On the Necessity and Sufficiency of Discrete-Time O'Shea–Zames–Falb Multipliers ⋆

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Abstract

This paper considers the robust stability of a discrete-time Lurye system consisting of the feedback interconnection between a linear system and a bounded and monotone nonlinearity. It has been conjectured that the existence of a suitable linear time-invariant (LTI) O'Shea–Zames–Falb multiplier is not only sufficient but also necessary. Roughly speaking, a successful proof of the conjecture would require: (a) a conic parameterization of a set of multipliers that describes exactly the set of nonlinearities, (b) a lossless S-procedure to show that the non-existence of a multiplier implies that the Lurye system is not uniformly robustly stable over the set of nonlinearities, and (c) the existence of a multiplier in the set of multipliers used in (a) implies the existence of an LTI multiplier. We investigate these three steps, showing the current bottlenecks for proving this conjecture. In addition, we provide an extension of the class of multipliers which may be used to disprove the conjecture.

Key words: Multipliers, Lurye systems, nonlinearities, robust stability, S-procedure

1 Introduction

A class of noncausal linear time-invariant (LTI) multipliers preserving the positivity of monotone nonlinearities was proposed by O'Shea (1967) for continuous-time and by O'Shea & Younis (1967) for discrete-time. In continuous-time, Zames & Falb (1968) produced a rigorous treatment of noncausal LTI multipliers to show that the existence of a suitable multiplier is a sufficient condition for the stability of the Lurye system with monotone and bounded nonlinearities. In discrete-time, similar development was produced by Willems & Brockett (1968), but they extended the original class by including linear time-varying (LTV) multipliers. Nowadays, both continuous-time and discrete-time LTI classes of multipliers are referred to as O'Shea–Zames–Falb (OZF), or just Zames–Falb multipliers.

Although the continuous-time class of OZF multipliers has attracted more attention (see Carrasco et al. (2016) for a recent overview), its discrete-time counterpart has attracted significant attention in the past years. Convex searches leading to numerical criteria have been proposed by Fetzer & Scherer (2017), Carrasco et al. (2020), Turner & Drummond (2021), and it is worth highlighting its role in convergence analysis of optimisation algorithms, e.g. Lessard et al. (2016), Freeman (2018), Zhang, Seiler & Carrasco (2022), Michalowsky et al. (2021), Lee & Seiler (2020), Khong et al. (2022). The efficiency of the searches for discrete-time OZF multipliers has generated questions on the conservatism of the sufficient condition with OZF multipliers, e.g. investigations into phase limitations by Megretski (1995), Wang et al. (2017), and limits derived from duality theory by Jönsson & Laiou (1996), Zhang, Carrasco & Heath (2022). In Carrasco et al. (2016), it was conjectured that the existence of a suitable OZF multiplier is, in fact, both necessary and sufficient for robust stability (see Section 3 for a formal statement). The conjecture, also known as Carrasco conjecture, has already underpinned the discovery of the first second-order counterexample to the discrete-time Kalman conjecture (Heath et al. (2015))

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and also motivated a systematic construction of destabilizing nonlinearities by Seiler & Carrasco (2020). Relevant studies of sufficient and necessary robust stability conditions involving other classes of multipliers can be located in Khong & van der Schaft (2018), Khong & Kao (2020, 2021).

The resolution to the Carrasco conjecture would be interesting regardless of the answer. If the conjecture were true, it would solve a problem which has been considered for more than 60 years, e.g. Kalman (1957). On the other hand, if the conjecture were false, it would mean that our best known criterion for slope-restricted nonlinearities by Carrasco et al. (2020) is conservative and further research is required to reduce this conservativeness. It is worth highlighting that classically it has been suggested that the conservativeness of the multiplier approach is due to the lack of a class of multipliers that tightly characterises the class of nonlinearities (see Figure 8 in Jönsson (2001*a*)). Here, we show that the LTV class of multipliers in Willems & Brockett (1968) is an exact characterization. Hence, if the Carrasco conjecture were false, it would demonstrate a different source of conservativeness.

This paper analyses the Carrasco conjecture. Firstly, we show that a class of LTV multipliers closely related to the one proposed in Willems & Brockett (1968) can tightly characterise the class of monotone nonlinearities, and it can be parameterized in terms of conic combinations. Secondly, we show a necessity condition for an extension of the class of nonlinearities. Thirdly, we show that the existence of a suitable LTV multiplier for an LTI system implies the existence of a suitable LTI multiplier, which belongs to the class of OZF multipliers. Our analysis identifies that the requirement of a countably infinite class of multipliers may imply an inherent conservatism, as the current version of the lossless S-procedure is limited to a finite number of constraints. The use of a finite, although arbitrarily large, number of constraints in the S-procedure becomes the main bottleneck to resolving the Carrasco conjecture. As a result, the conjecture remains unsolved. As a means to disprove the Carrasco conjecture, we introduce a set of nonlinear multipliers and show that it also tightly characterises monotone nonlinearities. The Carrasco conjecture would thus be potentially disproved if one could find *an* LTI system that cannot be characterised by any OZF multiplier but can be characterised by a nonlinear multiplier we proposed.

The structure of the paper is as follows. Section 2 describes the notation and provides the preliminaries of the paper. Section 3 provides a formal problem statement. Section 4 provides the main technical results of the paper. Initially, we identify a class of nonlinearities that includes all monotone nonlinearities, over which the robust stability of the Lurye system is ensured by the existence of a suitable finite-impulse-response LTI multiplier. The same condition is shown to be necessary for the uniform boundedness of the Lurye system when the set of nonlinearities is replaced by a relevant relation set. As an intermediate step, we use a wider class of LTV multipliers, and show the Lurye system with a periodic LTV plant is robustly stable against the set of monotone nonlinearities if there exists a suitable LTV multiplier. Then, for LTI plants, we show that the existence of a suitable LTV multiplier is necessary and sufficient for the existence of a suitable LTI multiplier. Moreover, the links between the identified set of nonlinearities and the set of monotone nonlinearities are established through the sets of LTV multipliers that characterise them, whereby the discrete-time version of the classical Zames–Falb theorem is recovered. Finally, Section 5 extends the class of multipliers by using nonlinear multipliers which may be useful to disprove the discrete-time Carrasco conjecture.

2 Notation and Preliminaries

Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_0^+$, $\mathbb{Z}_0^-$, and $\mathbb{Z}^+$ denote the sets of real numbers, complex numbers, integers, non-negative integers, non-positive integers, and positive integers, respectively. We use $\mathrm{Re}\{\lambda\}$ and $|\lambda|$ to denote the real part and the magnitude of a complex number λ, respectively. The set of all conic combinations of elements in $\mathcal{B}$ is $\{\sum_{i=1}^n \alpha_i B_i : \alpha_i \geq 0, B_i \in \mathcal{B}\}$. Given a set $\mathcal{G}$, we denote its closure as $\mathbf{cl}\,\mathcal{G}$.

Define ℓ_2 as the set of (two-sided) discrete-time signals $u : \mathbb{Z} \to \mathbb{R}^n$ where $\sum_{k\in\mathbb{Z}} u_k^\top u_k < \infty$. This forms an inner product space with $\langle u, w\rangle := \sum_{k\in\mathbb{Z}} u_k^\top w_k$ and corresponding norm $\|u\| := [\langle u, u\rangle]^{1/2}$. We will also use one-sided signals $\ell_2^{0+} := \{f \in \ell_2 : f_i = 0, \forall i < 0\}$. Two important operations, defined for any $\tau \in \mathbb{Z}$, are the truncation $P_\tau : \ell_2 \to \ell_2$ and (rightward) shift $S_\tau : \ell_2 \to \ell_2$. The truncation operator is defined by $(P_\tau u)_k := u_k$ for $k \leq \tau$ and $(P_\tau u)_k := 0$ for $k > \tau$. The shift operation is defined by $(S_\tau u)_k := u_{k-\tau}$. Further define the two-sided truncation operator $P_{-\tau,\tau}$ as $(P_{-\tau,\tau} u)_k := u_k$ for $k = -\tau, \ldots, \tau$ and $(P_{-\tau,\tau} u)_k := 0$ otherwise. The one-sided extended space is $\ell_{2e}^{0+} := \{f : \mathbb{Z} \to \mathbb{R}^n : P_\tau f \in \ell_2^{0+}, \forall \tau \in \mathbb{Z}_0^+\}$. Consider finite sequences of real numbers $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$. The sequences are similarly ordered if $v_i < v_j$ implies that $w_i \leq w_j$. The sequences are unbiased if $v_i w_i \geq 0$ for $1 \leq i \leq n$. The definition of similarly ordered, unbiased ℓ_2 sequences are analogous to those for finite sequences. A finite-dimensional matrix $M \in \mathbb{R}^{n\times n}$ is said to be doubly hyperdominant if $m_{ij} \leq 0$ for $i \neq j$ and $\sum_{i=1}^n m_{ij} \geq 0$ and $\sum_{j=1}^n m_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$. The definition of doubly hyperdominance for doubly-infinite matrices are analogous to that for finite-dimensional matrices.

A system (also called operator) H is modeled as an operator that maps an input sequence u to an output

$y := Hu$. H is said to be linear if $H(u+v) = Hu + Hv$ and $H(cu) = c\,Hu$ for any $u, v \in \ell_2$ and $c \in \mathbb{R}$.

For a system $M : \ell_2 \to \ell_2$, the induced-ℓ_2 norm is defined as:

$$\|M\| := \sup_{u \in \ell_2, u \neq 0} \frac{\|Mu\|}{\|u\|}.$$

M is said to be bounded if $\|M\| < \infty$. The set of all bounded, linear operators mapping from ℓ_2 to ℓ_2 is denoted as $\mathcal{L}(\ell_2, \ell_2)$. The system $M \in \mathcal{L}(\ell_2, \ell_2)$ can be represented by a doubly-infinite matrix of real numbers $\{m_{ij}\}_{i,j \in \mathbb{Z}}$ such that $y = Mu$ is defined by $y_i = \sum_{j \in \mathbb{Z}} m_{ij} u_j$ for $i \in \mathbb{Z}$. This infinite sum exists for all $u \in \ell_2$ and the resulting sequence belongs to ℓ_2. For a system $M \in \mathcal{L}(\ell_2, \ell_2)$, its adjoint system M^* is defined as the system whose matrix representation is the transpose of that of M. A system $M \in \mathcal{L}(\ell_2, \ell_2)$ is said to be self-adjoint if $M = M^*$. M is said to be time-invariant if $MS_\tau = S_\tau M$, for all $\tau \in \mathbb{Z}$. M is time-invariant if and only if its matrix representation is Toeplitz, i.e. $m_{i+l,j+l} = m_{i,j}$ for all $i, j, l \in \mathbb{Z}$. Time-invariance of M implies the matrix representation is uniquely defined by $\overline{m}_i := m_{l+i,l}$ and the response $y = Mu$ is given by the convolution $y_k = \sum_{l \in \mathbb{Z}} \overline{m}_{k-l} u_l$. The diagonals of the Toeplitz matrix $\{\overline{m}_i\}_{i \in \mathbb{Z}}$ are the impulse response coefficients for the LTI system $M \in \mathcal{L}(\ell_2, \ell_2)$. An important fact is that LTI systems in $\mathcal{L}(\ell_2, \ell_2)$ have an equivalent transfer function representation. Specifically, let $\mathbb{L}_\infty$ denote the set of complex functions $\widehat{M}$ satisfying $\operatorname{ess\,sup}_{\omega \in [0, 2\pi)} \bar{\sigma}(\widehat{M}(e^{j\omega})) < \infty$, where $\bar{\sigma}$ denotes the largest singular value (Clarke 2013, P.7). If $M \in \mathcal{L}(\ell_2, \ell_2)$ is time-invariant, then it has a transfer function $\widehat{M} \in \mathbb{L}_\infty$ such that $y = Mu$ is equivalent to multiplication in the frequency domain: $\widehat{y}(\omega) = \widehat{M}(e^{j\omega})\widehat{u}(\omega)$ where $\widehat{u}(\omega) := \sum_{n \in \mathbb{Z}} u_n e^{-j\omega n}$ and $\widehat{y}(\omega) := \sum_{n \in \mathbb{Z}} y_n e^{-j\omega n}$. For $T \in \mathbb{Z}^+$, M is said to be T-periodic if its matrix representation satisfies $m_{i+T,j+T} = m_{i,j}$ for all $i, j \in \mathbb{Z}$.

A system $G : \ell_{2e}^{0+} \to \ell_{2e}^{0+}$ is said to be causal if $P_\tau G P_\tau = P_\tau G$ for all $\tau \in \mathbb{Z}_0^+$. A system $G : \ell_{2e}^{0+} \to \ell_{2e}^{0+}$ is said to be T_0-periodic if $GS_{\tau T_0} = S_{\tau T_0} G$ for all $\tau \in \mathbb{Z}_0^+$ and is said to be time-invariant if it is periodic with $T_0 = 1$. A causal system $G : \ell_{2e}^{0+} \to \ell_{2e}^{0+}$ is said to be bounded if

$$\|G\| := \sup_{\tau \in \mathbb{Z}^+; 0 \neq P_\tau u \in \ell_2^{0+}} \frac{\|P_\tau G u\|}{\|P_\tau u\|} = \sup_{0 \neq u \in \ell_2^{0+}} \frac{\|Gu\|}{\|u\|} < \infty.$$

The set of all bounded, linear operators from ℓ_{2e}^{0+} to ℓ_{2e}^{0+} is denoted $\mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$. Let $\mathbb{H}_\infty$ denote the subspace of $\mathbb{L}_\infty$ that is analytic in the unit disk. If $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ is both time-invariant and causal then it has a transfer function $\widehat{G} \in \mathbb{H}_\infty$. See Dahleh & Diaz-Bobillo (1994) for more details on linear discrete-time operators.

A nonlinearity $\phi : \ell_{2e}^{0+} \to \ell_{2e}^{0+}$ is memoryless if there exists $N : \mathbb{R} \to \mathbb{R}$ such that $(\phi(v))_i = N(v_i)$ for all $i \in \mathbb{Z}_0^+$. The memoryless nonlinearity ϕ is bounded if there exists a constant $C > 0$ such that $|N(x)| \leq C|x|$ $\forall x \in \mathbb{R}$. Note that boundedness implies that $N(0) = 0$. The memoryless nonlinearity ϕ is monotone if $x_1 \geq x_2$ implies $N(x_1) \geq N(x_2)$. For the sake of simplicity, we are not considering the case where the nonlinearity is also required to be odd, but a parallel development may be possible.

3 Problem Formulation

Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ model a discrete-time system that is single-input single-output, LTI, causal and bounded. We consider the Lurye system of G in feedback with a causal bounded nonlinearity $\phi : \ell_{2e}^{0+} \to \ell_{2e}^{0+}$ as shown in Figure 1. The Lurye system $[G, \phi]$ is defined as:

$$\begin{aligned} v &= Gw + e \\ w &= \phi v. \end{aligned} \tag{1}$$

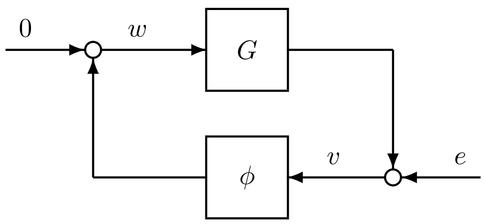


Figure 1. Lurye system

Note that the external signal at the input of G has been set to zero. This is done without loss of generality as the effect of a non-zero external signal at the input of G can be lumped with e due to the assumption that G is linear, causal and bounded.

Well-posedness and stability of the Lurye system are defined next.

Definition 1 $[G, \phi]$ *is* <u>*well-posed*</u> *if for any* $e \in \ell_{2e}^{0+}$ *there exist* $v, w \in \ell_{2e}^{0+}$ *that satisfy* (1) *and depend causally on* e.

Definition 2 $[G, \phi]$ *is* <u>*stable*</u> *if it is well-posed and there exists* $\gamma > 0$ *such that*

$$\sup_{\tau \in \mathbb{Z}^+,\ 0 \neq P_\tau e \in \ell_2^{0+}} \frac{\|P_\tau w\|}{\|P_\tau e\|} \leq \gamma.$$

Let $\mathcal{S}_0$ be a set of nonlinearities that map 0 to 0.

Definition 3 *$[G, \phi]$ is* *uniformly robustly stable over $\mathcal{S}_0$* *if $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}_0$ and there exists $\gamma > 0$ such that*

$$\sup_{\phi \in \mathcal{S}_0} \sup_{\tau \in \mathbb{Z}^+,\ 0 \neq P_\tau e \in \ell_2^{0+}} \frac{\|P_\tau w\|}{\|P_\tau e\|} \leq \gamma.$$

Two classes of nonlinearities will be considered in this work. In particular, the set of all nonlinearities that are memoryless, bounded, monotone is denoted $\mathcal{S}$, i.e.,

$$\mathcal{S} := \{\phi : \ell_2 \to \ell_2 \ : \ \phi \text{ is memoryless, bounded, and monotone}\}.$$

A related set that consists of the input-output pairs generated by $\mathcal{S}$ is defined as [1]

$$\mathcal{G}_1 := \{(v, w) \in \ell_2 : w = \phi v,\ \phi \in \mathcal{S}\}$$

In Section 4.2 we will introduce a second related set of nonlinearities denoted $\mathcal{S}^{T,B}$.

There is a large literature on robust stability of Lurye systems and details can be found in Desoer & Vidyasagar (2009), Willems (1970). The plant $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ is time-invariant and causal. Thus it has an equivalent transfer function representation $\widehat{G}(e^{j\omega}) \in \mathbb{H}_\infty$. For continuous-time systems, it is shown by Khong & Su (2021) that uniform robust stability over the set of monotone nonlinearities is guaranteed by the existence of an LTI O'Shea–Zames–Falb multiplier satisfying a certain frequency domain inequality. In what follows, two sets of multipliers for the discrete-time counterpart are introduced.

Definition 4 *A class of LTV multipliers $\mathcal{M}_{LTV} \subset \mathcal{L}(\ell_2, \ell_2)$ is given by the set of linear operators satisfying the following conditions:*

(1) The associated matrix $M = [m]_{ij}$ is a doubly hyperdominant matrix with zero excess, i.e., $m_{ij} \leq 0, \forall i \neq j$ and $\sum_{i \in \mathbb{Z}} m_{ij} = 0, \forall j \in \mathbb{Z}$, $\sum_{j \in \mathbb{Z}} m_{ij} = 0, \forall i \in \mathbb{Z}$.
(2) For all $\epsilon > 0$ there exists $n = n(\epsilon)$ such that in each row or each column the sum of n entries with largest absolute values is at most ϵ.

[1] Although ϕ is defined to map ℓ_{2e}^{0+} to ℓ_{2e}^{0+} for the Lurye system in (1), $\mathcal{G}_1 \subset \ell_2$ is well-defined since $\phi \in \mathcal{S}$ is memoryless and bounded.

Remark 5 *It is known that the finite-dimensional doubly hyperdominant matrices with zero excess are precisely the convex combinations of permutation matrices subtracted from I of the same dimension (Willems 1970, Th. 3.7). The same continues to hold for infinite-dimensional matrices satisfying the second condition in Definition 4, by which the infinite convex combinations converge in operator norm; see the proof of Lemma 6 in the next section and Isbell (1955). Note also that the set $\mathcal{M}_{LTV}$ defined above is a subset of the class of LTV multipliers introduced by Willems & Brockett (1968), which is defined as the set of linear operators satisfying the first condition in Definition 4 without the zero-excess constraint.*

The class of LTI multipliers $\mathcal{M}_{\mathrm{LTI}} \subset \mathcal{L}(\ell_2, \ell_2)$ is defined as

$$\mathcal{M}_{\mathrm{LTI}} := \{M \in \mathcal{M}_{\mathrm{LTV}} : M \text{ is LTI}\}. \tag{2}$$

The class of LTI multipliers $\mathcal{M}_{\mathrm{LTI}}$ is a subset of the class of OZF multipliers in O'Shea & Younis (1967). In particular, the class of OZF multipliers is the same as the set of all LTI elements in the class of LTV multipliers introduced by Willems & Brockett (1968). It must be highlighted that to the best of the authors' knowledge all stability criteria in the literature are obtained using LTI multipliers.

Conjecture 1 *[Carrasco Conjecture (Carrasco et al. (2016))] Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded. Assume the Lurye system $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}$. The feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}$ if and only if there exists $M \in \mathcal{M}_{LTI}$ such that*

$$\mathrm{Re}\left\{\widehat{M}(e^{j\omega})\widehat{G}(e^{j\omega})\right\} < 0, \forall \omega \in [0, 2\pi]. \tag{3}$$

The sufficiency of Conjecture 1 is a straightforward application of the multiplier results in Willems & Brockett (1968) and using the discrete-time counterpart of either the classical passivity results, e.g. see Section 9.3 in Desoer & Vidyasagar (2009), or the IQC theorem in Megretski & Rantzer (1997). The main results of this paper analyse the necessity direction in Conjecture 1.

4 Main Results

The main results are given in four different subsections. Firstly, we show an exact characterisation of monotone nonlinearities involving conic parameterization. Secondly, we define a larger set of nonlinearities which will be used in the necessity results. We show a limiting argument that relates this set with the set of monotone nonlinearities. Thirdly, we provide a necessary condition based on the lossless S-procedure that involves a finite number of constraints. The same condition is shown to be sufficient for robust stability against the larger set of nonlinearities. Finally, we show the equivalence between LTV and LTI multipliers.

4.1 Characterization of Monotone Nonlinearities

In this subsection, we show that the set of input-output pairs of all nonlinearities in $\mathcal{S}$ can be tightly characterised by the set of time-varying multipliers $\mathcal{M}_{\mathrm{LTV}}$, i.e., the closure of $\mathcal{G}_1$ is equal to the set $\mathcal{G}_2$ defined as

$$\mathcal{G}_2 := \{(v, w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \, \forall M \in \mathcal{M}_{\mathrm{LTV}}\}.$$

By introducing a conic parameterization for $\mathcal{M}_{\mathrm{LTV}}$, we also show that $\mathcal{G}_1$ can be equivalently characterised by a strict subset of multipliers in $\mathcal{M}_{\mathrm{LTV}}$.

Let $\mathcal{P}$ denote the set of operators in $\mathcal{L}(\ell_2, \ell_2)$ whose matrix representation is a doubly-infinite permutation matrix. Define

$$\mathcal{C} := \{C \in \mathcal{L}(\ell_2, \ell_2) : C = I - P, P \in \mathcal{P}\}. \qquad (4)$$

Observe that each element in $\mathcal{C}$ has a matrix representation which is doubly hyperdominant with zero excess, i.e., the sum of each row and each column is zero.

The next lemma provides a useful representation for the set $\mathcal{M}_{\mathrm{LTV}}$.

Lemma 6 *The set of all conic combinations of elements in $\mathcal{C}$ is equal to $\mathcal{M}_{LTV}$.*

The proof of Lemma 6 is provided in the appendix.

Define the set of sequence pairs that can be characterised by all the multipliers in $\mathcal{C}$ as follows:

$$\mathcal{G}_3 := \{(v, w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \, \forall M \in \mathcal{C}\}.$$

The following theorem presents the main result in this subsection.

Theorem 7 *It holds that* $\mathbf{cl}\, \mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3$.

The proof of Theorem 7 is provided in the appendix. Noting that the elements in $\mathcal{S}$ are memoryless bounded and monotone, Theorem 7 remains true with $v, w \in \ell_2^{0+}$ since $\ell_2^{0+} \subset \ell_2$. That is, the following sets are equal

$$\begin{aligned} &\mathbf{cl}\, \{(v, w) \in \ell_2^{0+} : w = \phi v, \, \phi \in \mathcal{S}\} \\ &\{(v, w) \in \ell_2^{0+} : \langle Mv, w\rangle \geq 0, \, \forall M \in \mathcal{M}_{\mathrm{LTV}}\} \\ &\{(v, w) \in \ell_2^{0+} : \langle Mv, w\rangle \geq 0, \, \forall M \in \mathcal{C}\}. \end{aligned}$$

Observe that in Theorem 7, any element in $\mathcal{G}_1$ satisfies a nonlinear constraint, while that in $\mathcal{G}_2$ is quadratically constrained by an uncountable number of linear multipliers. The characterisation of $\mathcal{G}_3$, on the other hand, is the cleanest in that its elements are quadratically constrained by a countable number of linear multipliers. In essence, Theorem 7 demonstrates that sequences related by monotone nonlinearity may be characterised by a countable number of quadratic constraints. The fact that the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_2$ shows that the class $\mathcal{M}_{\mathrm{LTV}}$ provides an exact characterisation of the class of nonlinearities $\mathcal{S}$. This is relevant for the Carrasco conjecture since it shows that there is no conservativeness in the description of the nonlinearities by linear multipliers. Moreover, the fact that the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_3$ shows that we can rewrite the class $\mathcal{M}_{\mathrm{LTV}}$ as a conic combination, which is required for the use of the S-procedure.

4.2 A Larger Set of Nonlinearities

In this subsection, we introduce a larger set of nonlinearities which contains all nonlinearities in $\mathcal{S}$ as a subset. This set facilitates the proof of the necessity of the existence of a certain multiplier for robust stability. In the context of the Carrasco conjecture, this development is interesting since either it shows the current bottleneck to show its validity or it shows the source of conservatism of the theorem if the Carrasco conjecture were to be false.

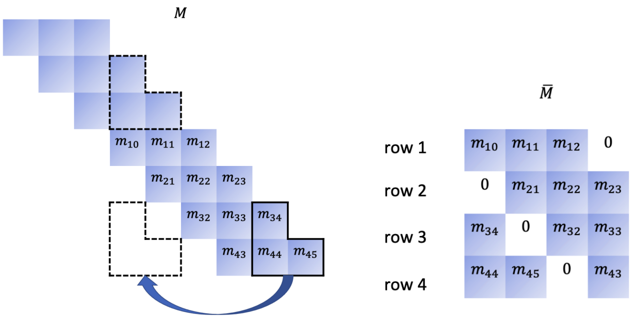
We start with defining a set of permutations. To this end, let $\pi(\cdot) : \mathbb{Z} \to \mathbb{Z}$ denote a permutation. For $T, B \in \mathbb{Z}^+$, define $\mathcal{P}_{T,B}$ as the set of $\pi(\cdot)$ that satisfies $\pi(k + T) = \pi(k) + T, \forall k \in \mathbb{Z}$ and $\pi(i) \neq j$, for all $i, j \in \mathbb{Z}$ such that $|i - j| > B$. In other words, $\mathcal{P}^{T,B}$ is the set of all operators P whose matrix representations are permutation matrices that are T-periodic and B-banded. Based upon the defined set of permutations, we define $\mathcal{G}^{T,B}$ as the set of (v, w) satisfying that the inner product $\langle v, w\rangle$ is not less than the inner products of every permuted v and w, i.e.,

$$\mathcal{G}^{T,B} := \{(v, w) \in \ell_2 : \sum_{k\in\mathbb{Z}} v_k w_k \geq \sum_{k\in\mathbb{Z}} v_{\pi(k)} w_k, \quad \forall \pi \in \mathcal{P}_{T,B}\}.$$

In what follows, we define a subset of $\mathcal{M}_{\mathrm{LTV}}$ that consists of periodic and banded elements. Given $T, B \in \mathbb{Z}^+$, let $\mathcal{M}_{\mathrm{LTV}}^{T,B}$ be defined as

$$\mathcal{M}_{\mathrm{LTV}}^{T,B} := \{M \in \mathcal{M}_{\mathrm{LTV}} : M \text{ is } T\text{-periodic}, \quad m_{ij} = 0, \, \forall |i - j| > B\}.$$

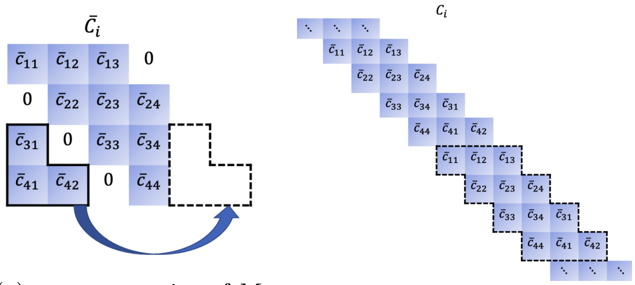
Recall from Lemma 6 that $\mathcal{M}_{\mathrm{LTV}}$ is equal to the set of all conic combinations of elements in $\mathcal{C}$ where $\mathcal{C}$ is defined



(a) a representation of M and the translation

(b) $\overline{M}$

Figure 2.



(a) a representation of M and the translation

(b) $\overline{M}$

Figure 3.

in (4). To derive an analogous result for the case with $\mathcal{M}^{T,B}_{\mathrm{LTV}}$, define

$$\begin{aligned}\mathcal{C}^{T,B} := \{C \in \mathcal{C} : C \text{ is } T\text{-periodic},\\ c_{ij} = 0,\ \forall |i-j| > B\}.\end{aligned}$$

Lemma 8 *Given T, B in $\mathbb{Z}^+$ with $T \geq 2B+1$, the set $\mathcal{M}^{T,B}_{LTV}$ is equal to the set of all conic combinations of elements in $\mathcal{C}^{T,B}$.*

PROOF. ($\supset$) This follows from Lemma 6 and the fact that the sum of T-periodic B-banded matrices is still T-periodic and B-banded.

($\subset$) For ease of exposition, we prove in the following using an concrete example with $T = 4$ and $B = 1$. Note that the same line of reasoning can be applied to prove for the general cases with any $T, B \in \mathbb{Z}^+$ satisfying $T \geq 2B + 1$.

Any $M \in \mathcal{M}^{4,1}_{\mathrm{LTV}}$ can be described by the banded matrix in Figure 2 (a). Therein, the shaded entries are possibly nonzero while the other entries are all zeros. Since M is 4-periodic, the associated matrix of M can be represented by any 4 consecutive rows, say row 1 to row 4. By definition, all rows and columns of M sum to zero. Hence, the sum of rows $1, 2, 3, 4$ and columns $0, 1, 2, 3$ are zeros. Now we translate horizontally the entries m_{34}, m_{44}, m_{45} to the left by four steps, as indicated in Figure 2 (a). Then apparently the row sum for rows $1, 2, 3, 4$ remain the same. Moreover, the fact that M is 4-periodic implies that the translated entries inside the dashed border in rows $3, 4$ can be equivalently obtained by translating the entries inside the dashed border in rows $-1, 0$ vertically downward by four steps. Therefore, the 4×4 matrix $\overline{M}$ as shown in Figure 2 (b), which is obtained by extracting rows $1, 2, 3, 4$ and columns $0, 1, 2, 3$ of Figure 2 (a) after translation, is doubly hyperdominant with zero excess. Next, let $\mathcal{P}_4$ denote the set of 4×4 permutation matrices. By the arguments of Lemma 6 tailored to finite-dimensional matrices, we have that $\overline{M}$ can be expressed as a conic combination of elements in $\mathcal{C}_4 := \{\overline{C} \in \mathbb{R}^{4\times 4} : \overline{C} = I_4 - P, P \in \mathcal{P}_4\}$. In other words, there exist $\alpha_i > 0, i = 1, \ldots, n$ with $n \leq 4!$ such that $\overline{M} = \sum_{i=1}^n \alpha_i \overline{C}_i$ with $\overline{C}_i \in \mathcal{C}_4$. Note that every off-diagonal entry in $\overline{C} \in \mathcal{C}_4$ takes values in $\{-1, 0\}$. Hence, if the ij-th entry in $\overline{M}$ is zero, then the ij-th entry in all $\overline{C} \in \{\overline{C}_1, \ldots, \overline{C}_n\}$ must be zero. Let $\overline{C}_i$ be any element in $\{\overline{C}_1, \ldots, \overline{C}_n\}$, depicted in Figure 3 (a).

By reversing the process in Figure 2, we translate horizontally the entries $\bar{c}_{31}, \bar{c}_{41}, \bar{c}_{42}$ in $\overline{C}_i$ to the right by four steps, as shown in Figure 3 (a), whereby a banded array is obtained. Next, by repeating the obtained banded array as shown in Figure 3 (b), we get a banded and periodic matrix C_i. Since $\overline{M} = \sum_{i=1}^n \alpha_i \overline{C}_i$, it follows from the operations described in Figures 2 and 3 that $M = \sum_{i=1}^n \alpha_i C_i$. It can also be observed that C_i is a permutation matrix that is 1-banded and 4-periodic, i.e., $C_i \in \mathcal{C}^{4,1}$. Hence, each element in $\mathcal{M}^{4,1}$ can be expressed as a conic combination of elements in $\mathcal{C}^{4,1}$. □

As illustrated in the proof above, the condition $T \geq 2B+1$ is used to find the conic basis of the same period as that of $\mathcal{M}^{T,B}_{\mathrm{LTV}}$. When $T < 2B+1$, define T_n as

$$T_n := \min_{n \in \mathbb{Z}^+} nT \ \text{ such that } nT \geq 2B+1.$$

Following the same line of argument in the preceding proof, it can be shown that each element in $\mathcal{M}^{T,B}_{\mathrm{LTV}}$ can be expressed as a conic combination of elements in $\mathcal{C}^{T_n,B}$. For instance, let $T = 2, B = 1$, then by definition $T_n = 2T = 4$. By taking 4 consecutive rows in $M \in \mathcal{M}^{2,1}_{\mathrm{LTV}}$, we can perform the same operations depicted in Figures 2 and 3 to show that each element in $\mathcal{M}^{2,1}$ can be expressed as a conic combination of elements in $\mathcal{C}^{4,1}$. As a result, for the case with $T < 2B+1$, one can replace T with T_n in $\mathcal{M}^{T,B}_{\mathrm{LTV}}$ and $\mathcal{C}^{T,B}_{\mathrm{LTV}}$, and then the result in Lemma 8 remains true.

The next theorem states that $\mathcal{G}^{T,B}$ can be tightly characterised by a positivity condition involving $\mathcal{M}^{T,B}_{\mathrm{LTV}}$ or $\mathcal{C}^{T,B}$.

Theorem 9 *Given T, B in $\mathbb{Z}^+$ with $T \geq 2B+1$, the set $\mathcal{G}^{T,B}$ is equal to $\{(v,w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \forall M \in \mathcal{M}_{LTV}^{T,B}\}$. The set $\mathcal{G}^{T,B}$ is also equal to $\{(v,w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \forall M \in \mathcal{C}^{T,B}\}$.*

PROOF. By definition, $\mathcal{G}^{T,B}$ can be expressed as

$$\mathcal{G}^{T,B} = \{(v,w) \in \ell_2 : \langle v, w\rangle \geq \langle Pv, w\rangle, \forall P \in \mathcal{P}^{T,B}\},$$

where $\mathcal{P}^{T,B}$ is the set of all operators P whose matrix representations are permutation matrices satisfying that $p_{ij} = p_{i+T,j+T}, \forall i,j \in \mathbb{Z}$ and $p_{ij} = 0, \forall |i-j| > B$. By definition, we have that the set $\{I - P : P \in \mathcal{P}^{T,B}\}$ is equal to $\mathcal{C}^{T,B}$. Therefore, we have

$$\mathcal{G}^{T,B} = \{(v,w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \forall M \in \mathcal{C}^{T,B}\}.$$

Then, it follows from Lemma 8 that the set of all conic combinations of elements in $\mathcal{C}^{T,B}$ is equal to $\mathcal{M}_{\text{LTV}}^{T,B}$. This, in turn, implies that

$$\mathcal{G}^{T,B} = \{(v,w) \in \ell_2 : \langle Mv, w\rangle \geq 0, \forall M \in \mathcal{M}_{\text{LTV}}^{T,B}\}.$$

$\square$

Since every $C \in \mathcal{C}^{T,B}$ has a finite bandwidth B and a finite period T, the number of elements in $\mathcal{C}^{T,B}$, denoted as $1 + N_{T,B}$, is finite. Denote

$$\mathcal{C}^{T,B} = \{0\} \cup \{C_1, C_2, \ldots, C_{N_{T,B}}\}. \tag{5}$$

For any $T, B \in \mathbb{Z}^+$, let $\mathcal{S}^{T,B}$ denote the set of all causal bounded nonlinearities $\phi : \ell_2 \to \ell_2$ such that

$$\begin{aligned}\mathcal{S}^{T,B} := \{\phi : \ell_2 \to \ell_2 \ :\ & \phi \text{ is causal, bounded and} \\ & \langle Mv, \phi v\rangle \geq 0, \forall v \in \ell_2,\ M \in \mathcal{M}_{\text{LTV}}^{T,B}\}.\end{aligned}$$

The links between the sets $\mathcal{S}$ and $\mathcal{S}^{T,B}$ are explained in the following lemma with its proof provided in the appendix.

Lemma 10 *For any $T, B \in \mathbb{Z}^+$, the set $\mathcal{S}$ is a subset of $\mathcal{S}^{T,B}$. Moreover, it holds that*

$$\mathcal{G}^{kT,B_1} \subset \mathcal{G}^{T,B_2}, \ \ \forall k \in \mathbb{Z}^+, B_1 \geq B_2$$

and

$$\bigcap_{n \in \mathbb{Z}^+} \mathcal{G}^{2^n,n} = \mathbf{cl}\,\mathcal{G}_1.$$

Again, it should be noted that Theorem 9 and Lemma 10 remain true with $v, w \in \ell_2^{0+}$ since $\ell_2^{0+} \subset \ell_2$.

Theorem 9 and Lemma 10 are significant since they show that $\mathcal{G}^{T,B}$ can be characterised by a finite number of quadratic constraints and that as $T, B \to \infty$ in a certain manner, we recover the uncertainty set $\mathcal{G}_1$. In effect, $\mathcal{G}^{T,B}$ is a larger uncertainty set than $\mathcal{G}_1$ against which we show the necessity of the existence of an appropriate LTI multiplier in the subsequent subsections. Whereas it is not known if the same is necessary for robustness against $\mathcal{G}_1$, the relation between $\mathcal{G}^{T,B}$ and $\mathcal{G}_1$ provides a justification for our endeavours and main results in this section.

4.3 Robust Stability with LTV Multipliers

In this subsection, we show that the existence of an appropriate LTV multiplier is necessary and sufficient for establishing the uniform robust stability of $[G, \phi]$. We start with the sufficiency direction for the case with $\phi \in \mathcal{S}^{T,B}$. The next result shows that the existence of an appropriate LTV multiplier in $\mathcal{M}_{LTV}^{T,B} \subset \mathcal{M}_{LTV}$ guarantees robustness against the uncertainty set $\mathcal{S}^{T,B} \supset \mathcal{S}$. Note that an element in $\mathcal{M}_{LTV}^{T,B}$ has more structure than those in $\mathcal{M}_{LTV}$, and its existence gives rise to robustness against a larger uncertainty set $\mathcal{S}^{T,B}$ than $\mathcal{S}$. In view of Lemma 10, the next result thus complements the known result that the existence of an LTV multiplier introduced by Willems & Brockett (1968) is sufficient for robustness against $\mathcal{S}$.

Theorem 11 *Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded. Let T, B be in $\mathbb{Z}^+$, and assume the Lurye system $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}^{T,B}$. The feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}^{T,B}$ if*

$$\begin{aligned}&\exists M \in \mathcal{M}_{LTV}^{T,B},\ \epsilon > 0 \\ &s.t.\ \ \langle MGw, w\rangle \leq -\epsilon\|w\|^2, \forall w \in \ell_2^{0+}.\end{aligned} \tag{6}$$

PROOF. The proof follows from existing results on integral quadratic constraints (IQCs) by Rantzer & Megretski (1997). Specifically, note from the definition of $\mathcal{S}^{T,B}$ that if $\phi \in \mathcal{S}^{T,B}$, then $\lambda\phi \in \mathcal{S}^{T,B}$ for all $\lambda \in [0,1]$. Then, by the definition of $\mathcal{S}^{T,B}$, we have

$$\left\langle \begin{bmatrix} v \\ \lambda\phi v \end{bmatrix}, \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} \begin{bmatrix} v \\ \lambda\phi v \end{bmatrix} \right\rangle = 2\langle Mv, \lambda\phi v\rangle \geq 0$$

for all $\phi \in \mathcal{S}^{T,B}, \lambda \in [0,1], v \in \ell_2^{0+}$, and all $M \in \mathcal{M}_{\text{LTV}}^{T,B}$. By hypothesis there exist $M \in \mathcal{M}_{\text{LTV}}^{T,B}$ and $\epsilon > 0$ such that $\langle MGw, w\rangle \leq -\epsilon\|w\|^2, \forall w \in \ell_2^{0+}$. Since G is bounded, it implies that there exist $M \in \mathcal{M}_{\mathrm{LTV}}^{T,B}$ and $\epsilon > 0$ such that

$$\left\langle \begin{bmatrix} Gw \\ w \end{bmatrix}, \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} \begin{bmatrix} Gw \\ w \end{bmatrix} \right\rangle \leq -\epsilon \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2.$$

Thus, it follows from the IQC theorem by (Khong 2021, Cor. IV.3) that for every $\phi \in \mathcal{S}^{T,B}$ the feedback system $[G, \phi]$ is stable. Uniform robust stability follows from the proof in (Khong & Su 2021, Th. 6). $\square$

Next, we consider the nonlinearity set $\mathcal{S}$, which is a subset of $\mathcal{S}^{T,B}$ according to Lemma 10. As with $\mathcal{S}^{T,B}$, we show in the following proposition that the uniform robust stability of $[G, \phi]$ over $\mathcal{S}$ can be ensured by the existence of a suitable LTV multiplier.

Proposition 12 *Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded. Assume the Lurye system $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}$. The feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}$ if*

$$\begin{aligned} &\exists M \in \mathcal{M}_{LTV},\ \epsilon > 0 \\ &s.t.\ \ \langle MGw, w\rangle \leq -\epsilon\|w\|^2, \forall w \in \ell_2^{0+}. \end{aligned} \tag{7}$$

PROOF. The claim follows from Theorem 7 and the same line of reasoning as in the proof of Theorem 11. $\square$

The sufficiency result in Proposition 12 is derived by using a modern IQC formulation as shown in the proof of Theorem 11 in contrast with the similar results in (Willems & Brockett 1968, Th. 7) which is based on classical multiplier theory. The IQC formulation will be generalised in Section 5 to consider an alternative sufficient condition involving nonlinear multipliers.

Before we can develop necessary conditions, we require a version of the lossless S-procedure dealing with LTV quadratic constraint. A time-invariant lossless S-procedure was presented by Jönsson (2001*b*). Here we provide a lossless S-procedure that involves time-varying quadratic forms based on the S-procedure lossless theorem by Jönsson (2001*b*).

Define the quadratic forms $\sigma_k : \ell_2^{0+} \to \mathbb{R}$ as

$$\sigma_k(f) = \langle f, \Pi_k f\rangle, \qquad k = 0, 1, \ldots, N,$$

where $\Pi_k : \ell_2 \to \ell_2, k = 0, 1, \ldots, N$.

Assumption 13 *Let $T, T_0 \in \mathbb{Z}^+$. Assume that*

- Π_0 *is bounded, linear, self-adjoint, and T_0-periodic;*
- Π_k, $k = 1, \ldots, N$ *are bounded, linear, self-adjoint, and T-periodic.*

Lemma 14 *Suppose the quadratic forms σ_k, k=0,1,...,N satisfy Assumption 13 and that there exists $f^* \in \ell_2^{0+}$ such that $\sigma_k(f^*) > 0$ for $k = 1, \ldots, N$. Then the following are equivalent:*

(i) *$\sigma_0(f) \leq 0$ for all $f \in \ell_2^{0+}$ that satisfy $\sigma_k(f) \geq 0, \forall k = 1, 2, \ldots, N$;*
(ii) *There exists $\alpha_k \geq 0$, $k = 1, \ldots, N$ such that*

$$\sigma_0(f) + \sum_{k=1}^{N} \alpha_k \sigma_k(f) \leq 0, \quad \forall f \in \ell_2^{0+}.$$

The proof of Lemma 14 is provided in the appendix.

Before presenting the main theorem, another supporting lemma is stated next.

Lemma 15 *Given a pair of sequences $\{v_1, v_2, \ldots, v_n\}$, $\{w_1, w_2, \ldots, w_n\}$, suppose $v_1 > v_2 > \cdots > v_n > 0$ and $w_1 > w_2 > \cdots > w_n > 0$. Then $\sum_{i,j=1}^{n} m_{ij} v_i w_j > 0$ for all nonzero $M = [m]_{ij} \in \mathbb{R}^{n\times n}$ that are doubly hyperdominant.*

The proof of Lemma 15 is provided in the appendix.

Theorem 16 *Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded, and let T, B in $\mathbb{Z}^+$ be given with $T \geq 2B + 1$. Suppose that $v, w, e \in \ell_{2e}^{0+}$ satisfy $v = Gw + e$. There exists $\gamma > 0$ such that*

$$\sup_{(v,w)\in\mathcal{G}^{T,B}\cap\ell_2^{0+}} \frac{\|w\|}{\|e\|} \leq \gamma \tag{8}$$

only if there exist $M \in \mathcal{M}_{LTV}^{T,B}$ and $\epsilon > 0$ such that

$$\langle MGw, w\rangle \leq -\epsilon\|w\|^2, \forall w \in \ell_2^{0+}.$$

PROOF. Suppose there exists $\gamma > 0$ such that (8) holds. Then, we have

$$\sigma_0(v, w) := \|w\|^2 - \gamma^2\|v - Gw\|^2 \leq 0$$

for all $(v, w) \in \ell_2^{0+}$ that are in $\mathcal{G}^{T,B}$. Define

$$\sigma_k(v, w) := \langle C_k v, w\rangle, k = 1, 2 \ldots, N_{T,B},$$

where $(v, w) \in \ell_2^{0+}$, $C_k \in \mathcal{C}^{T,B}$ is defined in (5) and $N_{T,B}$ is the number of nonzero elements in $\mathcal{C}^{T,B}$. By invoking Theorem 9, we then have

$$\begin{aligned} \sigma_0(v, w) \leq 0, \ \ &\text{for all } (v, w) \in \ell_2^{0+} \text{ such that} \\ &\sigma_k(v, w) \geq 0, k = 1, \ldots, N_{T,B}. \end{aligned} \tag{9}$$

Next, define v^*, w^* such that $v_k^* = w_k^* := \frac{1}{k+1}$, for $k = 0, 1, \ldots, T-1$ and $v_k^* = w_k^* = 0$ otherwise. It is clear that $(v^*, w^*) \in \ell_2^{0+}$, and we show in the following that $\langle Cv^*, w^* \rangle > 0$ for all nonzero $C \in \mathcal{C}^{T,B}$.

Let C be any nonzero element in $\mathcal{C}^{T,B}$. Then, we have that

$$\langle Cv^*, w^* \rangle = \left\langle \begin{bmatrix} 1 \\ \vdots \\ \frac{1}{T} \end{bmatrix}, \tilde{C} \begin{bmatrix} 1 \\ \vdots \\ \frac{1}{T} \end{bmatrix} \right\rangle,$$

where $\tilde{C} \in \mathbb{R}^{T \times T}$ is the corresponding principal submatrix of C. $\tilde{C}$ is nonzero as C is nonzero and T-periodic. Since every principle submatrix of a doubly hyperdominant matrix must be doubly hyperdominant. According to Lemma 15, one has that

$$\left\langle \begin{bmatrix} 1 \\ \vdots \\ \frac{1}{T} \end{bmatrix}, \tilde{C} \begin{bmatrix} 1 \\ \vdots \\ \frac{1}{T} \end{bmatrix} \right\rangle > 0.$$

Thus, $\langle Cv^*, w^* \rangle > 0$ for all nonzero $C \in \mathcal{C}^{T,B}$.

Noting that the operator defining the quadratic form σ_0 is time-invariant and the operators defining $\sigma_k, k = 1, \ldots, N_{T,B}$ are T-periodic, Assumption 13 can be easily verified. By invoking Lemma 14, (9) holds if and only if there exist $\alpha_k \geq 0, k = 1, \ldots, N_{T,B}$ such that

$$\sigma_0(v, w) + \sum_{k=1}^{N_{T,B}} \alpha_k \sigma_k(v, w) \leq 0, \ \forall (v, w) \in \ell_2^{0+}. \quad (10)$$

Now consider the subspace $\{(v, w) \in \ell_2^{0+} : v = Gw\}$, equation (10) implies that

$$\sum_{k=1}^{N_{T,B}} \langle \alpha_k C_k G w, w \rangle \leq -\|w\|^2.$$

The proof is completed by noting that $\sum_{k=1}^{N_{T,B}} \alpha_k C_k \in \mathcal{M}_{\text{LTV}}^{T,B}$. □

Remark 17 *It should be remarked that Theorems 11 and 16 hold also for the case with linear and periodic $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$. Specifically, for any $T_0 \in \mathbb{Z}^+$, the preceding proofs can be employed directly to show the same results in Theorems 11 and 16 but with T_0-periodic G. This can be observed by that the proof of Theorem 11 does not require G to be time-invariant, and the underlying lossless S-procedure for proving Theorem 16 allows for periodically time-varying quadratic form σ_0 as described in Lemma 14.*

Remark 18 *Recall from Lemma 10 that $\bigcap_{n \in \mathbb{Z}^+} \mathcal{G}^{2^n, n} = \mathbf{cl}\, \mathcal{G}_1$. Therefore, if Theorem 16 could be extended to the case with $(T, B) = \lim_{n\to\infty}(2^n, n)$, then it could be implied that $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}$ only if there exist $M \in \mathcal{M}_{LTV}$ and $\epsilon > 0$ such that $\langle MGw, w \rangle \leq -\epsilon \|w\|^2$ for all $w \in \ell_2^{0+}$. However, Theorem 16 can not be extended to the case with $(T, B) = \lim_{n\to\infty}(2^n, n)$ since the lossless S-procedure stated in Lemma 14 is no longer applicable. In particular, the number of quadratic forms $N_{T,B}$ needed to establish Theorem 16 will approach infinity as either T or B increases to infinity. By Lemma 10, an infinite number of quadratic forms are required to tightly characterise the set $\mathcal{S}$, which hinders the use of the lossless S-procedure. On the other hand, if there were an lossless S-procedure that allows for infinite number of quadratic forms, then the condition in Proposition 12 is also necessary whereby the discrete-time Carrasco conjecture can be proved based on the results in Section 4.4.*

4.4 Robust Stability with LTI Multipliers

By constraining the sets of LTV multipliers previously introduced to be LTI, we define the following sets of LTI multipliers:

$$\mathcal{M}_{\text{LTI}} := \{M \in \mathcal{M}_{\text{LTV}} : M \text{ is LTI}\}$$
$$\mathcal{M}_{\text{LTI}}^{B} := \{M \in \mathcal{M}_{\text{LTV}}^{T,B} : M \text{ is LTI}\}.$$

In this subsection, we show that LTV multipliers are "equivalent" [2] to LTI multipliers for the purpose of proving the stability of the Lurye system in (1). This means that an hypothetical sufficient and necessary stability result for the class $\mathcal{M}_{\text{LTV}}$ would not be affected if we restrict the condition to LTI multipliers, i.e. $\mathcal{M}_{\text{LTI}}$. As a straightforward application, the equivalence also holds between $\mathcal{M}_{\text{LTV}}^{T,B}$ and $\mathcal{M}_{\text{LTI}}^{B}$. Then, it follows that the necessary and sufficient conditions in Section 4.3 are preserved by limiting the search of a suitable multiplier to the subset of $\mathcal{M}_{\text{LTI}}^{B}$. It is worth highlighting that any Finite Impulse Response (FIR) multiplier belongs to class $\mathcal{M}_{\text{LTI}}^{B}$ for some $B \geq 0$.

The equivalence results between $\mathcal{M}_{\text{LTV}}$ and $\mathcal{M}_{\text{LTI}}$ is stated as follows:

Theorem 19 *Given any $\epsilon > 0$, there exists $M \in \mathcal{M}_{LTV}$ such that $\langle MGw, w \rangle \leq -\epsilon \|w\|^2$ for all $w \in \ell_2^{0+}$ if and only if there exists $\tilde{M} \in \mathcal{M}_{LTI}$ such that $\langle \tilde{M} Gw, w \rangle \leq -\epsilon \|w\|^2$ for all $w \in \ell_2^{0+}$.*

PROOF. ($\Leftarrow$) Sufficiency follows from the fact that $\mathcal{M}_{\text{LTI}}$ is a subset of $\mathcal{M}_{\text{LTV}}$.

[2] The equivalence between class of multipliers have been introduced by Carrasco et al. (2013).

($\Rightarrow$) To show necessity, assume that there exists $M \in \mathcal{M}_{\text{LTV}}$ such that $\langle MGw, w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$.

Note that if $w \in \ell_2^{0+}$ then $S_\tau w \in \ell_2^{0+}$ for all $\tau \in \mathbb{Z}_0^+$. Hence $\langle MGS_\tau w, S_\tau w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$ and $\tau \in \mathbb{Z}_0^+$. It follows from the shift-invariance of G that for all $\tau \in \mathbb{Z}_0^+$ and $w \in \ell_2^{0+}$,

$$\begin{aligned}\langle S_{-\tau}MS_\tau Gw, w\rangle &= \langle MS_\tau Gw, S_\tau w\rangle\\ &= \langle MGS_\tau w, S_\tau w\rangle \leq -\epsilon\|w\|^2.\end{aligned}$$

Thus, for all $\tau \in \mathbb{Z}_0^+$, the multiplier $S_{-\tau}MS_\tau$ has three useful properties: (i) $S_{-\tau}MS_\tau \in \mathcal{M}_{\text{LTV}}$, (ii) $\langle S_{-\tau}MS_\tau Gw, w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$, and (iii) $\|S_{-\tau}MS_\tau\| \leq \|M\|$.

Next define $M_N := \frac{1}{N}\sum_{\tau=0}^{N-1} S_{-\tau}MS_\tau$, and note that for all $N \in \mathbb{Z}^+$, M_N is doubly hyperdominant with zero excess. Boundedness of $S_{-\tau}MS_\tau$ for all $\tau \in \mathbb{Z}_0^+$ implies the sequence M_N is bounded. It follows from Theorems A.3.39 and A.3.52 in Curtain & Zwart (2012) that there exists a subsequence of M_N, denoted as M_{N_k}, that is weakly convergent. In other words, there is $\tilde{M} \in \mathcal{M}_{\text{LTV}}$ such that $\lim_{k\to\infty}\langle M_{N_k}v, w\rangle = \langle \tilde{M}v, w\rangle$, $\forall v, w \in \ell_2$.

Define $Y_N := S_{-1}M_NS_1 - M_N$. Then, it can be observed that $Y_N = \frac{1}{N}(S_{-N}MS_N - M)$. Since the sequence $S_{-N}MS_{-N}$ is uniformly bounded in the operator norm we have that Y_N converges strongly to zero. Hence, we have

$$\lim_{N\to\infty}\langle Y_N v, w\rangle = 0,\ \forall v, w \in \ell_2.$$

Considering the subsequence Y_{N_k} of Y_N, we further have

$$\lim_{k\to\infty}\left(\langle S_{-1}M_{N_k}S_1 v, w\rangle - \langle M_{N_k}v, w\rangle\right) = 0,\ \forall v, w \in \ell_2$$

which leads to

$$\lim_{k\to\infty}\langle S_{-1}M_{N_k}S_1 v, w\rangle = \lim_{k\to\infty}\langle M_{N_k}v, w\rangle,\ \forall v, w \in \ell_2.$$

The right-hand side of the above equation is $\langle \tilde{M}v, w\rangle$ while the left-hand side can be written as $\langle \tilde{M}S_1 v, S_1 w\rangle = \langle S_{-1}\tilde{M}S_1 v, w\rangle$. Thus, we have

$$\langle S_{-1}\tilde{M}S_1 v, w\rangle = \langle \tilde{M}v, w\rangle,\ \forall v, w \in \ell_2,$$

which implies $S_{-1}\tilde{M}S_1 = \tilde{M}$ (Young 1988, Th. 1.5(iv)), and therefore $\tilde{M} \in \mathcal{M}_{\text{LTI}}$.

Recall that $\langle S_{-\tau}MS_\tau Gw, w\rangle \leq -\epsilon\|w\|^2$ for all $\tau \in \mathbb{Z}_0^+$ and $w \in \ell_2^{0+}$. Therefore $\langle M_N Gw, w\rangle \leq -\epsilon\|w\|^2$ for all $N \in \mathbb{Z}_0^+$ and $w \in \ell_2^{0+}$. Since M_{N_k} weakly converges to $\tilde{M}$, it can now be concluded that $\langle \tilde{M}Gw, w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$, which completes the proof. □

A similar result to Theorem 19 has been recently established by Kharitenko & Scherer (2022) via a more complicated approach.

Remark 20 *By using the equivalence results by Carrasco et al. (2013) between $\mathcal{M}_{LTI}$ and the class of FIR OZF multipliers, we can conclude that the class $\mathcal{M}_{LTV}$ is equivalent to the class of FIR OZF multipliers for the purpose of proving the stability of the Lurye system. Hence, a complete search over the class of FIR OZF multipliers would be enough to test the suitability of any $M \in \mathcal{M}_{LTV}$.*

Combining Proposition 12 and Theorem 19 leads to the discrete-time version of the classical Zames–Falb theorem in Zames & Falb (1968) on the sufficiency of Conjecture 1.

Following the above result, the equivalence also follows between $\mathcal{M}_{\text{LTV}}^{T,B}$ and $\mathcal{M}_{\text{LTI}}^{B}$

Proposition 21 *Given any $\epsilon > 0$ and $T, B \in \mathbb{Z}^+$, there exists $M \in \mathcal{M}_{LTV}^{T,B}$ such that $\langle MGw, w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$ if and only if there exists $\tilde{M} \in \mathcal{M}_{LTI}^{B}$ such that $\langle \tilde{M}Gw, w\rangle \leq -\epsilon\|w\|^2$ for all $w \in \ell_2^{0+}$.*

PROOF. The result follows Theorem 19 but in this case we can provide a closed form to the equivalent LTI multiplier, i.e. for any $M \in \mathcal{M}_{\text{LTV}}^{T,B}$ the equivalent LTI multiplier is $\tilde{M} := \frac{1}{T}\sum_{\tau=0}^{T-1} S_{-\tau}MS_\tau$, and hence $\tilde{M} \in \mathcal{M}_{\text{LTI}}^{B}$. □

The above equivalence allows us to establish the equivalent result to Theorems 11 and 16 but reducing the statement to LTI multipliers.

Theorem 22 *Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded, and let T, B in $\mathbb{Z}^+$ be given with $T \geq 2B + 1$.*

(i) Assume the Lurye system $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}^{T,B}$. Then, the feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}^{T,B}$ if

$$\begin{aligned}&\exists M \in \mathcal{M}_{LTI}^{B}\ s.t.\\ &\operatorname{Re}\left\{\widehat{M}(e^{j\omega})\widehat{G}(e^{j\omega})\right\} < 0, \forall\omega \in [0, 2\pi].\end{aligned} \tag{11}$$

(ii) Suppose that $v, w, e \in \ell_{2e}^{0+}$ satisfy $v = Gw + e$. Then, (8) *holds only if* (11) *is satisfied.*

PROOF. : We first show part (i). Combining Theorem 11 and Proposition 21, and assuming well-posedness, we have that the feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}^{T,B}$ if there exist $M \in \mathcal{M}_{\text{LTI}}^{B}$ and $\epsilon > 0$ such that

$\langle (MG + \epsilon I)w, w\rangle \le 0$ for all $w \in \ell_2^{0+}$. The condition is equivalent to the existence of $M \in \mathcal{M}_{\rm LTI}^B$ and $\epsilon > 0$ such that $\left\langle \left((MG + \epsilon I) + (MG + \epsilon I)^*\right) w, w\right\rangle \le 0$ for all $w \in \ell_2^{0+}$. By the arguments in (Megretski & Treil 1993, Th. 3.1), this is then equivalent to $\left\langle \left((MG + \epsilon I) + (MG + \epsilon I)^*\right) w, w\right\rangle \le 0$ for all $w \in \ell_2$. Note that since $M \in \mathcal{M}_{\rm LTI}^B$ is bounded and LTI, it admits a transfer function representation $\widehat{M}(e^{j\omega}) \in \mathbb{L}_\infty$. Hence, the condition is equivalent to the existence of $M \in \mathcal{M}_{\rm LTI}^B$ and $\epsilon > 0$ such that $\mathrm{Re}\left\{\widehat{M}(e^{j\omega})\widehat{G}(e^{j\omega}) + \epsilon I\right\} \le 0$ for all $\omega \in [0, 2\pi]$. This is equivalent to the existence of an $M \in \mathcal{M}_{\rm LTI}^B$ such that (11) holds.

Part (ii) can be proved analogously by combining Theorem 16 and Proposition 21. $\square$

5 Robust Stability with Nonlinear Multipliers

In this subsection, we show that the uniform robust stability of $[G, \phi]$ over $\mathcal{S}$ can be also ensured by the existence of an appropriate nonlinear multiplier. The Lurye system is helpful since it allows us to analyse the stability of a nonlinear system by using linear tools. Hence, from a practical point of view, a nonlinear multiplier would remove the advantages of the Lurye structure. However, the following development may be helpful in showing a possible source of conservatism in the current stability results which use only linear multipliers, and hence it may be used to disprove the Carrasco conjecture.

Recall from Theorem 7 that the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_2$, i.e., the input-output pairs of nonlinearities in $\mathcal{S}$ can be tightly characterised by the set of multipliers $\mathcal{M}_{\rm LTV}$. Based on this, the sufficiency of the condition proposed in Proposition 12 is established by applying the IQC theorem. As far as sufficiency is concerned, one may propose a possibly less conservative condition for ensuring the robust stability of the Lurye system by looking for a richer set of multipliers than $\mathcal{M}_{\rm LTV}$ that also tightly characterises the set $\mathcal{S}$. This is shown in what follows.

Let Ψ consist of all bounded memoryless, possibly time-varying, nonlinear operators $\psi : \ell_2 \to \ell_2$ that are Lipschitz continuous and satisfy the sector condition $(\psi v)_k = N(v_k, k)$ with $N(x, k)x \ge 0$ and $N(0, k) = 0$, for all $k \in \mathbb{Z}, \forall x \in \mathbb{R}$. Let $\mathcal{S}_0$ denote the set of all bounded memoryless monotone nonlinearities that are Lipschitz continuous.

Lemma 23 *The closure of $\mathcal{G}_1$ is equal to the set $\mathcal{G}_5 := \{(v, w) \in \ell_2 : \langle M\phi_0 v, w\rangle + \langle \psi v, w\rangle \ge 0, \forall M \in \mathcal{M}_{LTV}, \forall \phi_0 \in \mathcal{S}_0, \forall \psi \in \Psi\}$.*

PROOF. ($\supset$) Given any $(v, w) \in \mathcal{G}_5$, by letting $\phi_0 = I, \psi = 0$, it then follows that $(v, w) \in \mathcal{G}_2$. Thus, the claim follows from Theorem 7.

($\subset$) Given any $(v, w) \in \mathcal{G}_1$, then (v, w) are similarly ordered and unbiased. Note that for all $\phi_0 \in \mathcal{S}_0$, the sequence pairs $(\phi_0 v, w)$ are also similarly ordered and unbiased. Therefore, we have that

$$\langle M\phi_0 v, w\rangle \ge 0,\ \forall M \in \mathcal{M}_{\rm LTV}.$$

Moreover, since (v, w) is unbiased, we have that $\psi(v_k, k)w_k \ge 0$ for all $\psi \in \Psi$ whereby

$$\langle \psi v, w\rangle \ge 0,\ \forall \psi \in \Psi.$$

Combining the two inequalities above gives that $(v, w) \in \mathcal{G}_5$, which completes the proof. $\square$

Theorem 24 *Let $G \in \mathcal{L}(\ell_{2e}^{0+}, \ell_{2e}^{0+})$ be LTI, causal and bounded. Assume the Lurye system $[G, \phi]$ is well-posed for all $\phi \in \mathcal{S}$. The feedback interconnection $[G, \phi]$ is uniformly robustly stable over $\mathcal{S}$ if there exist $M \in \mathcal{M}_{LTV}$, $\psi \in \Psi$, $\phi_0 \in \mathcal{S}_0$ and $\epsilon > 0$ such that*

$$\langle M\phi_0 G w, w\rangle + \langle \psi G w, w\rangle \le -\epsilon\|w\|^2, \forall w \in \ell_2^{0+}. \quad (12)$$

PROOF. Let $\vec{\delta}(\cdot, \cdot)$ denote the directed gap between two systems as defined by Georgiou & Smith (1997). A system $\Pi : \ell_2 \to \ell_2$ is said to be *incrementally L_2-bounded* if

$$\sup_{x,y\in\ell_2 : x\neq y} \frac{\|\Pi x - \Pi y\|}{\|x - y\|} < \infty.$$

Let ϕ be any element in $\mathcal{S}$. In what follows, we show by applying Theorem IV.2 in Khong (2021) that $[G, \phi]$ is stable if the condition in Theorem 24 is satisfied.

Firstly, it is clear that $\lambda \in [0, 1] \to \lambda\phi$ is continuous in the directed gap as $\vec{\delta}(\lambda_0\phi, \lambda_1\phi) \le |\lambda_1 - \lambda_0|\|\phi\|$ for all $\lambda_0, \lambda_1 \in [0, 1]$. Secondly, note that $[G, 0]$ is stable, and that by assumption $[G, \lambda\phi]$ is well-posed for all $\lambda \in [0, 1]$. Next, from Lemma 23 we have that

$$\left\langle \begin{bmatrix} v \\ \phi v \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M\phi_0 + \psi & 0 \end{bmatrix} \begin{bmatrix} v \\ \phi v \end{bmatrix} \right\rangle \ge 0$$

for all $v \in \ell_2^{0+}$, $M \in \mathcal{M}_{\rm LTV}$ and all $\phi_0 \in \mathcal{S}_0$, $\psi \in \Psi$. By hypothesis there exist $M \in \mathcal{M}_{\rm LTV}$, $\phi_0 \in \mathcal{S}_0$, $\psi \in \Psi$ and $\epsilon > 0$ such that (12) holds. Since G is bounded, it follows that there exist $M \in \mathcal{M}_{\rm LTV}$, $\phi_0 \in \mathcal{S}_0$, $\psi \in \Psi$ and $\widehat{\epsilon} > 0$ such that

$$\left\langle \begin{bmatrix} Gw \\ w \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M\phi_0 + \psi & 0 \end{bmatrix} \begin{bmatrix} Gw \\ w \end{bmatrix} \right\rangle \le -\widehat{\epsilon} \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2.$$

Since $M \in \mathcal{M}_{\mathrm{LTV}}$ is linear and $\phi_0 \in \mathcal{S}_0$ is memoryless and Lipschitz continuous, one can show $M\phi_0$ is incrementally ℓ_2-bounded. Similarly, $\psi \in \Psi$ being memoryless and Lipschitz leads to that ψ is also incrementally ℓ_2-bounded. Hence, $M\phi_0 + \psi(\cdot,\cdot)$ is incrementally ℓ_2-bounded. Now, by invoking Theorem IV.2 in Khong (2021) we obtain that $[G, \lambda\phi]$ is uniformly stable over $\lambda \in [0,1]$. Since ϕ can be any element in $\mathcal{S}$, it follows that $[G, \phi]$ is robustly stable for all $\phi \in \mathcal{S}$. Uniform robust stability follows from the proof by (Khong & Su 2021, Th. 6). $\square$

Remark 25 *The sufficient condition in Proposition 12 is at least as conservative as the condition in Theorem 24. This can be seen from that the condition in Proposition 12 can be recovered from the condition in Theorem 24 by fixing $\phi_0 = I$ and $\psi = 0$. It is noteworthy that this is the first time that a nonlinear multiplier is proposed to establish feedback stability of the Lurye system with monotone nonlinearities.*

Lemma 23 shows that the set $\mathcal{S}$ can be equivalently characterised also by the set of multipliers that involves both LTV multiplier in $\mathcal{M}_{\mathrm{LTV}}$ and nonlinear multipliers in Ψ and $\mathcal{S}_0$. It is shown in the preceding subsection that the existence of $M \in \mathcal{M}_{\mathrm{LTV}}$ is "equivalent" to the existence of $\mathcal{M}_{\mathrm{LTI}}$, but this is unlikely to hold for the nonlinear multipliers. In fact, the results in this work provide a possible direction to disprove the discrete-time Carrasco conjecture. That is to find a counterexample G such that there is no $M \in \mathcal{M}_{\mathrm{LTI}}$ satisfying (11) but there exist $M \in \mathcal{M}_{\mathrm{LTV}}, \phi_0 \in \mathcal{S}_0, \psi \in \Psi$ and $\epsilon > 0$ satisfying (12).

6 Conclusion

Motivated by the discrete-time Carrasco conjecture, this work studied both the necessity and sufficiency of a suitable LTI multiplier for uniform robust stability of discrete-time Lurye systems. First, it is shown that the set of monotone nonlinearities is tightly characterised by a set of LTV multipliers. Significantly, we show that a conic parameterization of LTV multipliers is possible. By restricting the set of LTV multipliers to be banded and periodic, we introduced a larger set of nonlinearities. Second, it is shown that the existence of a suitable banded and periodic LTV multiplier is sufficient for establishing the uniform robust stability of the Lurye system with time-varying periodic plant over the larger set of nonlinearities. The same condition is also shown to be necessary when the nonlinearity set is replaced by the relation set that is characterised by the same set of LTV multipliers. Third, when the plant is LTI, the existence of such a suitable LTV multiplier is shown to be equivalent to the existence of a suitable LTI multipliers.

The sufficiency direction in the second step above can be extended straightforwardly to the case that considers the set of monotone nonlinearities. This enables the recovery of the discrete-time Zames–Falb theorem. However, it remains unknown that if the necessity direction can be extended similarly. This leaves us an interesting future direction. If this can be done, then one can prove the converse of the discrete-time Zames–Falb theorem, and thus prove the discrete-time Carrasco conjecture. Another possible direction is to disprove the discrete-time Carrasco conjecture by searching for a counterexample such that a suitable nonlinear multiplier exists while no suitable LTI multipliers exist.

In continuous-time, the Carrasco conjecture remains open, and each of the three required steps poses interesting research challenges.

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A Proof of Lemma 6

Lemma 6 *The set of all conic combinations of elements in $\mathcal{C}$ is equal to $\mathcal{M}_{LTV}$.*

PROOF. First, we introduce two statements (C1) and (C2) that are used in the proof below.

(C1) for every $\delta > 0$ there exists $n = n(\delta)$ such that in each row or column the sum of the n largest entries is at least $1 - \delta$.

(C2) for every $\epsilon > 0$ there exists $n = n(\epsilon)$ such that in each row or column the sum of the n entries with largest absolute values is at most ϵ.

Recall that a matrix M is said to be doubly stochastic if all entries are non-negative and all rows and columns sum to 1. Next, define $\mathcal{A} := \{A \in \mathcal{L}(\ell_2, \ell_2) : A \text{ is doubly-infinite, doubly stochastic and satisfies (C1)}\}$. In what follows we show that

$$\mathcal{M}_{\text{LTV}} = \{d(I - A) \in \mathcal{L}(\ell_2, \ell_2) : d > 0, A \in \mathcal{A}\}. \quad \text{(A.1)}$$

To show ($\subset$), let D be any doubly-infinite doubly hyperdominant matrix with zero excess, and define $d := \max_{i,j} |d_{ij}|$. Then D can be expressed as $D = d(I - A)$ where $A := \frac{1}{d}(dI - D)$ is a doubly-infinite doubly stochastic matrix. If D satisfies (C2) then A satisfies (C1). To show ($\supset$), let d be any positive real number and A be any element in $\mathcal{A}$. It is clear that $d(I-A)$ is doubly hyperdominant with zero excess and satisfies (C2).

Note that $\mathcal{A}$ is equal to the convex closure of $\mathcal{P}$ (Isbell (1955)). According to the relationship between $\mathcal{A}$ and $\mathcal{M}_{\text{LTV}}$ given in (A.1), we have that $\mathcal{M}_{\text{LTV}}$ is equal to the set of all conic combinations of elements in $\{I - P : P \in \mathcal{P}\}$. By recalling the definition of $\mathcal{C}$ in (4), it thus follows that $\mathcal{M}_{\text{LTV}}$ is equal to the set of all conic combinations of elements in $\mathcal{C}$. □

B Proof of Theorem 7

To show the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_2$ in Theorem 7, we start by introducing an intermediate set $\mathcal{G}_0$ and presenting two supporting lemmas. Define

$$\mathcal{G}_0 := \{(v, w) \in \ell_2 : (v, w) \text{ is similarly ordered}\}.$$

It is worth mentioning that two sequences in ℓ_2 are similarly ordered if and only if they are similarly ordered and unbiased (Willems 1970, P.63). Therefore, $\mathcal{G}_0$ can be equivalently expressed as $\mathcal{G}_0 = \{(v, w) \in \ell_2 : (v, w) \text{ is similarly ordered unbiased}\}$.

The next lemma relates the input-output pairs of nonlinearities in $\mathcal{S}$ and similarly ordered sequences.

Lemma 26 *The closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_0$.*

PROOF. ($\subset$) This follows from the fact that any $(v, w) \in \ell_2$ satisfying $w = \phi v$ for a $\phi \in \mathcal{S}$ is necessarily similarly ordered and unbiased.

($\supset$) Let $(v, w) \in \ell_2$ be similarly ordered, then (v, w) is unbiased. Given any $\epsilon > 0$, there exists a finite $\tau > 0$ such that $(\bar{v}, \bar{w}) := (P_{-\tau,\tau}v, P_{-\tau,\tau}w)$ satisfy $\|v - \bar{v}\| \leq \epsilon/2$, and $\|w - \bar{w}\| \leq \epsilon/2$. The truncated sequences have a finite number of unique pairs coming from $(\bar{v}_k, \bar{w}_k)_{k=-\tau}^{\tau}$ and $(\bar{v}_k, \bar{w}_k) = (0, 0)$ otherwise. The truncated sequences are similarly ordered and unbiased so the data can be linearly interpolated by a monotone function. However, the function could be multi-valued and/or unbounded if the data contains points with $\bar{v}_i = \bar{v}_j$ but $\bar{w}_i \neq \bar{w}_j$. This issue is resolved by another perturbation to the data. Specifically, define $\widehat{w} := \bar{w}$ and define $\widehat{v}$ by perturbing $\bar{v}$ by a sufficiently small $\delta > 0$ as follows. If $\bar{v}_i = 0$ but $\bar{w}_i \neq 0$ for any $i \in \mathbb{Z}$ then define $\widehat{v}_i := \delta\bar{w}_i$. This preserves the point $(0, 0)$ and perturbs other pairs to lie along a line of slope δ. Similarly, suppose the sequence $\bar{v}$ has a non-zero value repeated N times: $\bar{v}_{i_1} = \cdots = \bar{v}_{i_N} \neq 0$ with $\bar{w}_{i_1} \leq \cdots \leq \bar{w}_{i_N}$ for some indices $\mathcal{I} := \{i_1, \ldots, i_N\}$. In this case, define $\widehat{v}_k = \bar{v}_{i_1} + \delta(\bar{w}_k - \bar{w}_{i_1})$ for $k \in \mathcal{I}$. Again, this perturbs the repeated pairs to lie along a line of slope δ. The perturbed sequences $(\widehat{v}, \widehat{w})$ can be linearly interpolated by a single-valued, monotone, bounded function $N : \mathbb{R} \to \mathbb{R}$ such that $N(0) = 0$ and $N(\widehat{v}_i) = \widehat{w}_i$ for all $i \in \mathbb{Z}$. Moreover, we have $\|\bar{w} - \widehat{w}\| = 0$ by definition and, if $\delta > 0$ is sufficiently small, then $\|\bar{v} - \widehat{v}\| \leq \epsilon/2$.

In summary, for any $\epsilon > 0$ and similarly ordered, unbiased $(v, w) \in \ell_2$, there exist a pair $(\widehat{v}, \widehat{w}) \in \ell_2$ such that: (i) $\|v - \widehat{v}\| \leq \epsilon$, (ii) $\|w - \widehat{w}\| \leq \epsilon$, and (iii) $\widehat{w} = \phi(\widehat{v})$ for some $\phi \in \mathcal{S}$. The inclusion holds since ϵ can be arbitrarily small. □

The next lemma states that similarly ordered sequences are tightly characterised by a positivity condition involving $\mathcal{M}_{\text{LTV}}$, i.e., $\mathcal{G}_0 = \mathcal{G}_2$.

Lemma 27 *The set $\mathcal{G}_0$ is equal to the set $\mathcal{G}_2$.*

PROOF. ($\subset$) This follows from Theorem 3.11 in Willems (1970) as $\mathcal{M}_{\text{LTV}}$ is a subset of the set consisting of all $M \in \mathcal{L}(\ell_2, \ell_2)$ whose associated matrix is doubly hyperdominant.

($\supset$) Assume $\langle Mv, w \rangle \geq 0$ for every $M \in \mathcal{M}_{\text{LTV}}$. First consider the multiplier M with the associated matrix defined by:

$$\begin{bmatrix} m_{kk} & m_{kl} \\ m_{lk} & m_{ll} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } m_{ij} = 0, \text{otherwise.}$$

By construction $M \in \mathcal{M}_{\text{LTV}}$. Moreover, $\langle Mv, w \rangle \geq 0$ can be rewritten as $(v_k - v_l)(w_k - w_l) \geq 0$. Thus, if $v_k < v_l$ then $w_k \leq w_l$ and the sequences v, w are similarly ordered. □

Theorem 7 *It holds that* $\mathbf{cl}\,\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3$.

PROOF. That the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_2$ can be obtained by combining Lemmas 26 and 27. To prove that the closure of $\mathcal{G}_1$ is equal to $\mathcal{G}_3$, it suffices to show

$\mathcal{G}_2 = \mathcal{G}_3$. If (v, w) is in $\mathcal{G}_2$, then $\langle Mv, w\rangle \geq 0$ for all $M \in \mathcal{M}_{\mathrm{LTV}}$. Since $\mathcal{C} \subset \mathcal{M}_{\mathrm{LTV}}$, this implies that (v, w) is in $\mathcal{G}_3$. Thus, $\mathcal{G}_2 \subset \mathcal{G}_3$. If (v, w) is not in $\mathcal{G}_2$, then $\langle Mv, w\rangle < 0$ for some $M \in \mathcal{M}_{\mathrm{LTV}}$. We can express such M as a conic combination of elements $M_i \in \mathcal{C}$ according to Lemma 6. Then, it follows that $\langle M_i v, w\rangle < 0$ for at least one $M_i \in \mathcal{C}$. Hence, (v, w) is not in $\mathcal{G}_3$. By contraposition $\mathcal{G}_3 \subset \mathcal{G}_2$. □

C Proof of Lemma 10

Lemma 10 *For any $T, B \in \mathbb{Z}^+$, the set $\mathcal{S}$ is a subset of $\mathcal{S}^{T,B}$. Moreover, it holds that*

$$\mathcal{G}^{kT,B_1} \subset \mathcal{G}^{T,B_2}, \ \ \forall k \in \mathbb{Z}^+, B_1 \geq B_2 \tag{C.1}$$

and

$$\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n} = \mathbf{cl}\,\mathcal{G}_1. \tag{C.2}$$

PROOF. First, we show that $\mathcal{S} \subset \mathcal{S}^{T,B}$ for any $T, B \in \mathbb{Z}^+$. Let ϕ be in $\mathcal{S}$. Since ϕ is memoryless bounded and monotone, any $(v, w) \in \ell_2$ satisfying $w = \phi v$ is necessarily similarly ordered and unbiased. Thus, for all $v \in \ell_2$, $(v, \phi v)$ satisfies that

$$\langle Mv, \phi v\rangle \geq 0, \forall M \in \mathcal{M}_{\mathrm{LTV}}.$$

As $\mathcal{M}^{T,B}_{\mathrm{LTV}} \subset \mathcal{M}_{\mathrm{LTV}}$ by definition, ϕ is in $\mathcal{S}^{T,B}$.

Next, the result in (C.1) follows directly from the fact that $\mathcal{M}^{kT,B_1}_{\mathrm{LTV}} \supset \mathcal{M}^{T,B_2}_{\mathrm{LTV}}$ when $k \in \mathbb{Z}^+, B_1 \geq B_2$.

Define the sequence $\mathcal{G}^{2^n,n}, n = 1, 2, \ldots$. According to (C.1), the sequence is monotone non-increasing, and thus its limit exists and given by $\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n}$. Then, we show in the following that $\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n} = \mathcal{G}_2$. To see $\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n} \supset \mathcal{G}_2$, suppose (v, w) is in $\mathcal{G}_2$. Then (v, w) satisfies $\langle Mv, w\rangle \geq 0$ for all $M \in \mathcal{M}_{\mathrm{LTV}}$. It follows that $\langle Mv, w\rangle \geq 0$ holds for all $M \in \mathcal{M}^{T,B}_{\mathrm{LTV}}$ and hence $(v, w) \in \mathcal{G}^{T,B}$ for all $T, B \in \mathbb{Z}^+$. Hence $(v, w) \in \bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n}$. To see $\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n} \subset \mathcal{G}_2$, suppose by contraposition that $(v, w) \notin \mathcal{G}_2$. Then it follows that there exist $M \in \mathcal{M}_{\mathrm{LTV}}$ and $\epsilon > 0$ such that $\langle Mv, w\rangle < -\epsilon$. One can always find an $M_1 \in \mathcal{M}^{T,B}$ for some large enough $T, B \in \mathbb{Z}^+$ such that $\langle (M_1 - M)v, w\rangle < \epsilon/2$. Then

$$\langle M_1 v, w\rangle = \langle (M_1 - M)v, w\rangle + \langle Mv, w\rangle \leq -\epsilon/2,$$

which implies that $(v, w) \notin \bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n}$. Thus, $\bigcap_{n\in\mathbb{Z}^+} \mathcal{G}^{2^n,n} = \mathcal{G}_2$ is proved. Equation (C.2) follows immediately from Theorem 7. □

D Proof of Lemma 14

Lemma 14 *Suppose the quadratic forms σ_k, k=0,1,...,N satisfy Assumption 13 and that there exists $f^* \in \ell_2^{0+}$ such that $\sigma_k(f^*) > 0$ for $k = 1, \ldots, N$. Then the following are equivalent:*

(i) $\sigma_0(f) \leq 0$ for all $f \in \ell_2^{0+}$ that satisfy $\sigma_k(f) \geq 0, \forall k = 1, 2, \ldots, N$;
(ii) There exists $\alpha_k \geq 0$, $k = 1, \ldots, N$ such that

$$\sigma_0(f) + \sum_{k=1}^{N} \alpha_k \sigma_k(f) \leq 0, \quad \forall f \in \ell_2^{0+}.$$

PROOF. That (ii) implies (i) is straightforward. To see that (i) implies (ii), define

$$\begin{aligned} \mathcal{K} &= \left\{(\sigma_0(f), \sigma_1(f), \ldots, \sigma_N(f)) : f \in \ell_2^{0+}\right\} \\ \mathcal{N} &= \left\{(n_0, n_1, \ldots, n_N) : n_k > 0 \ \forall k \in \{0, 1, \ldots, N\}\right\}. \end{aligned}$$

We show below that $\overline{\mathcal{K}}$, the closure of $\mathcal{K}$, is convex. Let $f_1, f_2 \in \ell_2^{0+}$,

$$\begin{aligned} k_1 &= (\sigma_0(f_1), \sigma_1(f_1), \ldots, \sigma_N(f_1)) \in \mathcal{K} \\ k_2 &= (\sigma_0(f_2), \sigma_1(f_2), \ldots, \sigma_N(f_2)) \in \mathcal{K}. \end{aligned}$$

Recall that the shift operator S_τ is defined by $(S_\tau f)_k = f_{k-\tau}$ for $\tau \in \mathbb{Z}_0^+$. For all $\lambda \in [0, 1]$,

$$\begin{aligned} &\sigma_k(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2) \\ &= \lambda\sigma_k(f_1) + (1-\lambda)\sigma_k(S_\tau f_2) + 2\sqrt{\lambda(1-\lambda)}\langle \Pi_k f_1, S_\tau f_2\rangle. \end{aligned}$$

Observe that $\langle \Pi_k f_1, S_\tau f_2\rangle \to 0, k = 0, 1, \ldots N$, as $\tau \to \infty$. Moreover, since $\Pi_k, k = 1, \ldots N$ is T-periodic and Π_0 is T_0-periodic, it follows that for all $\epsilon > 0$ there exists sufficiently large $\beta \in \mathbb{Z}^+$ such that with $\tau := \beta T T_0$,

$$\begin{aligned} \left|\langle \Pi_k f_1, S_\tau f_2\rangle\right| < \epsilon, \ \forall k = 0, 1, \ldots, N \quad \text{and} \\ \sigma_k(S_\tau f_2) - \sigma_k(f_2) = 0, \ \forall k = 0, 1, \ldots, N. \end{aligned}$$

Together, we have that

$$\begin{aligned} &\Big(\sigma_0(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2), \ldots, \sigma_N(\sqrt{\lambda} f_1 + \sqrt{1-\lambda} S_\tau f_2)\Big) \\ &\qquad \to \lambda k_1 + (1-\lambda) k_2 \end{aligned}$$

as $\beta \to \infty$. That is, $\overline{\mathcal{K}}$ is convex.

Since $\mathcal{N}$ is open and (i) implies that $\overline{\mathcal{K}} \bigcap \mathcal{N} = \emptyset$, it follows from the hyperplane separation theorem that there exists a hyperplane that separates $\overline{\mathcal{K}}$ and $\mathcal{N}$. That is, there exists a nonzero $N+1$-tuple $(c_0, c_1, \dots, c_N)$ such that

$$c_0 n_0 + c_1 n_1 + \cdots + c_N n_N > 0 \quad \forall (n_0, n_1, \dots, n_N) \in \mathcal{N} \tag{D.1}$$

and

$$c_0 \kappa_0 + c_1 \kappa_1 + \cdots + c_N \kappa_N \leq 0 \quad \forall (\kappa_0, \kappa_1, \dots, \kappa_N) \in \overline{\mathcal{K}}. \tag{D.2}$$

Note that (D.1) being true for all $(n_0, n_1, \dots, n_N) \in \mathcal{N}$ implies that $c_k \geq 0, k = 1, \dots, N$.

Finally, let $\kappa_k = \sigma_k(f^*)$ for $k = 0, 1, \dots, N$. Note that $\kappa_k > 0$ for $k = 1, \dots, N$ by hypothesis. It follows from (D.2) that $c_0 > 0$. Dividing (D.2) by c_0 and letting $\alpha_k = c_k/c_0$, $k = 1, \dots, N$ then yields (ii), as required. □

E Proof of Lemma 15

Lemma 15 *Given a pair of sequences* $\{v_1, v_2, \dots, v_n\}$, $\{w_1, w_2, \dots, w_n\}$, *and suppose* $v_1 > v_2 > \cdots > v_n > 0$ *and* $w_1 > w_2 > \cdots > w_n > 0$. *Then* $\sum_{i,j=1}^{n} m_{ij} v_i w_j > 0$ *for all nonzero* $M = [m]_{ij} \in \mathbb{R}^{n \times n}$ *that are doubly hyperdominant.*

PROOF. Since $v_1 > v_2 > \cdots > v_n$ and $w_1 > w_2 > \cdots > w_n$, it follows from the rearrangement inequality in (Hardy et al. 1952, Section 10.2) that

$$\sum_{i=1}^{n} v_i w_i > \sum_{i=1}^{n} v_i w_{\pi(i)}$$

for all permutation π except for $\pi(i) = i, i = 1, \dots, n$. That means $\sum_{i,j=1}^{n} [I-P]_{ij} v_i w_j > 0$ for all $P \in \{\mathcal{P}_n \backslash I\}$ where $\mathcal{P}_n$ denotes all $n \times n$ permutation matrices. According to the sufficiency proof of Theorem 3.7 in Willems (1970), given any doubly hyperdominant matrix M with zero excess, it can be written as $M = \sum_{i=1}^{n!} \beta_i (I - P_i)$, where $P_i \in \mathcal{P}_n$ and $\beta_i \geq 0, k = 1, \dots, n!$. Let $P_1 = I$. Note that M being nonzero implies that there exists at least one $i \in \{2, \dots, n!\}$ such that $\beta_i > 0$. Hence, $\sum_{i,j=1}^{n} m_{ij} v_i w_j > 0$ for all $M = [m]_{ij}$ that are nonzero and doubly hyperdominant with zero excess.

Now for any given doubly hyperdominant matrix $M \in \mathbb{R}^{n \times n}$, define $m_{i,n+1} := -\sum_{j=1}^{n} m_{ij}$, $m_{n+1,j} := -\sum_{i=1}^{n} m_{ij}$ for $i, j \leq n$, and $m_{n+1,n+1} := \sum_{i,j=1}^{n} m_{ij}$. Since the augmented matrix $M_+ := [m]_{ij}, i, j = 1, 2, \dots, n+1$ is a doubly hyperdominant with zero excess, by considering the sequence $\{v_1, v_2, \dots, v_n, 0\}$ $\{w_1, w_2, \dots, w_n, 0\}$, it then follows that $\sum_{i,j=1}^{n+1} m_{ij} v_i w_j = \sum_{i,j=1}^{n} m_{ij} v_i w_j > 0$. □