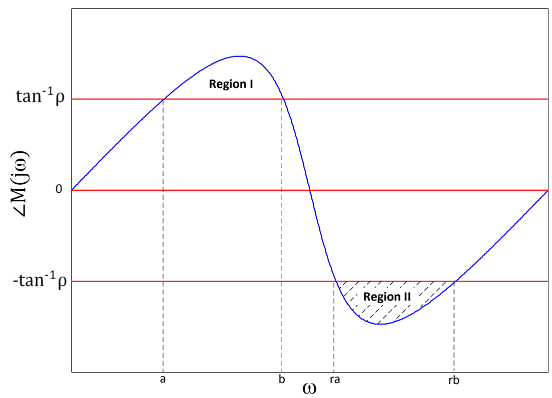


Fig. 3. Illustration of Theorem III.4 with the choice of interval from [46]: $\kappa=1$, $c=ra$ and $d=rb$. The result is given in terms of $\rho$ while the phase of the multiplier $M$ is $\tan^{-1}\rho$.

### A. Phase limitations

**Definition III.1** *Let $0<a<b<c<d$, $\kappa>0$, $\lambda>0$ and $\mu>0$. Define*

$$\rho^c=\sup_{t>0}\frac{|\psi(t)|}{\phi(t)}, \tag{22}$$

*and*

$$\rho^c_{odd}=\sup_{t>0}\frac{|\psi(t)|}{\tilde{\phi}(t)}, \tag{23}$$

*where*

$$\psi(t)=\frac{\lambda\cos(at)}{t}-\frac{\lambda\cos(bt)}{t}-\frac{\mu\cos(ct)}{t}+\frac{\mu\cos(dt)}{t}, \tag{24}$$

$$\phi(t)=\lambda(b-a)+\kappa\mu(d-c)+\phi_1(t), \tag{25}$$

*and*

$$\tilde{\phi}(t)=\lambda(b-a)+\kappa\mu(d-c)-|\phi_1(t)|, \tag{26}$$

*with*

$$\phi_1(t)=\frac{\lambda\sin(at)}{t}-\frac{\lambda\sin(bt)}{t}+\frac{\kappa\mu\sin(ct)}{t}-\frac{\kappa\mu\sin(dt)}{t}. \tag{27}$$

**Lemma III.2** *If $\lambda$ and $\mu$ are chosen such that*

$$\frac{\lambda}{\mu}=\frac{d^2-c^2}{b^2-a^2}, \tag{28}$$

*then $\rho^c$ and $\rho^c_{odd}$ in Definition III.1 are well-defined; that is to say $\rho^c<\infty$ and $\rho^c_{odd}<\infty$.*

*Proof:* See Appendix. ■

**Remark III.3** *The direct calculation of the ratios $\psi(t)/\phi(t)$ and $\psi(t)/\tilde{\phi}(t)$ is numerically ill-conditioned for small $t$ since, with the choice (28), we have $\psi(t)=0+O(t^3)$, $\phi(t)=0+O(t^2)$ and $\tilde{\phi}(t)=0+O(t^2)$, all as $t\to 0$. Nevertheless, the same construction ensures we can write*

$$\frac{\psi(t)}{\phi(t)}=\gamma t+O(t^3)\ \text{and}\ \frac{\psi(t)}{\tilde{\phi}(t)}=\gamma t+O(t^3)\ \text{as}\ t\to 0, \tag{29}$$

*with*

$$\gamma=-\frac{1}{4}\frac{\lambda(b^4-a^4)-\mu(d^4-c^4)}{\lambda(b^3-a^3)+\kappa\mu(d^3-c^3)}. \tag{30}$$

*We use this relation for small $t$ in the numerical examples below.*

**Theorem III.4 (Continuous-time phase limitations)** *Let $M$ be a continuous-time Zames-Falb multiplier. Suppose*

$$Im(M(j\omega)) > \rho Re(M(j\omega)) \text{ for all } \omega \in [a,b], \quad (31)$$

*and*

$$Im(M(j\omega)) < -\kappa\rho Re(M(j\omega)) \text{ for all } \omega \in [c,d], \quad (32)$$

*for some $\rho > 0$. Then under the conditions of Lemma III.2.*

(i) $\rho < \rho^c$ *if* $M \in \mathscr{M}^c$,

(ii) $\rho < \rho^c_{odd}$ *if* $M \in \mathscr{M}^c_{odd}$.

*Proof:* See Appendix. ∎

Lemma III.2 and Theorem III.4, with the choice $\kappa = 1$, $c = ra$, $d = rb$ and hence $\lambda/\mu = r^2$, are in [46]. An interpretation of Theorem III.4 with these values is illustrated in Fig. 3 (see also [45]). According to the constraints on the coefficients of continuous Zames-Falb multipliers, if the phase is simultaneously greater than $\tan^{-1}\rho$ on $\omega \in [a,b]$ (in Region I) and smaller than $-\tan^{-1}\rho$ on $\omega \in [ra,rb]$ (in Region II), then $\rho < \rho^c$ if $M \in \mathscr{M}^c$ and $\rho < \rho^c_{odd}$ if $M \in \mathscr{M}^c_{odd}$.

**Remark III.5** *It is straightforward to produce the counterpart of Theorem III.4 with (31) and (32) replaced by $Im(M(j\omega)) < -\rho Re(M(j\omega))$ for all $\omega \in [a,b]$ and $Im(M(j\omega)) > \kappa\rho Re(M(j\omega))$ for all $\omega \in [c,d]$ respectively.*

**Remark III.6** *In [46], Megretski uses a positive sign in the exponential of the Laplace transform:*

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{j\omega t} h(t)dt. \quad (33)$$

*This is the standard convention in the Physics literature (see for example [59]) but opposite to that used in [12]. The apparent discrepancy has no significant consequence for the analysis of phase limitations since if $M(s)$, with impulse response $m(t)$, is a Zames-Falb multiplier then $M(-s)$, with impulse response $m(-t)$, is also a Zames-Falb multiplier.*

It is natural to ask whether a phase limitation over a single frequency range can be constructed in a similar manner. This is not possible in continuous time, as any corresponding definition of $\rho^c$ or $\rho^c_{odd}$ would be unbounded as $t$ approaches 0. Loosely speaking, we can generate a multiplier in $\mathscr{M}^c$ with phase arbitrarially close to $\pm 90^\circ$ over an arbitrarily large frequency inteterval by selecting $h(t) = (1-\varepsilon)\delta(t - t^*)$ with $\varepsilon > 0$ arbitrarially close to 0 and $t^*$ arbitrarially close to 0. But we construct such phase limitations for discrete-time multipliers below, in Section IV.

## *B. Numerical example*

Here we illustrate Theorem III.4 with a numerical example. Let $a = 1.6$ and $b = 2.25$. Let $\kappa = 1$, $c = ra$ and $d = rb$ with $r = 2.1$. Then a sweep over time intervals followed by local numerical search gives

$$\rho^c \approx 0.6069, \ \tan^{-1}\rho^c \approx 31.25^\circ, \quad (34)$$

and

$$\rho^c_{odd} \approx 1.4928, \ \tan^{-1}\rho^c_{odd} \approx 56.18^\circ. \quad (35)$$

Now consider the multiplier

$$M(j\omega) = 1 - \int_{-\infty}^{\infty} e^{-j\omega t} h(t)dt \quad (36)$$

with $h(t) = -0.9\delta(t+1)$. Figure 4 shows that the relations (31) and (32), or equivalently

$$\angle M(j\omega) > \tan^{-1}\rho \text{ over the interval } [a,b], \quad (37)$$

and

$$\angle M(j\omega) < -\tan^{-1}\rho \text{ over the interval } [ra,rb], \quad (38)$$

are satisfied simultaneously for $\rho = \rho^c$ but not for $\rho = \rho^c_{odd}$. This is consistent with Theorem III.4 as $M \in \mathscr{M}^c_{odd}$ but $M \notin \mathscr{M}^c$. It is a counterexample to the false claim in [46] that the phase limitation of Theorem III.4 part (i) is applicable to the wider class $M \in \mathscr{M}^c_{odd}$.

**Remark III.7** *Both in this numerical example and at the end of Section III-A we consider multipliers where $h$ takes the form $h(t) = (1-\varepsilon)\delta(t-\tau)$ for some $\tau$. There is a close link with Theorem III.4. Specifically if $|\psi(\tau)|/\phi(\tau) = \rho^c$ and $\varepsilon \to 0$ then*

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt = \rho^c[\lambda(b-a) + \kappa\mu(d-c)] + \rho^c \int_{-\infty}^{\infty} \phi_1 h(t)\,dt. \quad (39)$$

*Compare (80) in the proof of Theorem III.4. Similarly if $|\psi(\tau)|/\tilde{\phi}(\tau) = \rho^c_{odd}$ then*

$$\int_{-\infty}^{\infty} \psi(t)h(t)dt = $$
$$= \rho^c_{odd}[\lambda(b-a) + \kappa\mu(d-c)] + \rho^c_{odd} \int_{-\infty}^{\infty} \phi_1 h(t)\,dt. \quad (40)$$

*We discuss the corresponding relations at greater length in Section IV-B for the discrete-time case.*

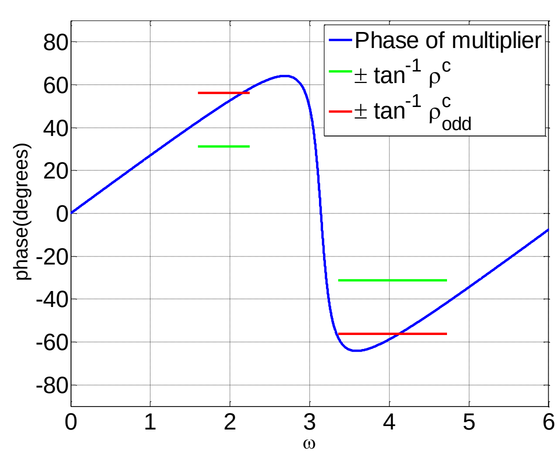


Fig. 4. Phase of the multiplier (36) and the continuous phase limitations $\pm\tan^{-1}\rho^c \approx \pm 31.25^\circ$ and $\pm\tan^{-1}\rho^c_{odd} \approx \pm 56.18$ evaluated on $[1.6, 2.25]$ and $[3.36, 4.725]$.

*C. Counterexamples to Kalman conjecture via phase limitations*

It is instructive to interpret the phase limitations of Theorem III.4 in a manner consistent with known results about the Kalman conjecture.

On the one hand, it is well-known that first, second and third order plants hold the Kalman conjecture [49]. The phase of such plants cannot reach both Regions I and II in Fig. 3. So the phase limitations cannot apply to these plants. First-order plants do not require a dynamic multiplier, second-order plants require a dynamics multiplier with a tunable zero and a pole at infinity, i.e. a Popov multiplier, and-third order plant requires both a tunable pole and zero, i.e. first order RL/RC multipliers. In all these cases, only a first-order multiplier is required, and we know that there is no phase limitation in the selection of such multipliers [39].

**Remark III.8** *Although the off-axis circle criterion is also based on RL/RC multipliers, it is not sufficient to show all third-order plants hold the Kalman conjecture. For example, the off-axis circle criterion with the plant*

$$G = \frac{s^2}{s^3 + 1.002s^2 + s + 0.998}, \tag{41}$$

*guarantees stability with* $k < 3.928$*, whereas the multiplier* $M = (s+1)/(s+\varepsilon)$ *guarantees stability for any positive* $k$ *with a sufficiently small value* $\varepsilon > 0$*.*

On the other hand the phase limitations may be applied to fourth-order plants, and these in turn may be counterexamples to the Kalman conjecture by: a) showing numerically that a phase limitation can be applied to a well-known plant, and b) showing that the Lur'e system with this plant and a slope-restricted nonlinearity may be unstable.

Specifically we will consider the phase limitation of Theorem III.4 part (i) with $\kappa = 1$, $c = ra$, and $d = rb$; that is to say the original result of [46] applied to $\mathscr{M}^c$. A particularly suitable example to show this limitation is O'Shea example [11], [45]:

$$G(s) = \frac{s^2}{(s^2 + 2\xi s + 1)^2}, \tag{42}$$

since the symmetry of the problem simplifies the selection of the parameters. In this example, O'Shea showed that there is a Zames-Falb multiplier for any $k$ if $\xi > 0.5$. The following result shows that it is not possible to reach an arbitrary large $k$ for any $\xi \leq 0.25$. For the case $\xi = 0.25$, the phase of $G(s)$ is above $177.98^\circ$ over the interval $[a,b]$ where $a = 0.02249$ and $b = 0.03511$; hence it is below $-177.98^\circ$ over the interval $[1/b, 1/a]$ by using the symmetry of the plant. Then a suitable Zames-Falb multiplier for this plant would require a phase below $-87.98^\circ$ over the interval $[a,b]$ and above $87.98^\circ$ over the interval $[1/b, 1/a]$. The phase limitation ensures that there is no Zames-Falb multiplier with such a phase characteristic, since $\tan^{-1}\rho^c \approx 87.79^\circ$. Strictly speaking, we have used the counterpart of Theorem III.4 mentioned in Remark III.5.

Although numerical reliability can be problematic in the discussion of the Kalman conjecture [57], simulations of the plant with asymmetrical saturation show a time evolution that does not appear to settle to zero, supporting the validity of Conjecture I.2. The simulation shown in Figure 5 has been run in MATLAB R2013, using the solver ode45, with maximum step size of 0.0001 s, and relative tolerance of $10^{-3}$. The nonlinearity $\phi$ is described by the nonlinear function

$$N(x) = \begin{cases} -1000 & x < -1; \\ 1000x & -1 \leq x \leq 0; \\ 0 & x > 0; \end{cases} \tag{43}$$

the input $g$ is given by

$$g(t) = \begin{cases} 100 & t \leq 20\text{s}; \\ 0 & t > 20\text{s}; \end{cases} \tag{44}$$

and $f(t) = 0$. The relevance of this counterexample to the Kalman conjecture is that we can show that there is no Zames-Falb multiplier with $h(t) \geq 0$ for the system. The asymmetry of the nonlinearity seems to be a key factor as simulations with symmetric saturations show stable behaviour. The importance of asymmetry in the stability of Lur'e systems with saturation has been discussed recently [36], [60].

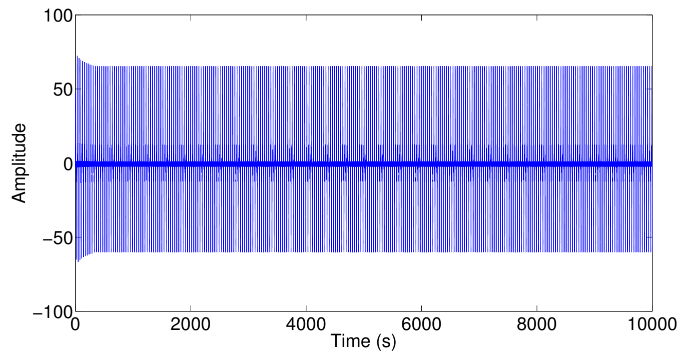


Fig. 5. Amplitude of the signal $v$ in a simulation of the feedback interconnection depicted in Fig. 2 where G is given by (42) with $\xi = 0.25$, $\phi$ is described by (43), $g(t)$ is given by (44), and $f(t) = 0$.

**Remark III.9** *The magnitude of the response in Fig. 5 is bounded. Since the plant G is stable and* $\phi$ *is sector-bounded it follows that if* $g \in \mathscr{L}_\infty$ *then all signals must be in* $\mathscr{L}_\infty$*.*

## IV. Discrete-time phase limitation

In this section we develop phase limitations for discrete-time Zames-Falb multipliers. Their derivation is in the spirit of Megretski's limitation [46] and Theorem III.4 for continuous-time multipliers. However their properties are simpler and consistent with the existence of second-order discrete-time counterexamples to the Kalman conjecture [50], [51]. In particular, and by contrast with their continuous-time counterparts, they are concerned with properties over a single interval $\omega \in [a,b]$.

It is worth highlighting that if a discrete-time multiplier preserves the positivity of all monotone and bounded nonlinearities then either it is a Zames-Falb multiplier or there exists a Zames-Falb multiplier with the same phase [14], [43]. Hence any phase limitation on the discrete-time Zames-Falb multipliers is also a limitation for any discrete-time multiplier.

### A. Phase limitations

**Definition IV.1** *Let* $0 \le a < b \le \pi$. *Define*

$$\rho^d = \max_{n\in\mathbb{Z}^+} \frac{|\psi_d(n)|}{\phi_d(n)}, \tag{45}$$

*and*

$$\rho^d_{odd} = \max_{n\in\mathbb{Z}^+} \frac{|\psi_d(n)|}{\tilde{\phi}_d(n)}, \tag{46}$$

*where*

$$\psi_d(n) = \frac{\cos(an)}{n} - \frac{\cos(bn)}{n}, \tag{47}$$
$$\phi_d(n) = (b-a) + \phi_{d,1}(n), \tag{48}$$

*and*

$$\tilde{\phi}_d(n) = (b-a) - |\phi_{d,1}(n)|, \tag{49}$$

*with*

$$\phi_{d,1}(n) = \frac{\sin(an)}{n} - \frac{\sin(bn)}{n}. \tag{50}$$

**Lemma IV.2** *Both* $\rho^d$ *and* $\rho^d_{odd}$ *in Definition IV.1 are well-defined; that is to say* $\rho^d < \infty$ *and* $\rho^d_{odd} < \infty$.

*Proof:* See Appendix. ■

**Theorem IV.3 (Discrete-time phase limitations)** *Let* $M$ *be a discrete-time Zames-Falb multiplier. Suppose*

$$Im(M(e^{j\omega})) > \rho Re(M(e^{j\omega})) \text{ for all } \omega \in [a,b], \tag{51}$$

*for some* $\rho > 0$. *Then*

(i) $\rho < \rho^d$ *if* $M \in \mathscr{M}^d$,
(ii) $\rho < \rho^d_{odd}$ *if* $M \in \mathscr{M}^d_{odd}$.

*Proof:* See Appendix. ■

An interpretation of Theorem IV.3 is illustrated in Fig. 6. According to the constraints on the coefficients of discrete-time Zames-Falb multipliers, if the phase is greater than $\tan^{-1}\rho$ on $\omega \in [a,b]$ (in Region A), then $\rho < \rho^d$ if $M \in \mathscr{M}^d$ and $\rho < \rho^d_{odd}$ if $M \in \mathscr{M}^d_{odd}$.

**Remark IV.4** *It is straightforward to produce the counterpart of Theorem IV.3 with* (51) *replaced by*

$$Im(M(e^{j\omega})) < -\rho Re(M(e^{j\omega})) \text{ for all } \omega \in [a,b]. \tag{52}$$

An algorithm for finding the phase limitation in Theorem IV.3 part (i) for a second order plant is given in [54]. For a given stable plant $G$ and a value of $k$ such that $0 < k < k_N$ the phase of an ideal multiplier is obtained as

$$\angle M_d = \begin{cases} \angle(G+1/k) - 90 & \text{if} \quad \angle(G+1/k) > 90 \\ \angle(G+1/k) + 90 & \text{if} \quad \angle(G+1/k) < -90 \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

Then the algorithm increases $k$ until the existence of such a multiplier can be discarded by using the limitation presented in Theorem IV.3.

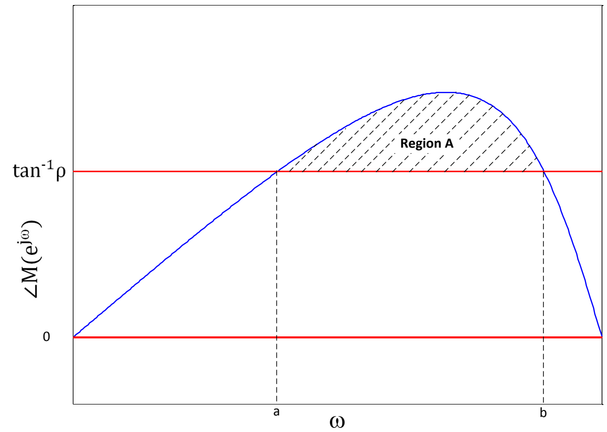


Fig. 6. Illustration of Theorem IV.3. The result is given in terms of $\rho$ while the phase of the multiplier $M$ is $\tan^{-1}\rho$.

### B. Integral bound and sparsely parametrized multipliers

Theorem IV.3 gives relative bounds on the real and imaginary parts of a Zames-Falb multiplier's frequency response over an interval $[a,b]$. It is straightforward to derive a closely related result in terms of the integrals over the same interval.

**Theorem IV.5** *Let* $M$ *be a discrete-time Zames-Falb multiplier. Suppose*

$$\int_a^b Im(M(e^{j\omega}))\,d\omega > \rho \int_a^b Re(M(e^{j\omega}))\,d\omega \tag{54}$$

*for some* $\rho > 0$. *Then*

(i) $\rho < \rho^d$ *if* $M \in \mathscr{M}^d$,
(ii) $\rho < \rho^d_{odd}$ *if* $M \in \mathscr{M}^d_{odd}$.

*Proof:* See Appendix. ■

**Remark IV.6** *Theorem IV.5 is stronger than Theorem IV.3 in the sense that condition (51) is sufficient for condition (54) but not necessary. Theorem IV.3 may be derived as a Corollary of Theorem IV.5 by applying the Mean Value Theorem [58].*

Theorem IV.5 gives a tight phase limitation in the sense that we can associate a set of sparsely parameterized multipliers with Theorem IV.5 as follows.

**Proposition IV.7**

(i) *For a given a and b, define the set* $\mathscr{N}^d \subset \mathbb{Z}$ *as the set of integers n such that* $\psi_d(n)/\phi_d(n) = \rho^d$. *Then multipliers of the form*

$$M(z) = 1 - \sum_{n\in\mathscr{N}^d} h_n z^{-n} \tag{55}$$

*with*

$$h_0 = 0,\ h_n \ge 0 \text{ and } \sum_{n\in\mathscr{N}([a,b])} h_n = 1-\varepsilon \tag{56}$$

*satisfy (54) with* $\rho$ *arbitrarily close to* $\rho^d$ *in the limit as* $\varepsilon \to 0$.

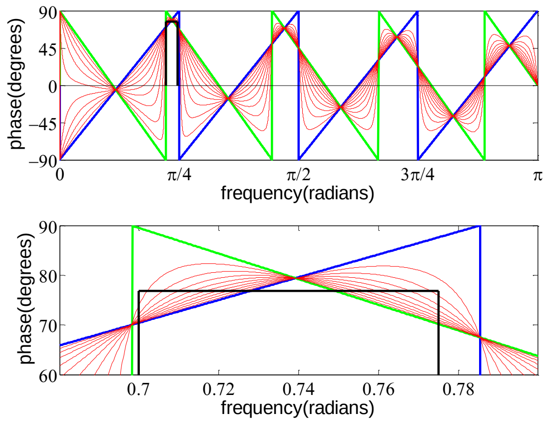


Fig. 7. Phases of the limiting cases $M(z) = 1 - z^8$, $M(z) = 1 - z^{-9}$ and linear combinations of the form $M(z) = 1 - \lambda z^8 - (1-\lambda)z^{-9}$ with $0 < \lambda < 1$. The phase limitation $\tan^{-1}\rho^d \approx 76.8^\circ$ over the interval $[a,b] = [0.7, 0.77501]$ is also shown. The top figure shows the phase over the frequency range from 0 to $\pi$ radians, while the bottom figure shows the same data in the frequency range from 0.68 to 0.8 radians.

(ii) *For a given a and b, define the set* $\mathcal{N}^d_{odd} \subset \mathbb{Z}$ *as the set of integers n such that* $\psi_d(n)/\tilde{\phi}_d(n) = \rho^d_{odd}$. *Then multipliers of the form*

$$M(z) = 1 - \sum_{n \in \mathcal{N}^d} h_n z^{-n} \tag{57}$$

*with*

$$h_0 = 0 \text{ and } \sum_{n \in \mathcal{N}([a,b])} h_n = 1 - \varepsilon \tag{58}$$

*satisfy (54) with* $\rho$ *arbitrarily close to* $\rho^d_{odd}$ *in the limit as* $\varepsilon \to 0$.

*Proof:* See Appendix. ■

**Remark IV.8** *It is, once again, straightforward to produce the counterpart of Theorem IV.5 with* (54) *replaced by*

$$\int_a^b Im(M(e^{j\omega}))\,d\omega < -\rho \int_a^b Re(M(e^{j\omega}))\,d\omega. \tag{59}$$

*Similarly for Theorem IV.7 with (i)* $\psi_d(n)/\phi_d(n) = -\rho^d$ *and (ii)* $\psi_d(n)/\tilde{\phi}_d(n) = -\rho^d_{odd}$.

As an illustrative example, suppose $a = 0.7$ and $b = 0.77501$ (approx.). Then

$$\tan^{-1}\rho^d \approx 76.8^\circ, \tag{60}$$

and

$$\mathcal{N}^d = \{-8, 9\}. \tag{61}$$

Fig 7 shows the phases of the limiting cases $M(z) = 1 - z^8$, $M(z) = 1 - z^{-9}$ and linear combinations of the form $M(z) = 1 - \lambda z^8 - (1-\lambda)z^{-9}$ with $0 < \lambda < 1$. It can be seen that the phases are near to $\tan^{-1}\rho^d$ over the interval $[a,b]$. However they always have values both above and below, indicating that Theorem IV.3 is not tight in the same sense as Theorem IV.5.

**Remark IV.9** *A similar analysis is possible for continuous-time multipliers. Compare Remark III.7.*

### C. Discrete-time counterparts of the off-axis circle criterion

The off-axis circle criterion [3] (Theorem II.10) is a useful frequency-based graphical stability test for continuous-time systems. It is sometimes assumed (e.g. in [52]) that its discrete-time counterpart is true. We state this as a conjecture:

**Conjecture IV.10** *Consider the feedback system in Fig. 2 with* $G \in \mathbf{RH}_\infty$, *and* $\phi$ *is slope-restricted in* $S[0,k]$. *Suppose that the Nyquist plot of the linear part of the system* $G(e^{j\omega})$ *lies entirely to the right of a straight line passing through the point* $(-\frac{1}{k} + \delta, 0)$ *where* $\delta > 0$ *and* $\phi$ *is monotonically increasing. Then the feedback interconnection* (2) *is* $\ell_2$*-stable.*

A geometrical interpretation of both Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems is given in Fig 8.

The phase-limitation on discrete-time Zames-Falb multipliers carries the implication that there can be no multiplier construction corresponding to that for RL/RC multipliers of [42] used to prove Theorem II.10. We summarise the argument as follows:

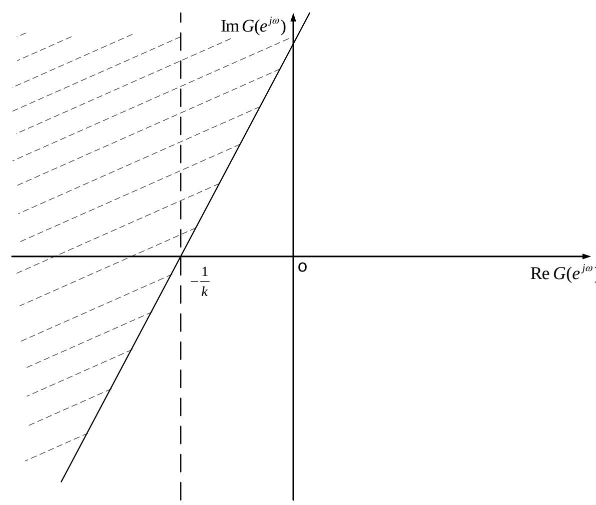


Fig. 8. Geometrical interpretation of the off-axis circle criterion considering the plant $G$ (Theorem II.10 for continuous-time systems and Conjecture IV.10 for discrete-time systems). The Theorem for continuous-time systems is true but the Conjecture for discrete-time systems is false in general.

1) Under the conditions of Conjecture IV.10 there is some $\theta$ in $(-90, 90)$ degrees such that the phase of $1 + kG$ always lies in the interval $(-90 - \theta, 90 - \theta)$ degrees. Hence an ideal LTI multiplier with constant phase $\theta$ would render the real part of $M(1 + kG)$ positive over all frequencies.
2) In their proof of the continuous off-axis circle criterion Cho and Narendra [3] show that it is possible to construct RL/RC multipliers whose phase is arbitrarily close to some constant $\theta$ degrees over an arbitrarily large interval. We show that for some values of $\theta$ this may not be possible for any discrete-time LTI multiplier.
3) If a discrete-time LTI multiplier preserves the positivity of a slope-restricted nonlinearity then there is a Zames-Falb multiplier with the same phase [14], [43], so we can

limit our set of multipliers to the class of LTI Zames-Falb multipliers.

4) If $\theta > \tan^{-1}(2/\pi) \approx 32.48°$ then Theorem IV.3 precludes any such construction of a Zames-Falb multiplier since if $a \to 0^+$ and $b \to \pi^-$ then $\rho^d \to \tan^{-1}(-2/\pi)$.

Hence the phase limitation can be used to invalidate Conjecture IV.10 when $\theta > \tan^{-1}(2/\pi)$. Smaller values can be obtained by using different values of $a$ and $b$. It follows that any counterpart of the off-axis circle criterion in discrete-time must take into account specific information about frequency intervals. This is true of the more limited result originally derived by Narendra and Cho [42] (Theorem II.11). In fact it can be shown that the counterexamples to the Kalman conjecture of [50] and [51] are also counterexamples to Conjecture IV.10.

*D. Finite search in discrete-time domain*

Here we provide a result which simplifies the numerical implementation. Although the definitions of $\rho^d$ and $\rho^d_{odd}$ are given with an infinite number of terms, they can be calculated using a finite number $n = n_N$ given in Lemma IV.11.

**Lemma IV.11** *Let $0 \leq a < b \leq \pi$, then*

$$\rho^d = \max_{1\leq n\leq n_N} \frac{|\psi_d(n)|}{\phi_d(n)}, \tag{62}$$

*and*

$$\rho^d_{odd} = \max_{1\leq n\leq n_N} \frac{|\psi_d(n)|}{\tilde{\phi}_d(n)}, \tag{63}$$

*with*

$$n_N = \lfloor \nu \rfloor, \tag{64}$$

*where*

$$\nu = \frac{2(b-a) - 2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}. \tag{65}$$

*Proof:* See Appendix. ∎

Suppose we wish to find a phase limitation over the interval $\omega \in [0.7, 0.75]$. Applying Lemma IV.11 we find $\nu = 55.2$ and hence $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all $n > n_N$, with

$$n_N = 55. \tag{66}$$

Hence it is sufficient to search over the integers $1 \leq n \leq 55$ for $\rho^d$. The numerical results shown in Fig. 9 demonstrate that $|\psi_d(n)|/\phi_d(n) < |\psi_d(1)|/\phi_d(1)$ for all $n > 18$. In fact the maximum occurs at $n = -9$.

*E. Numerical example*

Let us consider the negative feedback interconnection between the plant

$$G(z) = \frac{z}{z^2 - 1.8z + 0.81}, \tag{67}$$

and a slope-restricted nonlinearity. This second-order plant is a counterexample to the discrete-time Kalman conjecture as there is a periodic solution when $k = 2.1$ [50], and the Nyquist value is $k_N = 3.61$. Using the algorithm of [24] we find there

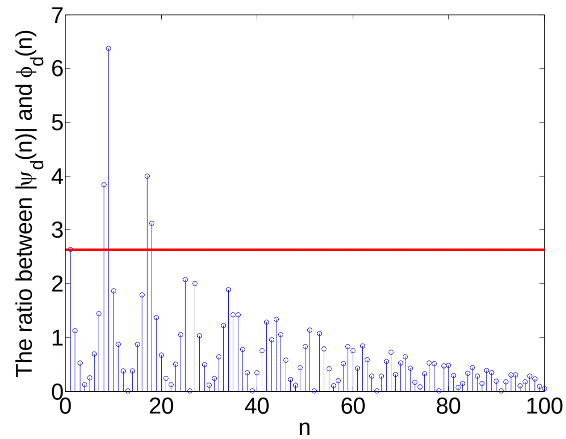


Fig. 9. The value of $f_d(n)$ with different value of $n$

exists a Zames-Falb multiplier for non-odd nonlinearities when $\hat{k}_{ZF} = 1.3028$.

Using the phase limitation result given in Theorem IV.3 part (i), it is possible to show that there is no Zames-Falb multiplier for any $k > k_{PL} = 1.4603$. Fig. 10 illustrates that the phase limitation results indicate there can be no appropriate Zames-Falb multiplier when $k = 1.5$. The phase limitation is given by $\tan^{-1}\rho^d = 66.7137°$, where $\rho^d$ is obtained using Definition IV.1 with $a = 0.7198$ and $b = 0.8996$. By contrast, Fig. 11 shows that this limitation is not active when $k = \hat{k}_{ZF}$; this is expected since we have been able to find a suitable Zames-Falb multiplier for this value of the gain. A complete list of slope restriction results of $G(z)$ in (67) is given in Table II. The result of the reduced off-axis circle criterion $k_R$ shows conservativeness compared to all the other results in the Table. The (false) result from the direct discrete-time counterpart of the off-axis circle criterion is greater than the slope obtained by phase limitation, i.e. $k_O > k_{PL}$; this demonstrates that Conjecture IV.10 is false.

Finally, using combination of deadzone and saturation as nonlinearity, we are able to find periodic solution with $\hat{k}_C = 1.3666$. These results are consistent with Conjecture I.2, i.e. $\hat{k}_{ZF} < \hat{k}_C < k_{PL}$.

TABLE II
RESULTS OF DIFFERENT SLOPE RESTRICTIONS (NON-ODD NONLINEARITY)

| $k$ | $k_{RO}$ | $\hat{k}_{ZF}$ | $\hat{k}_C$ | $k_{PL}$ | $k_O$ | $k_N$ |
|---|---|---|---|---|---|---|
| Result | 0.8962 | 1.3028 | 1.3666 | 1.4603 | 3.61 | 3.61 |

## V. CONCLUSIONS

In this paper we have demonstrated the connection between phase limitations of Zames-Falb multipliers and the Kalman conjecture.

In continuous time, we have generalised a limitation proposed by Megretski, clarified its remit and illustrated its effect with a numerical example. In particular we show it can be applied to a fourth-order plant where the resulting numerical implementation shows instability. It remains open which

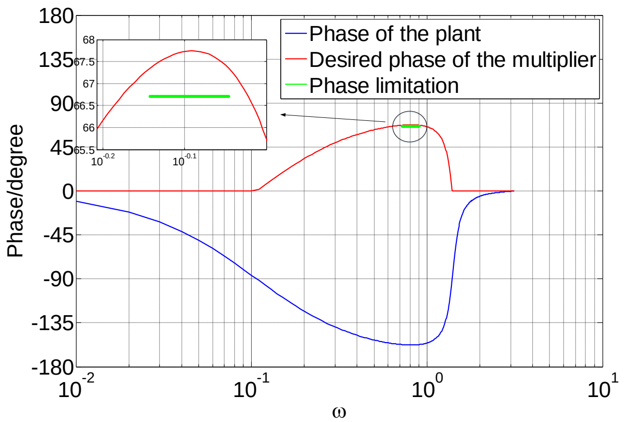


Fig. 10. Phase of $(1+1.5G)$, desired phase of the multiplier and the phase limitation

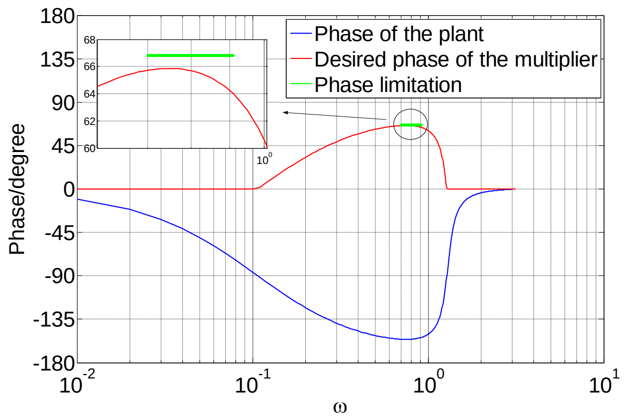


Fig. 11. Phase of $(1+1.3028G)$, desired phase of the multiplier and the phase limitation

choice of intervals $[a,b]$ and $[c,d]$ and scaling parameter $\kappa$ in Theorem III.4 provides most insight.

Motivated by this connection and recent results on the Kalman conjecture in discrete time, we have derived a more simple phase limitation for discrete-time Zames-Falb multipliers. Numerical results in discrete time are easier to obtain and we show that the slope restriction obtained by using phase limitation theorems can be about 40% of the Nyquist value even for some second-order examples. Thus the phase limitation can be directly useful when forming benchmarks for searches over Zames-Falb multipliers. Further, the phase limitation can be used to show there can be no direct counterpart in discrete time (Conjecture IV.10) to the off-axis circle criterion for continuous-time systems (Theorem II.10).

Based on the results of this paper, we propose Conjecture I.2, which seems to be compatible with current state-of-the-art knowledge and results for both continuous and discrete-time domains.

There is plenty of scope for future work. It seems possible that the phase limitations might be used to provide a more computationally efficient search fo appropriate multipliers. Phase limitations for the class of Zames-Falb multipliers available when the nonlinearity is quasi-odd [36] require further research.

## VI. Acknowledgements

We would like to thank the anonymous reviewers for their helpful suggestions.

## VII. Appendix

### A. Proof of Lemma III.2

Both the functions

$$f_1(\omega) = \omega - \frac{\sin \omega t}{t} \tag{68}$$

and

$$f_2(\omega) = \omega + \frac{\sin \omega t}{t} \tag{69}$$

are monotone non-decreasing in $\omega$ when $t>0$. It follows that $\phi(t)>0$ and $\tilde{\phi}(t)>0$ when $t>0$. In addition

$$\lim_{t\to\infty} \phi(t) = \lambda(b-a)+\kappa\mu(d-c) > 0, \tag{70}$$

and

$$\lim_{t\to\infty} \tilde{\phi}(t) = \lambda(b-a)+\kappa\mu(d-c) > 0. \tag{71}$$

Finally

$$\begin{aligned}\phi_1(t) &= -[\lambda(b-a)+\kappa\mu(d-c)] \\ &\quad +\lambda\frac{(b^3-a^3)t^2}{6}+\kappa\mu\frac{(d^3-c^3)t^2}{6}+O(t^4) \text{ as } t\to 0,\end{aligned} \tag{72}$$

and

$$\begin{aligned}\psi(t) &= \lambda\frac{(b^2-a^2)t}{2}-\mu\frac{(d^2-c^2)t}{2} \\ &\quad -\lambda\frac{(b^4-a^4)t^3}{24}+\mu\frac{(d^4-c^4)t^3}{24}+O(t^5) \text{ as } t\to 0,\end{aligned} \tag{73}$$

so the choice (28) ensures

$$\lim_{t\to 0}\frac{|\psi(t)|}{\phi(t)} = 0, \tag{74}$$

and

$$\lim_{t\to 0}\frac{|\psi(t)|}{\tilde{\phi}(t)} = 0. \tag{75}$$

### B. Proof of Theorem III.4

Suppose (31) and (32) hold for some multiplier $M(j\omega) = 1-H(j\omega)$. Then

$$\text{Im}(M(j\omega)) = \int_{-\infty}^{\infty} \sin(\omega t)h(t)\,dt, \tag{76}$$

and

$$\text{Re}(M(j\omega)) = 1-\int_{-\infty}^{\infty} \cos(\omega t)h(t)\,dt, \tag{77}$$

where $h$ in the impulse response of $H$. Hence integrating (31) and (32) over their respective intervals gives

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(at)-\cos(bt)}{t}h(t)\,dt &> \\ \rho(b-a)+\rho\int_{-\infty}^{\infty}&\frac{\sin(at)-\sin(bt)}{t}h(t)\,dt,\end{aligned} \tag{78}$$

and

$$\int_{-\infty}^{\infty}\frac{\cos(ct)-\cos(dt)}{t}h(t)\,dt < \\ -\kappa\rho(d-c)-\kappa\rho\int_{-\infty}^{\infty}\frac{\sin(ct)-\sin(dt)}{t}h(t)\,dt. \quad (79)$$

Summing the two inequalities, multiplied by $\lambda$ and $-\mu$ respectively, gives

$$\int_{-\infty}^{\infty}\psi(t)h(t)dt > \rho[\lambda(b-a)+\kappa\mu(d-c)]+\rho\int_{-\infty}^{\infty}\phi_1 h(t)\,dt. \quad (80)$$

(i) If $M\in\mathscr{M}^c$ then $\|h\|_1<1$ and $h(t)\geq 0$ for all $t$. So

$$\rho[\lambda(b-a)+\kappa\mu(d-c)] > \\ \int_{-\infty}^{\infty}\rho[\lambda(b-a)+\kappa\mu(d-c)]h(t)\,dt. \quad (81)$$

and hence we can write (80) as

$$\int_{-\infty}^{\infty}(\psi(t)-\rho\phi(t))h(t)\,dt>0. \quad (82)$$

But, since $\phi$ is an even function and $\psi$ is an odd function,

$$\psi(t)-\rho^c\phi(t)\leq 0 \text{ for all } t. \quad (83)$$

Further, since $\phi$ is non-negative,

$$\psi(t)-\rho\phi(t)\leq 0 \text{ for all } t \text{ when } \rho\geq\rho^c. \quad (84)$$

Hence $\rho<\rho^c$.

(ii) If $M\in\mathscr{M}^c_{odd}$ then we can only say $\|h\|_1<1$. Nevertheless,

$$\rho[\lambda(b-a)+\kappa\mu(d-c)] > \\ \int_{-\infty}^{\infty}\rho[\lambda(b-a)+\kappa\mu(d-c)]|h(t)|\,dt. \quad (85)$$

and hence (80) leads to

$$\int_{-\infty}^{\infty}(|\psi(t)|-\rho\tilde{\phi}(t))|h(t)|\,dt>0. \quad (86)$$

But, since $\tilde{\phi}$ is also an even function and (as before) $\psi$ is an odd function,

$$|\psi(t)|-\rho^c_{odd}\tilde{\phi}(t)\leq 0 \text{ for all } t. \quad (87)$$

Further, since $\tilde{\phi}$ is non-negative,

$$|\psi(t)|-\rho\tilde{\phi}(t)\leq 0 \text{ for all } t \text{ when } \rho\geq\rho^c_{odd}. \quad (88)$$

Hence $\rho<\rho^c_{odd}$.

### C. Proof of Lemma IV.2

The result is immediate following a similar argument to the proof of Lemma III.2. In particular, as $\phi_d$ and $\tilde{\phi}_d$ are evaluated for discrete values of $n\geq 1$, their limiting behaviour as $n\to 0$ need not be considered.

### D. Proof of Theorem IV.3

Suppose (51) holds for some multiplier $M(e^{j\omega})=1-H(e^{j\omega})$. Then

$$\text{Im}(M(e^{j\omega}))=\sum_{n=-\infty}^{\infty}\sin(\omega n)h_n \quad (89)$$

and

$$\text{Re}(M(e^{j\omega}))=1-\sum_{n=-\infty}^{\infty}\cos(\omega n)h_n, \quad (90)$$

where $h$ is the impulse response of $H$. Hence integrating (51) over the interval $[a,b]$ gives

$$\sum_{n=-\infty}^{\infty}\frac{\cos(an)-\cos(bn)}{n}h_n > \\ \rho(b-a)+\rho\sum_{n=-\infty}^{\infty}\frac{\sin(an)-\sin(bn)}{n}h_n. \quad (91)$$

(i) If $M\in\mathscr{M}^d$ then $h_0=0$, $\|h\|_1<1$ and $h_n\geq 0$ for all $n$. So

$$\rho(b-a)>\sum_{n=-\infty}^{\infty}\rho(b-a)h_n, \quad (92)$$

and hence we can write (91) as

$$\sum_{n=-\infty}^{\infty}(\psi_d(n)-\rho\phi_d(n))h_n>0. \quad (93)$$

But, since $\phi_d$ is an even function and $\psi_d$ is an odd function,

$$\psi_d(n)-\rho^d\phi_d(n)\leq 0 \text{ for all } n\geq 1. \quad (94)$$

Further, since $\phi_d$ is non-negative,

$$\psi_d(n)-\rho\phi_d(n)\leq 0 \text{ for all } n\geq 1 \text{ when } \rho\geq\rho^d. \quad (95)$$

Hence $\rho<\rho^d$.

(ii) If $M\in\mathscr{M}^d_{odd}$ then we can only say $h_0=0$ and $\|h\|_1<1$. Nevertheless,

$$\rho(b-a)>\sum_{n=-\infty}^{\infty}\rho(b-a)|h_n|, \quad (96)$$

and hence (91) leads to

$$\sum_{n=-\infty}^{\infty}(|\psi_d(n)|-\rho\tilde{\phi}_d(n))|h_n|>0. \quad (97)$$

But, since $\tilde{\phi}_d$ is also an even function and (as before) $\psi_d$ is an odd function,

$$|\psi_d(n)|-\rho^d_{odd}\tilde{\phi}_d(n)\leq 0 \text{ for all } n\geq 1. \quad (98)$$

Further, since $\tilde{\phi}_d$ is non-negative,

$$|\psi_d(n)|-\rho\tilde{\phi}_d(n)\leq 0 \text{ for all } n\geq 1 \text{ when } \rho\geq\rho^d_{odd}. \quad (99)$$

Hence $\rho<\rho^d_{odd}$.

### E. Proof of Theorem IV.5

Substituting

$$\mathrm{Im}(M(e^{j\omega})) = \sum_{n=-\infty}^{\infty} \sin(\omega n) h_n \tag{100}$$

and

$$\mathrm{Re}(M(e^{j\omega})) = 1 - \sum_{n=-\infty}^{\infty} \cos(\omega n) h_n, \tag{101}$$

into (54) leads to (91). The proof is then identical to that of Theorem IV.3.

### F. Proof of Proposition IV.7

(i) Let $M_n(z) = 1 - z^{-n}$ with $n \in \mathcal{N}([a,b])$. Then

$$\mathrm{Im}(M_n(e^{j\omega})) = \sin(\omega n), \tag{102}$$
$$\mathrm{Re}(M_n(e^{j\omega})) = 1 - \cos(\omega n). \tag{103}$$

Integrating over the interval yields

$$\int_a^b \mathrm{Im}(M(e^{j\omega}))\, d\omega = \rho^c \int_a^b \mathrm{Re}(M(e^{j\omega}))\, d\omega. \tag{104}$$

Furthermore, if

$$M(z) = 1 - \sum_{n \in \mathcal{N}([a,b])} \lambda_n z^{-n}, \tag{105}$$

with

$$\lambda_n \geq 0 \text{ and } \sum_{n \in \mathcal{N}([a,b])} \lambda_n = 1, \tag{106}$$

then we may write

$$M(z) = \sum_{n \in \mathcal{N}([a,b])} M_n(z). \tag{107}$$

The proof follows straightforwardly.

(ii) Similar.

### G. Proof of Lemma IV.11

Let

$$\begin{aligned} \varepsilon = \frac{|\psi_d(1)|}{\phi_d(1)} &= \frac{|\cos a - \cos b|}{b - a - (\sin b - \sin a)} \\ &= -\frac{\cos b - \cos a}{b - a - (\sin b - \sin a)} = -\frac{\psi_d(1)}{\phi_d(1)}, \end{aligned} \tag{108}$$

where we have used that $(x - \sin x)$ is a monotonically increasing function; and

$$\begin{aligned} \nu = \frac{2}{(b-a)} \frac{1+\varepsilon}{\varepsilon} &= \frac{2 - 2\psi_d(1)/\phi_d(1)}{-(b-a)\psi_d(1)/\phi_d(1)} \\ &= \frac{2(b-a) - 2\sin b + 2\sin a - 2\cos b + 2\cos a}{(a-b)(\cos b - \cos a)}. \end{aligned} \tag{109}$$

For $n > \nu$, we know $(b-a)n - 2 > 0$. In addition,

$$\varepsilon(b-a)n > 2 + 2\varepsilon, \tag{110}$$

so

$$\frac{|\psi_d(n)|}{\phi_d(n)} = \frac{|\cos(bn) - \cos(an)|}{(b-a)n - [\sin(bn) - \sin(an)]} < \frac{2}{(b-a)n - 2} < \varepsilon. \tag{111}$$

As a result,

$$\frac{|\psi_d(n)|}{\phi_d(n)} < \frac{|\psi_d(1)|}{\phi_d(1)} \qquad \forall n > \nu. \tag{112}$$

Finally, it is easy to check that

$$\frac{|\psi_d(1)|}{\tilde{\phi}_d(1)} = \frac{|\psi_d(1)|}{\phi_d(1)} \tag{113}$$

and hence the same relation holds for $|\psi_d(n)|/\tilde{\phi}_d(n)$.

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