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Multipliers for nonlinearities with monotone bounds

William P. Heath, *Member, IEEE,* Joaquin Carrasco, *Member, IEEE,* and Dmitry A. Altshuller,

***Abstract*—We consider Lurye (sometimes written Lur'e) systems whose nonlinear operator is characterised by a possibly multivalued nonlinearity that is bounded above and below by monotone functions. Stability can be established using a subclass of the Zames-Falb multipliers. The result generalises similar approaches in the literature. Appropriate multipliers can be found using convex searches. Because the multipliers can be used for multivalued nonlinearities they can be applied after loop transformation. We illustrate the power of the new mutlipliers with two examples, one in continuous time and one in discrete time: in the first the approach is shown to outperform available stability tests in the literature; in the second we focus on the special case for asymmetric saturation with important consequences for systems with non-zero steady state exogenous signals.**

***Index Terms*—Lure systems, quasi-monotone, quasi-odd, asymmetry, Zames-Falb multiplier.**

I. INTRODUCTION

We are concerned with the input-output stability of the Lurye system given by

$$y_1 = Gu_1,\ y_2 = \phi u_2,\ u_1 = r_1 - y_2 \text{ and } u_2 = y_1 + r_2. \tag{1}$$

Let $\mathscr{L}_2$ be the space of finite energy Lebesgue integrable signals and let $\mathscr{L}_{2e}$ be the corresponding extended space (see for example [1]). The Lurye system is said to be stable if $r_1, r_2 \in \mathscr{L}_2$ implies $u_1, u_2, y_1, y_2 \in \mathscr{L}_2$.

Assumption 1. *The Lurye system (1) is assumed to be well-posed with $G : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ linear time invariant (LTI) causal and stable, with $\phi : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ some nonlinear causal operator and with $r_1(t) = r_2(t) = 0$ for $t < 0$.*

A function $\alpha : \mathbb{R} \to \mathbb{R}$ is said to be monotone if $\alpha(x_1) \geq \alpha(x_2)$ for all $x_1 \geq x_2$. It is said to be bounded if there exists $C \geq 0$ such that $|\alpha(x)| \leq C|x|$ for all $x \in \mathbb{R}$[1]. It is said to be odd if $\alpha(-x) = -\alpha(x)$ for all $x \in \mathbb{R}$. It is said to be slope-restricted on $[0, s]$ if $0 \leq (\alpha(x_1) - \alpha(x_2))/(x_1 - x_2) \leq s$ for all $x_1 \neq x_2$. If the nonlinear operator ϕ can be characterised by the monotone and bounded function $\alpha : \mathbb{R} \to \mathbb{R}$ in the sense that $y(t) \triangleq (\phi u)(t) = \alpha(u(t))$, then the Zames-Falb multipliers may be used to determine stability [4], [5], [1], [6] and local stability [7]. In this case we call α the nonlinearity that characterises the nonlinear operator ϕ. Further results may be obtained if the nonlinearity is odd, if it is slope-restricted or if it is both odd and slope-restricted [5], [1].

In this note we consider a more general class of causal nonlinear operators. The nonlinearity that characterises the operator ϕ is the set of pairs $(u(t), y(t))$ such that $y(t) = (\phi u)(t)$ with $u \in \mathscr{L}_{2e}$. This set need not correspond to a monotone or single-valued function. Instead, we say the nonlinearity is bounded below and above in the following sense.

W. P. Heath and J. Carrasco are with the Control Systems Centre, Department of Electrical and Electronic Engineering, University of Manchester, UK. e-mail: william.heath@manchester.ac.uk, joaquin.carrasco@manchester.ac.uk. The late D. A. Altshuller was with Dassault Systems, USA.

[1]Here the term "bounded" is not used in the standard sense it is used for functions (e.g. [2]); rather it is used in a sense consistent with the notion of bounded operators (e.g.[3]). We use the terms "bounded below" and "bounded above" in a further sense below.

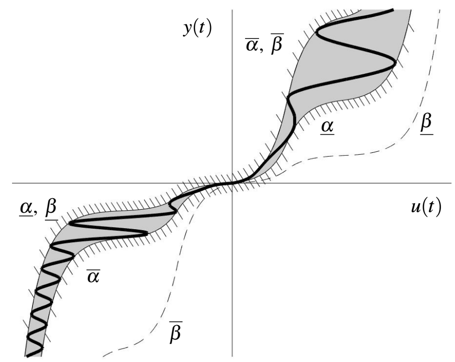

Fig. 1. The nonlinearity (i.e. the map from $u(t)$ to $y(t) = (\phi u)(t)$) is bounded below and above by the monotone and bounded functions $\underline{\alpha}$ and $\overline{\alpha}$ respectively. In addition the nonlinearity is bounded below and above by the monotone, bounded and odd functions $\underline{\beta}$ and $\overline{\beta}$ respectively. For the specific case illustrated the functions $\underline{\beta}$ and $\overline{\beta}$ are constructed as follows: when $u(t) < 0$ set $\underline{\beta}(u(t)) = \underline{\alpha}(u(t))$ and $\overline{\beta}(u(t)) = -\overline{\alpha}(-u(t))$; when $u(t) \geq 0$ set $\underline{\beta}(u(t)) = -\underline{\alpha}(-u(t))$ and $\overline{\beta}(u(t)) = \overline{\alpha}(u(t))$.

Assumption 2 (Fig 1)**.** *Let $\phi : \mathscr{L}_{2e} \to \mathscr{L}_{2e}$ be a causal operator and let $y(t) = (\phi u)(t)$. There are assumed to exist monotone and bounded functions $\underline{\alpha} : \mathbb{R} \to \mathbb{R}$ and $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ such that for all $t \in \mathbb{R}$ with $u(t) \neq 0$,*

$$0 \leq \frac{\underline{\alpha}(u(t))}{u(t)} \leq \frac{y(t)}{u(t)} \leq \frac{\overline{\alpha}(u(t))}{u(t)}. \tag{2}$$

If $u(t) = 0$ then $y(t) = 0$. We say the nonlinearity is bounded below by $\underline{\alpha}$ and above by $\overline{\alpha}$. There are also assumed to exist monotone, bounded and odd functions $\underline{\beta} : \mathbb{R} \to \mathbb{R}$ and $\overline{\beta} : \mathbb{R} \to \mathbb{R}$ such that the nonlinearity is bounded below by $\underline{\beta}$ and above by $\overline{\beta}$.

Remark 1. *For a given $u \in \mathscr{L}_{2e}$ the values of $y(t)$ remain uniquely determined even though the characterising nonlinearity may be multivalued. The Lurye problem is sometimes restricted to memoryless, but possibly time-varying, nonlinearities [8]. Our class includes both dynamic and time-varying operators; nevertheless the conditions of Assumption 2 exclude many common nonlinearities. In the terminology of [9] condition (2) is an instantaneous condition.*

We will often make the following further assumption.

Assumption 3. *Given Assumption 2, there is assumed to be some finite $A \geq 1$ such that $\overline{\alpha}$ is bounded above by $A\underline{\alpha}$. Similarly, there is assumed to be some (possibly infinite) $B \geq A$ such that $\overline{\beta}$ is bounded above by $B\underline{\beta}$.*

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If $A=1$ then the nonlinearity is single-valued and monotone. If $A=B=1$ then the nonlinearity is odd. If $A=1$ and $1<B<\infty$ we say the nonlinearity is quasi-odd.

If $1<A<\infty$ we say the nonlinearity is quasi-monotone. If $1<A=B<\infty$ we say the nonolinearity is quasi-monotone with odd bounds. If $1<A<B<\infty$ we say the nonolinearity is quasi-monotone with quasi-odd bounds.

Remark 2. *In the literature [10], [11] "quasimonotone" has a wider and more general definition than that for quasi-monotone adopted here. In [12] "quasi-monotone-and-odd" is used for the case where, in our terminology, the nonlinearity is quasi-monotone with odd bounds. Our terminology is slightly different to that in [13].*

Our main result, Theorem 1, is to derive a subclass of the Zames-Falb multipliers that preserves the positivity of such nonlinearities. The original results of Zames and Falb [5] for nonlinearities characterisd by either monotone and bounded or monotone, bounded and odd functions can be recovered as special cases with $A=1$ and respectively where either B is ignored or $B=1$. A generalisation for quasi-odd nonlinearities follows immediately (Corollary 1).

A similar approach is taken by [14] and [12] for quasi-monotone nonlinearities with odd bounds. In [14] a specific stiction model is considered. Corollary 2 generalises the results of [14] in two senses: firstly it allows more general bounds on the nonlinearity; secondly it allows the nonlinearity to be multivalued. For the specific application of [14] our results are the same. Corollary 2 provides a less conservative result than that of [12]; a similar improvement is noted by [10] without proof, where the result is generalised to time-periodic, but not more general, nonlinearities.

We extend our results to the case where the bounds on the nonlinearity are also slope-restricted (Theorem 2) by applying loop transformation techniques. A single-valued function need not be single-valued after loop transformation (see [1]) so our relaxation of the standard assumption that the nonlinearity be single-valued [5], [14], [12] is necessary. Once again the original results of [5] can be derived as special cases and we state the counterpart of Corollary 1 under loop transformation as Corollary 3.

Our development is for continuous time multipliers. Corresponding results for discrete-time systems can be derived similarly and are briefly stated in Appendix A. In Appendix B we show how the convex search for Zames-Falb multipliers [15] and for their discrete-time counterparts [16] can be modified to search for the multipliers of this paper.

We illustrate the stability results with two examples. The first, in continuous time, is similar (though not identical) to an example in [12]. It illustrates some of the subtleties that arise with loop transformations and how the new stability criteria can provide better results than those in the literature. The second example, in discrete time, illustrates how the new results can be applied to Lurye systems with asymmetric saturation. This offers insight to the behaviour of (for example) anti-windup systems with exogenous signals with non-zero steady state values. This example was discussed in [13] where some technical results were also presented without proof.

II. MULTIPLIERS

In our development we will exploit the Hahn decomposition [17] of a signal. If $x\in\mathscr{L}_{2e}$ its Hahn decomposition is $x=x_+-x_-$ where $x_+(t)=\max(x(t),0)$ for all $t\in\mathbb{R}$. We begin by establishing the following inequalities.

Lemma 1. *Under the conditions of Assumptions 2 and 3, for all $u\in\mathscr{L}_2$ and for all $\tau\in\mathbb{R}$,*

$$-B\int_{-\infty}^{\infty}u(t)y(t)\,dt\leq\int_{-\infty}^{\infty}u(t+\tau)y(t)\,dt\leq A\int_{-\infty}^{\infty}u(t)y(t)\,dt. \tag{3}$$

Proof. Since $\underline{\alpha}$ is monotone and bounded, and since $\underline{\beta}$ is monotone, bounded and odd, it follows (e.g. [1], p205) that for any $u\in\mathscr{L}_2$,

$$\int_{-\infty}^{\infty}u(t+\tau)\underline{\alpha}(u(t))\,dt\leq\int_{-\infty}^{\infty}u(t)\underline{\alpha}(u(t))\,dt, \tag{4}$$

and

$$\left|\int_{-\infty}^{\infty}u(t+\tau)\underline{\beta}(u(t))\,dt\right|\leq\int_{-\infty}^{\infty}u(t)\underline{\beta}(u(t))\,dt. \tag{5}$$

Let $u=u_+-u_-$ and $y=y_+-y_-$ be the Hahn decompositions of u and y respectively. Then for any $t,\tau\in\mathbb{R}$,

$$\begin{aligned}u(t+\tau)y(t)&=[u_+(t+\tau)-u_-(t+\tau)][y_+(t)-y_-(t)],\\&\leq u_+(t+\tau)y_+(t)+u_-(t+\tau)y_-(t),\\&\leq u_+(t+\tau)\overline{\alpha}(u_+(t))+u_-(t+\tau)\overline{\alpha}(u_-(t)),\\&\quad\text{by Assumption 2,}\\&\leq Au_+(t+\tau)\underline{\alpha}(u_+(t))+Au_-(t+\tau)\underline{\alpha}(u_-(t)),\\&\quad\text{by Assumption 3.}\end{aligned} \tag{6}$$

Hence

$$\begin{aligned}&\int_{-\infty}^{\infty}u(t+\tau)y(t)\,dt\\&\leq A\int_{-\infty}^{\infty}[u_+(t)\underline{\alpha}(u_+(t))+u_-(t)\underline{\alpha}(u_-(t))]\,dt\text{ by (4),}\\&=A\int_{-\infty}^{\infty}u(t)\underline{\alpha}(u(t))\,dt,\\&\leq A\int_{-\infty}^{\infty}u(t)y(t)\,dt\text{ by Assumption 2.}\end{aligned} \tag{7}$$

Furthermore, for any $t,\tau\in\mathbb{R}$,

$$\begin{aligned}|u(t+\tau)y(t)|&=|u(t+\tau)|\,|y(t)|,\\&\leq|u(t+\tau)|\,|\overline{\beta}(u(t))|\text{ by Assumption 2,}\\&\leq B|u(t+\tau)|\,|\underline{\beta}(u(t))|\text{ by Assumption 3,}\\&=B|u(t+\tau)|\,\underline{\beta}(|u(t)|)\text{ since }\underline{\beta}\text{ is odd.}\end{aligned} \tag{8}$$

Hence

$$\begin{aligned}\left|\int_{-\infty}^{\infty}u(t+\tau)y(t)\,dt\right|&\leq\int_{-\infty}^{\infty}|u(t+\tau)y(t)|\,dt,\\&\leq B\int_{-\infty}^{\infty}|u(t+\tau)|\,\underline{\beta}(|u(t)|)\text{ by (8),}\\&\leq B\int_{-\infty}^{\infty}|u(t)|\,\underline{\beta}(|u(t)|)\,dt\text{ by (5),}\\&=B\int_{-\infty}^{\infty}u(t)\underline{\beta}(u(t))\,dt\text{ since }\underline{\beta}\text{ is odd,}\\&\leq B\int_{-\infty}^{\infty}u(t)y(t)\,dt\text{ by Assumption 2.}\quad\square\end{aligned} \tag{9}$$

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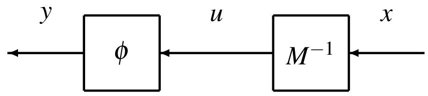

Fig. 2. The proof of Theorem 1 establishes positivity from x to y where $x(t) = (m * u)(t)$. Note that M^{-1} exists because M belongs to a subclass of the Zames-Falb multipliers.

Define $\mathbf{H}$ as the set of generalized functions $h(\cdot)$ of the form

$$h(t) = \sum_i h_i \delta(t - t_i) + h_a(t), \tag{10}$$

with $t_i \neq 0$, $h_a(0) = 0$, $h_i \in \mathbb{R}$ for all i and $h_a(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. In addition, define the norm (c.f. [1], [8][2]):

$$\|h\|_H \triangleq \sum_i |h_i| + \int_{-\infty}^{\infty} |h_a(t)|\, dt < \infty. \tag{11}$$

Define $\mathbf{H_p}$ as the subset of $\mathbf{H}$ where $h_i \geq 0$ for all i and $h_a(t) \geq 0$ for all $t \in \mathbb{R}$. We establish the following generalization of the Zames-Falb theorem.

Theorem 1 (quasi-monotone or quasi-odd nonlinearity)**.** *Under the conditions of Assumptions 1, 2, and 3, let H_+ and H_- be noncausal convolution operators whose respective impulse responses are $h_+ \in \mathbf{H_p}$ and $h_- \in \mathbf{H_p}$ satisfying*

$$A\|h_+\|_H + B\|h_-\|_H < 1. \tag{12}$$

Let $M = 1 - H_+ + H_-$. Then for any $u \in \mathscr{L}_2$,

$$\int_{-\infty}^{\infty} (Mu)(t)\,(\phi u)(t)\, dt \geq 0. \tag{13}$$

Furthermore the continuous-time Lurye system (1) is stable provided there exists $\varepsilon > 0$ such that

$$Re[M(j\omega)G(j\omega)] \geq \varepsilon \text{ for all } \omega \in \mathbb{R}. \tag{14}$$

Proof. Let $y = \phi u$ and let m be the impulse response[3] of M. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} (Mu)(t)\,(\phi u)(t)\, dt &= \int_{-\infty}^{\infty} (m * u)(t) y(t)\, dt \\
&= \int_{-\infty}^{\infty} u(t) y(t)\, dt - \int_{-\infty}^{\infty} (h_+ * u)(t) y(t)\, dt \\
&\quad + \int_{-\infty}^{\infty} (h_- * u)(t) y(t)\, dt \\
&\geq (1 - A\|h_+\|_H - B\|h_-\|_H) \int_{-\infty}^{\infty} u(t) y(t)\, dt \\
&\quad \text{by Lemma 1,} \\
&\geq 0 \text{ provided (12) holds.}
\end{aligned} \tag{15}$$

Since M belongs to a subclass of the Zames-Falb multipliers, M^{-1} exists. This establishes the positivity of the map from x to y where $u = M^{-1}x$ (see Fig 2). Similarly an appropriate factorization of M is guaranteed (see also [18]). Stability then follows from standard multiplier theory (see e.g. [1]). □

Remark 3. *The positivity result of Theorem 1 is sufficient to establish stability using classical theory [1]. Nevertheless it is straightforward to establish stability via the integral quadratic constraint (IQC) theory of [19]. Specifically it follows immediately from Lemma 1 that*

$$\tau \int_{-\infty}^{\infty} (Mu)(t)\,(\phi u)(t)\, dt \geq 0 \text{ for any } \tau \geq 0. \tag{16}$$

Thus the homotopy argument of [19] can be used to establish stability.

Similarly the stability result of Theorem 1 can be established via the theory of delay-integral-quadratic constraints [20], [21], [22], [10]. Specifically Lemma 1 establishes the time-domain quadratic forms used to give the frequency domain stability criterion of Theorem 1. The relation between the IQC theory of [19] and delay-integral-quadratic constraints is discussed in [23].

Remark 4. *We can write $M = 1 - H$ where H is a noncausal convolution operator whose impulse response is $h \in \mathbf{H}$. The Hahn decomposition of h is $h = h_+ - h_-$.*

III. Special cases

A. Quasi-odd nonlinearities

If the nonlinearity is time-invariant, bounded and monotone we can set $\overline{\alpha} = \underline{\alpha}$ so $A = 1$. The Zames-Falb theorem for such nonlinearities follows immediately by setting $h_- = 0$. The Zames-Falb theorem for monotone, bounded and odd nonlinearities also follows immediately when $A = B = 1$.

The Zames-Falb theorem for odd nonlinearities allows a much wider class of multipliers, but if the nonlinearity is quasi-odd then it cannot be used. Yet the Zames-Falb theorem for more general nonlinearities is independent of the value B. This immediately suggests an intermediate result.

Corollary 1 (quasi-odd nonlinearity)**.** *Under the conditions of Assumptions 1, 2 and 3 with $A = 1$, let H_+ and H_- be noncausal convolution operators whose impulse responses are respectively $h_+ \in \mathbf{H_p}$ and $h_- \in \mathbf{H_p}$ satisfying*

$$\|h_+\|_H + B\|h_-\|_H < 1. \tag{17}$$

Let $M = 1 - H_+ + H_-$. Then the positivity condition (13) holds and the Lurye system of Fig 1 is stable provided (14) holds.

Proof. Immediate from Theorem 1 when $A = 1$. □

B. Quasi-monotone nonlinearities with odd bounds

Both [14] and [12] consider single-valued non-monotone nonlinearities with odd bounds. In our terminology $\underline{\alpha} = \underline{\beta}$, $\overline{\alpha} = \overline{\beta}$ and $A = B > 1$.

In [14] a time-invariant stiction nonlinearity is given as $\alpha(u(t)) = ku(t)/\varepsilon$ when $|u(t)|$ is small, and $1 \leq |\alpha(u(t))| \leq 1 + \delta$ when $|u(t)|$ is large. This nonlinearity can be bounded below by $\underline{\beta}(u(t)) = \text{sign}(u(t)) \times \min(k|u(t)|/\varepsilon, 1)$ and above by $B\underline{\beta}$ with $B = 1 + \delta$. In [14] a stability condition is given equivalent to choosing $M = 1 - H$ where H has impulse response $h \in \mathbf{H}$ and $(1 + \delta)\|h\|_H < 1$.

In [12] a nonlinearity is bounded below by $(1 - D)\hat{\beta}$ and above by $(1 + D)\hat{\beta}$ for some monotone, bounded and odd "skeleton" $\hat{\beta} : \mathbb{R} \to \mathbb{R}$. Hence $B = (1 + D)/(1 - D)$. In [12] a

[2] The notations $\|\cdot\|_{\mathscr{A}}$ and $\|\cdot\|_{\mathbf{A}}$ are used in [1] and [8] respectively.

[3] There is a typo in [1] that we repeat throughout [13]. The impulse response of M is $m(t) = \delta(t) - h_+(t) + h_-(t)$.

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stability condition is given equivalent to choosing $M = 1 - H$ where H has impulse response $h \in \mathbf{H}$ and $(1+D)^2/(1-D)^2\|h\|_H < 1$. It is observed in [10] that it is sufficient to require $(1+D)/(1-D)\|h\|_H < 1$. The result in [10] also extends to time-periodic (but not more general) nonlinearities.

Both the results of [14] and [12], as well as the latter's refinement in [10], can be expressed as a corollary of Theorem 1 with $A = B$, and with the further generalisation that the nonlinearity need be neither memoryless nor time-invariant.

Corollary 2 (quasi-monotone nonlinearity with odd bounds, c.f. [14], [12], [10])**.** *Under the conditions of Assumptions 1, 2 and 3 with $A = B$, let H be a noncausal convolution operator whose impulse response is $h \in \mathbf{H}$ satisfying*

$$B\|h\|_H < 1. \tag{18}$$

Let $M = 1 - H$. Then the positivity condition (13) holds and the Lurye system of Fig 1 is stable provided (14) holds.

Proof. Setting $h = h_+ - h_-$ to be the Hahn decomposition of h with $h_+ \in \mathbf{H_p}$ and $h_- \in \mathbf{H_p}$, together with (18) are sufficient for (12) in Theorem 1. □

IV. Loop transformation

In classical multiplier analysis [5], [1] it is standard to apply loop transformations (Fig 3) when the nonlinearity is slope-restricted. Similarly we may apply loop transformations when a nonlinearity has slope-restricted bounds.

Lemma 2. *Under the conditions of Assumption 2, suppose $\underline{\alpha}$ and $\overline{\alpha}$ are both, in addition, slope-restricted on $[0,s]$. Let $k > s$. Define $\underline{\alpha}_k$ as the map from $u(t) - \underline{\alpha}(u(t))/k$ to $\underline{\alpha}(u(t))$ and define $\overline{\alpha}_k$ similarly. Then $\underline{\alpha}_k$ and $\overline{\alpha}_k$ are both monotone and bounded. Furthermore the map from $u(t) - y(t)/k$ to $y(t)$ is bounded below by $\underline{\alpha}_k$ and above by $\overline{\alpha}_k$.*

A similar statement follows if we define $\underline{\beta}_k$ and $\overline{\beta}_k$ similarly. In addtion, $\underline{\beta}_k$ and $\overline{\beta}_k$ are both odd.

Proof. It is well-known that $\underline{\alpha}_k$ and $\overline{\alpha}_k$ are monotone and bounded, and that $\underline{\beta}_k$ and $\overline{\beta}_k$ are monotone, bounded and odd [5], [1].

Suppose (without loss of generality) that $u(t) > 0$. Then $0 \leq \underline{\alpha}(u(t)) \leq y(t) \leq \overline{\alpha}(u(t))$. Similarly $u(t) - \underline{\alpha}(u(t))/k \geq u(t) - y(t)/k \geq u(t) - \overline{\alpha}(u(t))/k \geq 0$. Hence

$$0 \leq \frac{\underline{\alpha}(u(t))}{u(t) - \underline{\alpha}(u(t))/k} \leq \frac{y(t)}{u(t) - y(t)/k} \leq \frac{\overline{\alpha}(u(t))}{u(t) - \overline{\alpha}(u(t))/k}. \tag{19}$$

The result for $\underline{\beta}_k$ and $\overline{\beta}_k$ follows similarly. □

Remark 5. *Even when the mapping from $u(t)$ to $y(t)$ is single-valued, the mapping from $u(t) - y(t)/k$ to $y(t)$ need not be [1]. In particular the slope of the mapping from $u(t)$ to $y(t)$ may exceed k. The results of [14], [12] cannot be applied with loop transformations without either further restrictions or the generalisation in Theorem 1 to possibly multivalued nonlinearities.*

We require a counterpart to Assumption 3.

Assumption 4. *Given Assumption 2, suppose in addition $\underline{\alpha}$ and $\overline{\alpha}$ are both slope-restricted on $[0,s]$. Let $\underline{\alpha}_k$ and $\overline{\alpha}_k$ be defined as in Lemma 2 with $k > s$. Then there is assumed to be some finite $A_k \geq 1$ such that $\overline{\alpha}_k$ is bounded above by $A_k\underline{\alpha}_k$.*

Similarly suppose in addition $\underline{\beta}$ and $\overline{\beta}$ are also both slope restricted on $[0,s]$. Let $\underline{\beta}_k$ and $\overline{\beta}_k$ also be defined as in Lemma 2 with $k > s$. Then there is assumed to be some (possibly infinite) $B_k \geq A_k$ such that $\overline{\beta}_k$ is bounded above by $B_k\underline{\beta}_k$.

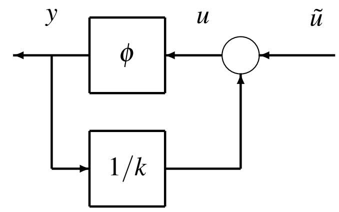

Fig. 3. Loop transformation. The map from $\tilde{u}(t) = u(t) - y(t)/k$ to $y(t)$ may be multivalued, even if the map from $u(t)$ to $y(t)$ is single-valued. Similarly the ratios of bounds on the nonlinearity (i.e. A and B for the map from $u(t)$ to $y(t)$ and A_k and B_k for the map from $u(t) - y(t)/k$ to $y(t)$) are not necessarily preserved under loop transformation.

Lemma 3. *Under the conditions of Assumptions 2 and 4, for all $u \in \mathscr{L}_2$ and for all $\tau \in \mathbb{R}$,*

$$\begin{aligned} -B_k \int_{-\infty}^{\infty} (u(t) - y(t)/k)\, y(t)\, dt \leq \\ \int_{-\infty}^{\infty} (u(t+\tau) - y(t+\tau)/k)\, y(t)\, dt \leq \\ A_k \int_{-\infty}^{\infty} (u(t) - y(t)/k)\, y(t)\, dt. \end{aligned} \tag{20}$$

Proof. Immediate from Lemmas 1 and 2 and Assumption 4. □

Theorem 2 (quasi-monotone or quasi-odd nonlinearity with slope-restricted bounds)**.** *Under the conditions of Assumptions 1, 2 and 4, let H_+ and H_- be noncausal convolution operators whose respective impulse responses are $h_+ \in \mathbf{H_p}$ and $h_- \in \mathbf{H_p}$ satisfying*

$$A_k\|h_+\|_H + B_k\|h_-\|_H < 1. \tag{21}$$

Let $M = 1 - H_+ + H_-$. Then for any $u \in \mathscr{L}_2$

$$\int_{-\infty}^{\infty} M(u - \phi u/k)(t)\,(\phi u)(t)\, dt \geq 0, \tag{22}$$

and the Lurye system of Fig 1 is stable provided there exists $\varepsilon > 0$ such that

$$Re\left[M(j\omega)(1 + kG(j\omega))\right] \geq \varepsilon \text{ for all } \omega \in \mathbb{R}. \tag{23}$$

Proof. Similar to that of Theorem 1. □

There is no guarantee that A_k (or B_k) is small, even when A (or B) is. In fact it is straightforward to construct examples where A_k (or B_k) can be arbitrarily large. By the same token, there are cases where $A_k = A$ (and where $B_k = B$). Here we consider such a case for A_k where the nonlinearity is monotone.

Lemma 4. *Suppose the nonlinearity is monotone. Then $A_k = A = 1$.*

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Proof. We may set $y(t) = \alpha u(t)$ for some monotone α and $\overline{\alpha} = \underline{\alpha} = \alpha$. Hence $\overline{\alpha}_k = \underline{\alpha}_k$. □

Corollary 3 (quasi-odd nonlinearity with slope-restricted bounds)**.** *Under the conditions of Theorem 2, suppose the nonlinearity is in addition monotone. Then (21) may be replaced by the condition*

$$\|h_+\|_H + B_k\|h_-\|_H < 1. \quad (24)$$

Proof. Immediate from Theorem 2 and Lemma 4. □

V. Example with deadzone and monotone slope-restricted bounds

In this section and the next we illustrate the practical applicability of the multipliers. The example in this section is continuous-time while the example in the next is discrete-time. In both cases we exploit loop transformation.

In this section we give an example of a class of nonlinearity with deadzone where Theorem 2 gives better results than the circle criterion. It is similar in spirit to an example in [12] but differs in that the deadzone need not be symmetric, the bounds need not be symmetric, the nonlinearity itself need be neitehr memoryless nor time-invariant and we apply loop transformation. The example illustrates how the values A_k and B_k may differ from A and B and hence the set of multipliers available if we apply Theorem 2 may be smaller than the set available if we apply Theorem 1.

A. Nonlinearity with deadzone and monotone slope-restricted bounds

Suppose the nonlinearity is bounded by

$$\underline{\alpha}(u(t)) = \begin{cases} s_{n1}(u(t)+d_n) & \text{for} \quad u(t) < -d_n \\ 0 & \text{for} \quad -d_n \le u(t) \le d_p \\ s_{p1}(u(t)-d_p) & \text{for} \quad u(t) > d_p \end{cases} \quad (25)$$

and

$$\overline{\alpha}(u(t)) = \begin{cases} s_{n2}(u(t)+d_n) & \text{for} \quad u(t) < -d_n \\ 0 & \text{for} \quad -d_n \le u(t) \le d_p \\ s_{p2}(u(t)-d_p) & \text{for} \quad u(t) > d_p \end{cases} \quad (26)$$

with $0 < s_{n1} < s_{n2}$, $0 < s_{p1} < s_{p2}$ and $d_n > 0$, $d_p > 0$ (Fig 4). It follows that

$$A = \max\left(\frac{s_{n2}}{s_{n1}}, \frac{s_{p2}}{s_{p1}}\right), \quad (27)$$

and

$$B = \frac{\max(s_{n2}, s_{p2})}{\min(s_{n1}, s_{p1})} \text{ when } d_n = d_p, \quad (28)$$

but there is no finite B when $d_n \neq d_p$.

Both $\underline{\alpha}$ and $\overline{\alpha}$ are monotone and slope-restricted on $[0,s]$ with $s = \max(s_{n2}, s_{p2})$. If we apply a loop transformation with $k > s$ the new bounds $\underline{\alpha}_k$ and $\overline{\alpha}_k$ of Lemma 2 are given by

$$\underline{\alpha}_k(u(t)) = \begin{cases} \tilde{s}_{n1}(u(t)+d_n) & \text{for} \quad u(t) < -d_n \\ 0 & \text{for} \quad -d_n \le u(t) \le d_p \\ \tilde{s}_{p1}(u(t)-d_p) & \text{for} \quad u(t) > d_p \end{cases} \quad (29)$$

and

$$\overline{\alpha}_k(u(t)) = \begin{cases} \tilde{s}_{n2}(u(t)+d_n) & \text{for} \quad u(t) < -d_n \\ 0 & \text{for} \quad -d_n \le u(t) \le d_p \\ \tilde{s}_{p2}(u(t)-d_p) & \text{for} \quad u(t) > d_p \end{cases} \quad (30)$$

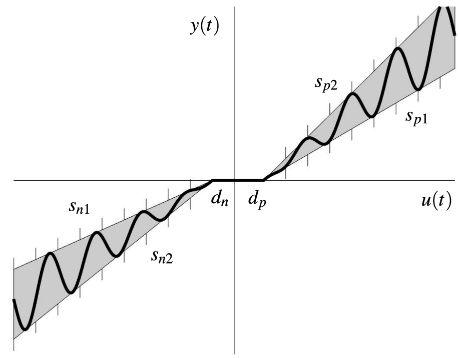

Fig. 4. Nonlinearity bounds for the example of Section V. The analysis of [14], [12] and [10] cannot be used when $d_n \neq d_p$. We provide a specific example where Theorem 2 gives better results than the circle criterion.

with

$$\tilde{s}_{n1} = \frac{ks_{n1}}{k - s_{n1}},\ \tilde{s}_{n2} = \frac{ks_{n2}}{k - s_{n2}},$$
$$\tilde{s}_{p1} = \frac{ks_{p1}}{k - s_{p1}} \text{ and } \tilde{s}_{p2} = \frac{ks_{p2}}{k - s_{p2}}. \quad (31)$$

Hence

$$A_k = \max\left(\frac{s_{n2}}{s_{n1}}\frac{k - s_{n1}}{k - s_{n2}}, \frac{s_{p2}}{s_{p1}}\frac{k - s_{p1}}{k - s_{p2}}\right) \quad (32)$$

and B_k can be found similarly when $d_n = d_p$.

B. Stability criteria

Now consider the continuous-time Lurye system (1). The circle criterion can be used to establish stability provided $\text{Re}[G(j\omega)] > -1/\max(s_{n2}, s_{p2})$ for all ω. If the nonlinearity itself is time-varying the Popov criterion cannot be used, and similalry if the nonlinearity is not monotone then the Zames-Falb criterion cannot be used. If $d_n \neq d_p$ then there is no finite B so none of the criteria of [14], [12] or [10] can be used to establish stability. However either Theorem 1 or Theorem 2 may be used. Here we illustrate the use of Theorem 2.

C. Specific example

As a specifc example, let G be the resonant system with delay

$$G(s) = e^{-s/5}\frac{1}{s^2 + 0.3s + 1} \quad (33)$$

and let $s_{n1} = s_{p1} = 0.5$ and $s_{n2} = s_{p2} = 0.6$.

For this example the circle criterion fails to establish stability since $\min(\text{Re}\ [G(j\omega)]) \approx -1.8195 < -1/0.6$. Similarly Theorem 1 fails to stablish stability directly since the phase of G drops to $-\infty$ as $\omega \to \infty$.

However, if we apply a loop transformation with $k = 1$ we obtain $\tilde{s}_{n1} = \tilde{s}_{p1} = 1$, $\tilde{s}_{n2} = \tilde{s}_{p2} = 3/2$ and hence $\tilde{R}_m = 3/2$. Define the multiplier $M_\varepsilon(s) = 1 - (2/3 - \varepsilon)e^{-0.7s}$ with $\varepsilon \ge 0$. We find $-90^o < \angle M_0(j\omega)(1 + G(j\omega)) < 90^o$ for all ω. If follows by continuity that there is an $\varepsilon > 0$ such that Theorem 2 with multiplier M_ε establishes stability.

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VI. Example with asymmetric saturation

A. Asymmetric saturation

One of our motivations is to study asymmetric saturation. This is of high practical importance as it corresponds to the case with odd bounds but constant offset (due to non-zero setpoint or disturbance). It is possible for a Lurye system to be stable with symmetric saturation but unstable with asymmetric saturation [24], [25] and the behaviour of Lurye systems with asymmetric saturation continues to be of interest [26]. In this example stability is guaranteed with exogenous signals whose steady-state is small but exhibits cycling with exogenous signals whose steady state is large.

Define the asymmetric saturation with gain s as:

$$\text{sat}_{s,-m,n}(u(t)) = \begin{cases} -m \text{ for } u(t) < \frac{-m}{s}, \\ su(t) \text{ for } \frac{-m}{s} \leq u(t) \leq \frac{n}{s} \\ n \text{ for } \frac{n}{s} < u(t) \end{cases} \tag{34}$$

where $s > 0$, $m > 0$ and $n > 0$ (Fig 5).

The nonlinearity is monotone so $A = 1$. Define

$$\underline{B} = \min\{m,n\} \text{ and } \overline{B} = \max\{m,n\}. \tag{35}$$

The nonlinearity is bounded below by $\underline{\beta} = \text{sat}_{s,-\underline{B},\underline{B}}$ and above by $\overline{\beta} = \text{sat}_{s,-\overline{B},\overline{B}}$. Hence it is quasi-odd with $B = \overline{B}/\underline{B}$.

Corollary 4 (asymmetric saturation)**.** *Under the conditions of Theorem 2, suppose ϕ is given by the asymmtetric saturation (34). Then (21) may be replaced by the condition*

$$\|h_+\|_H + B_k\|h_-\|_H < 1 \text{ where } B_k = \overline{B}/\underline{B}, \tag{36}$$

where $\overline{B}$ and $\underline{B}$ are given by (35). Furthermore we may allow $k = s$.

Proof. Since $\underline{\beta}$ and $\overline{\beta}$ are slope-restricted on $[0,s]$ we can apply Corollary 3. Let $k = s+\varepsilon$ with $\varepsilon > 0$. The map from $u(t) - \text{sat}_{s,-m,n}(u(t))/k$ to $\text{sat}_{s,-m,n}(u(t))$ is $\text{sat}_{1/\varepsilon,-m,n}(u(t))$. Hence $\overline{\beta}_k$ is bounded above by $B_k\underline{\beta}_k$. A limiting argument for $\varepsilon \to 0$ can be made following [27]. □

Remark 6. *It is straightforward to show that (21) may be replaced by (36) for any saturation function $\text{sat}_{s,-\mu,\nu}$ with $s > 0$, $\underline{B} \leq \mu \leq \overline{B}$ and $\underline{B} \leq \nu \leq \overline{B}$ where $\overline{B}$ and $\underline{B}$ are given by (35).*

B. Set-points and disturbances

Suppose ϕ in the Lurye system (1) is memoryless and characterised by the symmetric saturation nonlinearity $\text{sat}_{1,-m,m}$ for some $m > 0$. It may be of interest to analyse the behaviour when the exogenous signals r_1 and/or r_2 are step functions and non-zero in steady state. In particular, suppose the system is stable without saturation (i.e. when ϕ is replaced by a unit gain) and the signal u_2 tends to some $u_s \in \mathbb{R}$ in steady state with $|u_s| < m$. Under what circumstances can we guarantee u_2 tends to the same value when there is saturation? Our definition of input-output stability for the Lurye system requires $r_1, r_2 \in \mathscr{L}_2$, so it cannot be applied directly in this case.

But the question is equivalent to asking whether the system is stable when we renormalise our variables so that r_1 and r_2 are both in $\mathscr{L}_2$ and u_2 tends to zero when there is no saturation. In this case the saturation becomes $\text{sat}_{1,-m-u_s,m-u_s}$. Hence we can apply Corollary 4 to test for stability with $B_k = (m+|u_s|)/(m-|u_s|)$. Observe in particular that the value of B_k is dependent on the magnitudes of the exogenous signals.

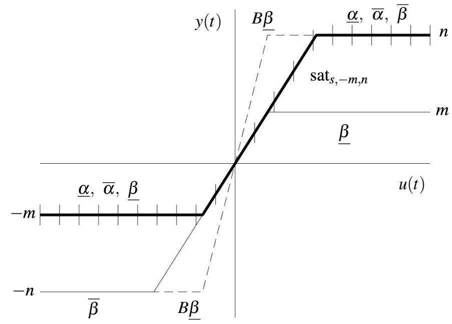

Fig. 5. Asymmetric saturation. The nonlinearity is monotone so $\underline{\alpha} = \overline{\alpha}$ and $A = 1$. Where $u(t) < 0$ we have $\underline{\alpha}(u(t)) = \overline{\alpha}(u(t)) = \underline{\beta}(u(t))$ and where $u(t) > 0$ we have $\underline{\alpha}(u(t)) = \overline{\alpha}(u(t)) = \overline{\beta}(u(t))$. Both $\underline{\beta}$ and $\overline{\beta}$ are slope-restricted on $[0,s]$. The bound $\overline{\beta}$ is itself bounded above by $B\underline{\beta}$.

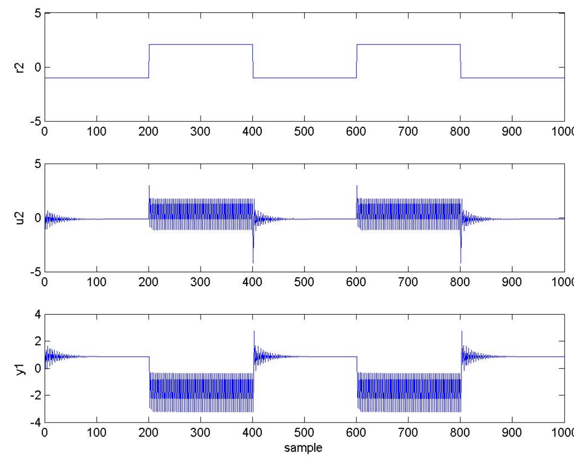

Fig. 6. Stability changes with size of exogenous signal. For low level, $r_2 = -1$ giving steady state $-25/171$ for u_2. For high level, $r_2 = 2.1$ giving steady state $35/114$ for u_2. Since the saturation is ± 1 the effective values of B_k are $(1+25/171)/(1-25/171) = 98/73 \approx 1.3425$ and $(1+35/114)/(1-35/114) = 149/79 \approx 1.886$.

C. Specific example

Here we illustrate the result for asymmetric saturation with a discrete-time example (see Appendix A). Consider the Lurye system (1) where ϕ is characterised by the saturation function

$$\alpha(u(t)) = \text{sat}_{1,-1,1}(u(t)). \tag{37}$$

and let G be the discrete-time transfer function

$$G(z) = \frac{2z+0.92}{z(z-0.5)}. \tag{38}$$

By classical analysis [28], [29] this is only guaranteed stable when the exogenous signals are zero in steady state.

We find $M(1+G)$ is positive with $M(z) = 0.596z + 1 + 0.022z^{-2} - 0.093z^{-3}$. Corollary 4 implies the loop is stable

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TABLE I
VALUES OF B_k AND CORRESPONDING STEADY STATE VALUES FOR u_2 AND r_2 IN THE EXAMPLE.

B_k	$\lvert u_2\rvert$	$\lvert r_2\rvert$	Stable	Comment
1	0	0	Yes	By classical analysis
1.343	0.146	1	Yes	Low level in simulation
1.467	0.189	1.295	Yes	Corollary 3
1.586	0.227	1.55	No	Three-period limit cycle [25]
1.886	0.307	2.1	No	High level in simulation

provided $B_k < 1.467$. The multipliers were found using a convex search as discussed in Appendix B. Corresponding steady-state values of exogenous signal r_2 and input to the saturation u_2 are given in Table I.

We know [25] there is a three-period limit cycle when $B_k = 436/275 \approx 1.586$. Fig 6 shows the signals r_2, u_2 and y_1 when r_2 is switched between -1 and 2.1 every 200 samples. Corresponding values of B_k are shown in Table I. The loop is guaranteed stable when $r_2 = -1$, but shows a three-period limit cycle when $r_2 = 2.1$.

VII. CONCLUSION

We have provided a generalisation of Zames-Falb muliplier theory for both quasi-monotone nonlinearities and quasi-odd nonlinearities. Both the classical results [5], [1] and the generalisations of [14], [12] can be stated as special cases. We have also provided the counterpart results for discrete-time systems in Appendix A. The results follow classical multiplier analysis [1] but exploit the Hahn decomposition [17] of the impulse response $h = h_+ - h_-$ of the operator H where the multiplier is $M = 1 - H$.

Whereas the generalisations of [14], [12] are focused on non-monotone nonlinearities, we also consider nonlinearities that are monotone and quasi-odd. In this case we provide a result (Corollary 1) that we illustrate via an example with asymmetric saturation (Section VI). Our results may be applied to time-varying and multivalued nonlinearites and hence accommodate loop transformation. Unlike the classical results of [5], the set of available multipliers M may be reduced after loop transformation; this is illustrated in the example of Section V.

In Appendix B we indicate how modifications of existing search algorithms can provide convex searches for the new class of multipliers. Such a search for discrete-time multipliers is used in the example of Section VI where multiplier theory can be used to test stability according to the magnitude of exogenous signals in steady state.

APPENDIX A
DISCRETE-TIME RESULTS

The discrete-time counterparts of the Zames-Falb multipliers were proposed in [28], [29]. Applications of the discrete-time Zames–Falb multipliers range from input-constrained model predictive control [30], [31] to first order numerical optimization algorithms [32], [33]. Although they are defined similarly to the continuous-time Zames-Falb multipliers, their properties are significantly different [34], [35].

Here, for completeness, we state the discrete-time counterpart of Theorem 1. Define $\mathbf{h_p}$ as the set of sequences in ℓ where $h_k \geq 0$ for all $k \in \mathbb{Z}$ and $h_0 = 0$.

Theorem 3 (discrete-time, quasi-monotone or quasi-odd nonlinearity)**.** *Under the discrete-time counterparts of the conditions of Assumptions 1 and 2, let H_+ and H_- be noncausal convolution operators whose respective impulse responses are $h_+ \in \mathbf{h_p}$ and $h_- \in \mathbf{h_p}$ satisfying*

$$A\|h_+\|_1 + B\|h_-\|_1 < 1. \tag{39}$$

Let $M = 1 - H_+ + H_-$. Then for any $u \in \ell_2$

$$\sum_{k=-\infty}^{\infty} (Mu)_k(\phi u)_k \geq 0. \tag{40}$$

Furthermore the discrete-time Lurye system of Fig 1 is stable provided

$$Re\left[M(e^{j\omega})G(e^{j\omega})\right] > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{41}$$

Proof. Similar to Theorem 1. □

Discrete-time counterparts to Corollaries 1 and 2 follow straightforwardly as do counterparts to Theorem 2 and Corollaries 3 and 4.

APPENDIX B
CONVEX SEARCHES

Our construction relies on the Hahn decomposition of the impulse response $h = h_+ - h_-$ with $h_+(t) \geq 0$ and $h_-(t) \geq 0$ for all $t \in \mathbb{R}$ (continuous time) or $[h_+]_k \geq 0$ and $[h_-]_k \geq 0$ for all $k \in \mathbb{Z}$ (discrete time). It follows that any search method for Zames-Falb multipliers (or their discrete equivalents) that exploits the characterisation of the impulse response h as the sum of basis functions can be easily modified to search for the multipliers of this paper. This is the case with Chen and Wen's LMI search [15] and the convex FIR search for discrete-time multipliers reported in [16].

Specifically, suppose a search algorithm constructs an impulse response h as

$$h = \sum_{i=1}^{N} \lambda_i h_i, \tag{42}$$

where each $h_i \in \mathbf{H_p}$ satisfies $h_i(0) = 0$, $h_i(t) \geq 0$ for all $t \in \mathbb{R}$ and $\|h_i\|_H = 1$ (continuous time) or where each $h_i \in \mathbf{h_p}$ satisfies $[h_i]_0 = 0$, $[h_i]_k \geq 0$ for all $k \in \mathbb{Z}$ and $\|h_i\|_1 = 1$ (discrete time). Then multipliers for monotone and bounded nonlinearities can be parameterised with the convex constraints

$$\lambda_i \geq 0 \text{ for } i = 1, \ldots, N \text{ and } \sum_{i=1}^{N} \lambda_i < 1. \tag{43}$$

Similarly multipliers for monotone, bounded and odd nonlinearities can be parameterised with the convex constraint

$$\sum_{i=1}^{N} |\lambda_i| < 1. \tag{44}$$

These can be modified to construct an impulse response as $h = h_+ - h_-$ with

$$h_+ = \sum_{i=1}^{N} \lambda_{i+} h_i \text{ and } h_- = \sum_{i=1}^{N} \lambda_{i-} h_i, \tag{45}$$

with each h_i defined as before. The appropriate convex constraints are then

$$\lambda_{i+} \geq 0 \text{ and } \lambda_{i-} \geq 0 \text{ for } i = 1, \ldots, N, \tag{46}$$

and

$$A \sum_{i=1}^{N} \lambda_{i+} + B \sum_{i=1}^{N} \lambda_{i-} < 1. \tag{47}$$

In particular, both the continuous-time search of [15] and the discrete-time search of [16] may be modified in this way to give LMI-based convex searches. We use such a modified discrete-time search in the example of Section VI.

Dedication

We dedicate this paper to our late collaborator and co-author Dmitry Altshuller. Had he lived this paper would surely have had a different flavour. We have preserved his spelling of Lurye throughout. But he would have prefered the development in terms of delay integral quadratic contsraints [20], [21], [22], [10]; although such development is straightforward, we have not resolved some minor technical details, and prefer to retain the classical analysis with which we are more comfortable. In addition, Dmitry proposed the development in the more elegant framework of Fourier analysis on locally Abelian compact groups [36], [37]; for the time-being this will have to remain as an exercise for the reader. We miss working with Dmitry.

WPH and JC.

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