Conditions for the equivalence between IQC and graph separation stability results

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ABSTRACT
This paper provides a link between time-domain and frequency-domain stability results in the literature. Specifically, we focus on the comparison between stability results for a feedback interconnection of two nonlinear systems stated in terms of frequency-domain conditions. While the Integral Quadratic Constrain (IQC) theorem can cope with them via a homotopy argument for the Lurye problem, graph separation results require the transformation of the frequency-domain conditions into truncated time-domain conditions. To date, much of the literature focuses on "hard" factorizations of the multiplier, considering only one of the two frequency-domain conditions. Here it is shown that a symmetric, "doubly-hard" factorization is required to convert both frequency-domain conditions into truncated time-domain conditions. By using the appropriate factorization, a novel comparison between the results obtained by IQC and separation theories is then provided. As a result, we identify under what conditions the IQC theorem may provide some advantage.

KEYWORDS
IQC theorem, graph separation, multipliers factorizations.

1. Motivation

Classical multiplier theory is a well known technique to reduce the conservatism of absolute stability criteria (Desoer and Vidyasagar, 1975; Zames and Falb, 1968). Frequency-domain and time-domain conditions are combined, and the canonical factorization of the multiplier is the essential tool to ensure that time-domain properties can be recovered from the frequency-domain conditions (Carrasco, Heath, and Lanzon, 2012; Goh, 1996; Goh and Safonov, 1995; Jönsson, 1996).

The IQC theorem by Megretski and Rantzer (1997) uses only frequency-domain inequalities and provides a shortcut to avoid conditions on the existence of factorizations by using a homotopy argument in their proof. However the original IQC framework was developed using time-domain constraints by Yakubovich (1965, 1967, 1971), so Megretski and Rantzer (1997) have coined the terms *soft* and *hard* IQC[1] to establish the connection between their IQC theorem and Yakubovich's work. Loosely speaking:

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[1]It has been shown in (Seiler, 2015; Seiler, Packard, and Balas, 2010) that the same IQC can be either hard or soft depending on the factorization used to convert from frequency to time-domain; therefore the terms hard and soft factorizations terminology must be introduced.

- an IQC is *hard* when the time-domain version of the constraint holds for any finite time interval $[0,T]$;
- an IQC is *soft* when the time-domain version of the constraint holds for the interval $[0,\infty)$ but need not be satisfied on finite time intervals.

It may appear that a hard factorization is equivalent to the canonical factorization in the classical multiplier theory. In other words, one may think that a hard factorization of an IQC is sufficient to convert frequency-domain stability conditions to equivalent time-domain conditions (Goh, 1996; Seiler et al., 2010). However, it has been shown that hard factorizations are not enough to establish such equivalence (Seiler, 2015; Veenman and Scherer, 2013). The equivalence between IQC and the so-called dissipative inequality is shown in Seiler (2015). The term hard factorization is still used, and then an extra condition is imposed on the solution of an LMI involving the LTI system.

The graph separation framework (Georgiou and Smith, 1997; Safonov, 1980; Teel, 1996) can be seen as a generalisation of the classical multiplier theory and uses truncated time-domain conditions to obtain stability result. Recently, Carrasco and Seiler (2015) have shown that it is possible to establish a counterpart of the IQC theorem using the graph separation framework. However, they rely on results in (Seiler, 2015) and require one of the two systems in the interconnection to be LTI.

This paper builds on the results presented in (Seiler, 2015) and Carrasco and Seiler (2015). The main contribution of this paper is the development of the counterpart of Lemma 2 in (Seiler, 2015). With this new result, we can establish the equivalence between frequency-domain conditions and truncated time-domain conditions from a pure input-output point of view, without involving LMIs; hence the definition of the factorization does not require one of the systems to be LTI. This approach provides new insights, in particular, we are able to establish a formal comparison between stability results using IQC and graph separation theories for the feedback interconnection of two nonlinear systems. The current state-of-the-art in the use of the IQC theorem requires one of the system to be LTI. In the spirit of Jönsson (2011), Corollary 6.2 in this paper relaxes this assumption. Additional details on this technical point point are given in Remark 1 below.

The structure of the paper is as follows. Sections 2 and 3 provides the IQC theorem and discusses classical hard and soft factorizations as defined by Megretski and Rantzer (1997). Section 4 states a new factorization, the so-called doubly-hard factorization, and characterises this factorization for a class of multipliers. Section 5 demonstrates that not all hard factorizations are doubly-hard factorizations. Section 6 develops two results for the stability of the feedback interconnection of two systems, one using the IQC theorem, and another using the graph separation result by Teel (1996). Finally, Section 7 gives the conclusions of the paper. We use the same notation as in (Megretski and Rantzer, 1997).

2. IQC theorem

Definitions and results related with the IQC framework are given in this section.

Definition 2.1. A stable and causal system $\Delta : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$ is said to satisfy the IQC defined by a bounded, measurable Hermitian-valued function $\Pi : j\mathbb{R} \to \mathbb{C}^{(m+l)\times(m+l)}$ if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta u}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{1}$$

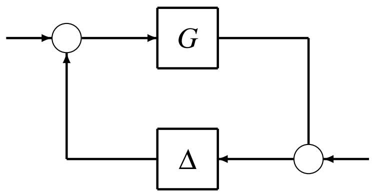

Figure 1. Lurye problem

for any $u \in \mathscr{L}_2^m[0,\infty)$.

Theorem 2.2. *Let $G \in \mathbf{RH}_\infty^{m\times l}$, let $\Delta : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$ be a bounded causal operator, and let $\Pi : j\mathbb{R} \to \mathbb{C}^{(m+l)\times(m+l)}$ be a bounded, measurable Hermitian-valued function. Assume that:*

(1) for every $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed;
(2) for every $\tau \in [0,1]$, the IQC defined by Π is satisfied by $\tau\Delta$;
(3) there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < -\varepsilon I \qquad \forall \omega \in \mathbb{R}. \tag{2}$$

Then, the feedback interconnection of G and Δ in Fig. 1 is stable.

The multiplier Π is normally defined as a block 2-by-2 matrix, i.e.

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}, \tag{3}$$

where Π_{11} is $m \times m$ and Π_{22} is $l \times l$. Then $\Pi(j\omega)$ is called a *positive-negative multiplier* if there exists $\varepsilon > 0$ such that $\Pi_{11}(j\omega) \geq \varepsilon I_m$ and $\Pi_{22}(j\omega) \leq -\varepsilon I_l$ $\forall \omega \in \mathbb{R}$.

In this note we restrict our attention to positive-negative rational multipliers $\Pi \in \mathbf{RL}_\infty^{(m+l)\times(m+l)}$.

3. Hard and soft factorizations

The IQC in Equation 1 can be expressed in the time-domain and this leads to a characterization of the IQC as soft or hard. Specifically, let $\Pi(j\omega) = \Psi^\top(-j\omega)M\Psi(j\omega)$ where Ψ is a causal and stable transfer function. Such factorizations are not unique but can be computed with state-space methods (Scherer and Weiland, 2000). With some abuse of notation we will use the same notation for the transfer function and its corresponding stable operator. The IQC-factorization (Ψ, M) is said to be *soft* if

$$\int_0^\infty \left(\Psi \begin{bmatrix} u \\ \Delta u \end{bmatrix}\right)^\top M \left(\Psi \begin{bmatrix} u \\ \Delta u \end{bmatrix}\right) dt \geq 0, \tag{4}$$

for any $u \in \mathscr{L}_2^m[0,\infty)$.[2] The frequency-domain constraint of Inequality (1) implies the time-domain soft constraint of Inequality (4) by Parseval's theorem. The factorization is said to be *hard* if

$$\int_0^T \left(\Psi \begin{bmatrix} u \\ \Delta u \end{bmatrix}\right)^\top M \left(\Psi \begin{bmatrix} u \\ \Delta u \end{bmatrix}\right) dt \geq 0, \tag{5}$$

for any $u \in \mathscr{L}_{2e}^m[0,\infty)$ and any $T > 0$. This condition for a hard factorization is more restrictive. Specifically, all factorizations of Π are soft but only certain factorizations are hard. It is now clear that the factorization step, i.e. $\Pi = \Psi^{\sim} M \Psi$, is a key point as the same Π can have hard and soft factorizations. These are called (Ψ, M)-hard and (Ψ, M)-soft factorizations (Seiler, 2015; Seiler et al., 2010). The terms *complete* and *conditional* IQCs by Megretski (2010) are generalizations of hard and soft IQCs. The hard/soft terminology will be used here.

There are simple sufficient conditions for the existence of a hard factorization (Goh, 1996). For positive-negative multipliers, it is always possible to find a hard factorization $(\Psi, J_{m,l})$:

$$\Psi = \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \text{ and } J_{m,l} = \begin{bmatrix} I_m & 0 \\ 0 & -I_l \end{bmatrix} \tag{6}$$

where $\Psi_{11}, \Psi_{11}^{-1}, \Psi_{22}$ are stable rational transfer functions. This ensures that the truncation of the IQC will preserve its sign (Goh, 1996). This fact is shown via a simple argument as

$$\int_0^\infty \begin{bmatrix} \Psi_{11} u \\ (\Psi_{21} + \Psi_{22}\Delta) u \end{bmatrix}^\top J_{m,l} \begin{bmatrix} \Psi_{11} u \\ (\Psi_{21} + \Psi_{22}\Delta) u \end{bmatrix} dt = \\ \|\Psi_{11} u\|^2 - \|(\Psi_{21} + \Psi_{22}\Delta) u\|^2 \geq 0. \tag{7}$$

Any input u can be truncated on $[0,T]$ and extended over the positive real line by selecting an artificial input $z([T,\infty))$ such that $\Psi_{11}\tilde{u}(t) = 0$ for all $t > T$ where the piecewise input $\tilde{u}$ is defined by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \leq T, \\ z(t) & \text{if } t > T. \end{cases} \tag{8}$$

The pair $(\tilde{u}, \Delta\tilde{u})$ satisfies the infinite horizon constraint of Inequality (7). Hence, by construction, the pair $(u, \Delta u)$ satisfies the constraint over the finite horizon $[0,T]$. The key point in this construction is the stability of Ψ_{11}^{-1} since it ensures that the artificial input $\tilde{u}$ belongs to $\mathscr{L}_2$ and $\|\Psi_{11} u\|_T = \|\Psi_{11}\tilde{u}\|_T$.

It may initially appear that this truncation is sufficient to complete a dissipativity (or graph separation) proof for stability. However, the role of the second IQC condition seems under-appreciated in the literature. Specifically Equation (2) in the IQC theorem is equivalent to the following second time-domain IQC condition (because both G and Ψ are LTI):

$$\int_0^\infty \left(\Psi \begin{bmatrix} Gu \\ u \end{bmatrix}\right)^\top M \left(\Psi \begin{bmatrix} Gu \\ u \end{bmatrix}\right) dt < -\varepsilon \|u\|^2. \tag{9}$$

All operators in this IQC condition are stable LTI systems and hence the condition can be

[2]Note that the dependence of time has been suppressed in Equation 4 for simplicity. More precisely, this soft IQC is $\int_0^\infty y^\top(t) M y(t) dt \geq 0$ where $y := \Psi \left[\begin{smallmatrix} u \\ \Delta u \end{smallmatrix}\right]$. The time dependence will similarly be dropped in other time-domain IQCs.

checked via an equivalent frequency-domain condition. However, this does not imply that the sign of this inequality will be preserved under finite-horizon truncations in the time-domain. In particular, the key difficulty is observed if we use the triangular factorization along with the truncation arguments introduced above:

$$-\int_0^\infty \begin{bmatrix} \Psi_{11}Gu \\ (\Psi_{21}G+\Psi_{22})u \end{bmatrix}^\top J_{m,l} \begin{bmatrix} \Psi_{11}Gu \\ (\Psi_{21}G+\Psi_{22})u \end{bmatrix} dt = \\ \|(\Psi_{21}G+\Psi_{22})u\|^2 - \|\Psi_{11}Gu\|^2 > \varepsilon\|u\|^2. \quad (10)$$

We now see the difficulties in creating an extension of the input once a truncation $u([0,T])$ has been selected. The extension of the piecewise input on $[T,\infty)$ must cancel $(\Psi_{21}G+\Psi_{22})\tilde{u}$ for any time after the truncation. It may be possible in some cases, but in general this leads to piecewise $\tilde{u} \notin \mathscr{L}_2[0,\infty)$ since Ψ_{22}^{-1} is not stable. This problem is linked to the well known difficulties of applying feedback linearisation to non-minimum phase systems (Isidori, 2013).

4. Doubly-hard IQC factorization

It is possible to show that positive-negative multipliers have a more useful factorization for the purposes of stability analysis. It is shown by Seiler (2015) that J-spectral factorizations can be constructed for positive-negative multipliers, i.e. $\Pi(j\omega) = \Psi^\top(-j\omega)J_{m,l}\Psi(j\omega)$ where Ψ and Ψ^{-1} are both stable transfer functions. Moreover, this factorization allows us to ensure that the signs of both IQCs are preserved under truncation.

To the best of our knowledge this duality property of the factorization has been overlooked. The argument by Seiler (2015), where the J-spectral factorization is given, was based on storage function and dissipativity arguments. The factorization there was still referred to as a hard factorization with a focus on the IQC condition for Δ, but the second condition was established in terms of the resulting LMI. Here we propose a more symmetric and convenient definition where we do not require the construction of the LMI, so we are able to establish the properties of the factorization without invoking the linearity of one of the systems.

In the graph framework, it is standard to use the graph and the inverse graph (Safonov, 1980; Teel, 1996). The standard IQC notation uses the graph of the system Δ. To develop a symmetric formulation, we define the IQC over the inverse graph, henceforward "inverse-graph IQC" as follows:

Definition 4.1 (Inverse-graph IQC)**.** A stable and causal system $\Delta : \mathscr{L}_{2e}^l[0,\infty) \to \mathscr{L}_{2e}^m[0,\infty)$ is said to strictly satisfy the *inverse-graph IQC* defined by a bounded, measurable Hermitian-valued function $\Pi : j\mathbb{R} \to \mathbb{C}^{(m+l)\times(m+l)}$ if there exists $\varepsilon > 0$ such that

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{\Delta u}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{\Delta u}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix} d\omega \leq -\varepsilon\|u\|, \quad (11)$$

for any $u \in \mathscr{L}_2^l[0,\infty)$.

If Δ is linear, then (11) is equivalent to (2) by using Δ instead of G.

Then we can state the definition of the factorization which will lead to an equivalence between frequency-domain conditions and truncated time-domain conditions:

Definition 4.2 (Doubly-hard factorization)**.** For a given $\Pi : j\mathbb{R} \to \mathbb{C}^{(m+l)\times(m+l)}$, a factorization (Ψ, M) is said to be a *doubly-hard IQC factorization* if the following two conditions hold:

(1) for any bounded and causal $\Delta_1 : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$, the IQC condition

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta_1 u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta_1 u}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{12}$$

for all $u \in \mathscr{L}_2^m[0,\infty)$ implies that

$$\int_0^T \left(\Psi \begin{bmatrix} u \\ \Delta_1 u \end{bmatrix} \right)^\top M \left(\Psi \begin{bmatrix} u \\ \Delta_1 u \end{bmatrix} \right) dt \geq 0. \tag{13}$$

for any $u \in \mathscr{L}_{2e}^m[0,\infty)$ and any $T > 0$, and

(2) for any bounded and causal $\Delta_2 : \mathscr{L}_{2e}^l[0,\infty) \to \mathscr{L}_{2e}^m[0,\infty)$, the *inverse-graph IQC* condition

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{\Delta_2 u}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{\Delta_2 u}(j\omega) \\ \widehat{u}(j\omega) \end{bmatrix} d\omega \leq -\varepsilon \|u\|^2, \tag{14}$$

for all $u \in \mathscr{L}_2^l[0,\infty)$ implies that

$$\int_0^T \left(\Psi \begin{bmatrix} \Delta_2 u \\ u \end{bmatrix} \right)^\top M \left(\Psi \begin{bmatrix} \Delta_2 u \\ u \end{bmatrix} \right) dt \leq -\varepsilon \|u\|_T^2, \tag{15}$$

for any $u \in \mathscr{L}_{2e}^l[0,\infty)$ and any $T > 0$.

Finally, we show that the key property to obtain a doubly-hard factorization is the stability of both Ψ and Ψ^{-1}. This result requires Lemma 2 in Seiler (2015), and the development of a result for the inverse-graph condition (14).

Let

$$\Psi \sim \left[\begin{array}{c|cc} A & B_v & B_w \\ \hline C & D_v & D_w \end{array} \right]. \tag{16}$$

Define the functional J on $v \in \mathscr{L}_2[0,\infty)$, $w \in \mathscr{L}_2[0,\infty)$ and $x_0 \in \mathbb{R}^n$ as

$$J(v,w,x_0) = \int_0^{\infty} z(t)^\top M z(t) dt \tag{17}$$

subject to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_v v(t) + B_w w(t), \quad x(0) = x_0; \\ z(t) &= Cx(t) + D_v v(t) + D_w w(t). \end{aligned}$$

Define the upper value $\bar{J}(x_0)$ as

$$\bar{J}(x_0) := \inf_{v \in \mathscr{L}_2[0,\infty)} \sup_{w \in \mathscr{L}_2[0,\infty)} J(v,w,x_0),$$

and the lower value $\underline{\mathrm{J}}(x_0)$ as

$$\underline{\mathrm{J}}(x_0) := \sup_{w \in \mathscr{L}_2[0,\infty)} \inf_{v \in \mathscr{L}_2[0,\infty)} J(v,w,x_0).$$

Lemma 4.3 (Seiler (2015))**.** *Let Π be a multiplier and (Ψ, M) any factorization with Ψ stable. Assume Δ_1 is a causal bounded operator such that*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{18}$$

for any $v \in \mathscr{L}_2[0,\infty)$ and $w = \Delta_1 v$. Then for all $T \geq 0$, for all $v \in \mathscr{L}_2[0,\infty)$, and $w = \Delta_1 v$, the signal defined by $z = \Psi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}$ satisfies

$$\int_0^T z(t)^\top M z(t) dt \geq -\bar{J}(x(T)), \tag{19}$$

where $x(T)$ denotes the state of the system Ψ at the instant T when driven by the inputs (v,w) with null initial conditions.

Lemma 4.4. *Let Π be a multiplier and (Ψ, M) any factorization with Ψ stable. Assume Δ_2 is a causal bounded operator such that*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{w}(j\omega) \end{bmatrix} d\omega \leq -\varepsilon(\|v\|^2 + \|w\|^2), \tag{20}$$

for any $w \in \mathscr{L}_2[0,\infty)$ and $v = \Delta_2 w$. Then for all $T \geq 0$, for all $w \in \mathscr{L}_2[0,\infty)$, and $v = \Delta_2 w$, the signal defined by $z = \Psi \begin{bmatrix} v \\ w \end{bmatrix}$ satisfies

$$\int_0^T z(t)^\top M z(t) dt \leq -\varepsilon \|z\|_T - \underline{J}(x(T)), \tag{21}$$

where $x(T)$ denotes the state of the system Ψ at the instant T when driven by the inputs (v,w) with null initial conditions.

Proof: See Appendix

Theorem 4.5. *Given a positive-negative multiplier $\Pi \in \mathbf{RL}_\infty$, the factorization (Ψ, M) is doubly-hard if $\Psi, \Psi^{-1} \in \mathbf{RH}_\infty$.*

Proof. If $\Psi \in \mathbf{RH}_\infty$, Lemma 4.3 and Lemma 4.4 hold. Moreover, if the multiplier Π is positive-negative and $\Psi^{-1} \in \mathbf{RH}_\infty$ then $\bar{J}(x) = \underline{\mathrm{J}}(x) = 0$ for all $x \in \mathbb{R}^n$ (see Lemma 5 in (Seiler, 2015)). As a result the factorization (Ψ, M) is doubly-hard. $\square$

Therefore all J-spectral factorizations are doubly-hard factorizations.

5. Triangular factorization *vs* J-spectral factorization

This section provides a simple example highlighting the distinction between triangular and J-spectral factorizations. Consider a simple feedback interconnection of the static system $G = \frac{1}{2}$ and an operator Δ. Define a positive-negative multiplier $\Pi \in \mathbf{RL}_\infty^{2\times 2}$ by

$$\Pi(s) = \begin{bmatrix} 3 & \frac{-s+2}{s+1} \\ \frac{-s-2}{s-1} & \frac{-s^2+4}{s^2-1} \end{bmatrix} \tag{22}$$

Assume the interconnection of G and $\tau\Delta$ is well-posed for all $\tau \in [0,1]$. Also assume that $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$. It can be verified that $\left[\begin{smallmatrix} G \\ 1 \end{smallmatrix}\right]^\sim \Pi \left[\begin{smallmatrix} G \\ 1 \end{smallmatrix}\right] = -1.25 < 0$, i.e. G satisfies the IQC constraint with Π. Thus the frequency domain IQC conditions in Theorem 2.2 are satisfied and the feedback interconnection is stable.

As noted above, the factorization of Π is not unique. Here we construct two different factorizations. First, a stable triangular factorization (Ψ, M) of Π is given by:

$$M = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \text{ and } \Psi = \begin{bmatrix} 2 & 0 \\ 1 & \frac{s-2}{s+1} \end{bmatrix}. \tag{23}$$

Note that Ψ is stable but the (2,2) entry of Ψ is non-minimum phase. The multiplier satisfies the positive-negative conditions and hence it also has a J-spectral factorization $(\tilde{\Psi}, \tilde{J})$:

$$\tilde{J} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \text{ and } \tilde{\Psi} = \begin{bmatrix} -1.751 & \frac{0.4133s-1.508}{s+1} \\ -0.2554 & \frac{-1.082s-2.505}{s+1} \end{bmatrix}. \tag{24}$$

Note that for this factorization $\tilde{\Psi}$ and $\tilde{\Psi}^{-1}$ are both stable. Figure 2 shows the IQC evaluated on $[0,T]$ versus the finite horizon time T for the input signal $u(t) = 0.458\sin(t)$ for $t \in [0,10]$ and $u(t) = 0$ otherwise. The coefficient 0.458 is selected to normalize the signal $\|u\|_2 = 1$. As $t \to \infty$, both IQCs converge to -1.25. This value is consistent with the constraint $\left[\begin{smallmatrix} G \\ 1 \end{smallmatrix}\right]^\sim \Pi \left[\begin{smallmatrix} G \\ 1 \end{smallmatrix}\right] = -1.25 < 0$. Thus both factorizations satisfy the time-domain constraint as $t \to \infty$. However, the lower triangular factorization goes positive on the approximate interval $[0, 2.8]$. Thus the lower triangular factorization can violate the constraint over finite horizons. On the other hand, the J-spectral factorization remains negative and hence satisfies the constraint over all finite horizons.

It can be shown that lower triangular factorizations have a $(2,2)$ entry that is non-minimum phase in general. Specifically, if Ψ is lower triangular and $\Pi = \Psi^\sim J \Psi$ then the entries of Ψ satisfy:

$$\begin{aligned} \Pi_{11} &= \Psi_{11}^\sim \Psi_{11} - \Psi_{21}^\sim \Psi_{21} \\ \Pi_{12} &= -\Psi_{21}^\sim \Psi_{22} \\ \Pi_{22} &= -\Psi_{22}^\sim \Psi_{22} \end{aligned}$$

These conditions imply that if Π_{12} has poles in the left half plane then Ψ_{22} must be non-minimum phase. Specifically, if Π is a positive-negative multiplier then there exists $\varepsilon > 0$ such that $-\Pi_{22}(j\omega) \geq \varepsilon I \; \forall \omega \in R$. Hence it can be factorized as $-\Pi_{22} = H^\sim H$ where $H \in RH$ and H^{-1} is anti-stable. In other words H is stable and anti-minimum phase. This factorization can be constructed from the normal stable, minimum phase spectral factorization (Youla, 1961).

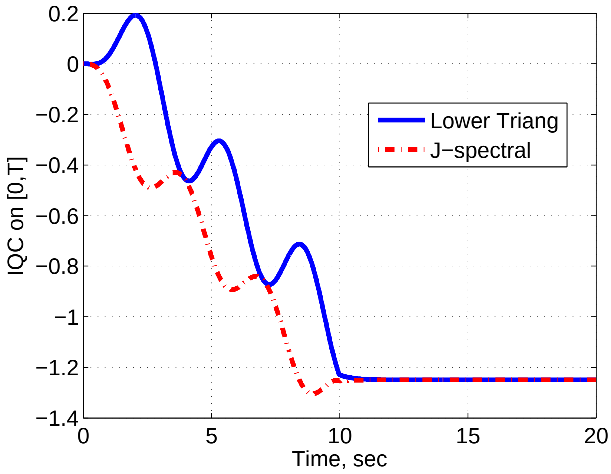

Figure 2. $\int_0^T \left(\Psi \begin{bmatrix} G \\ 1 \end{bmatrix} u\right)^T M \left(\Psi \begin{bmatrix} G \\ 1 \end{bmatrix} u\right) dt$ versus time T.

Next, let $\{p_i\}_{i=1}^N$ denote the poles of Π_{12} in the left half plane. Define Ψ_{21} and Ψ_{22} as

$$\Psi_{22}(s) := H(s) \left(\prod_{i=1}^{N} \frac{s+\bar{p}_i}{s-p_i} \cdot I_m \right) \tag{25}$$

$$\Psi_{21}(s) := \left(-\Pi_{12}(s)\Psi_{22}^{-1}(s)\right)^{\sim} \tag{26}$$

By construction, Ψ_{22} is stable and anti-minimum phase. The inclusion of the Blaschke products[3] in the definition of Ψ_{22} does not impact the value of $\Psi_{22}^{\sim}\Psi_{22}$ on the imaginary axis. Thus $\Pi_{22} = \Psi_{22}^{\sim}\Psi_{22}$ on the imaginary axis by construction of H. This choice of Ψ_{22} is required to ensure that $\Pi_{12}\Psi_{22}^{-1}$ is anti-stable and hence Ψ_{21} is stable. Moreover, $\Psi_{21}^{\sim}\Psi_{22} = -\Pi_{12}$. A stable, stably invertible Ψ_{11} can then be constructed from a spectral factorization of $\Pi_{11} + \Psi_{21}^{\sim}\Psi_{21}$. In this construction, any LHP poles of Π_{12} appear as RHP zeros in Ψ_{22}.

6. Comparison between IQC and graph separation results

6.1. *Stability results*

In this section we develop two stability results for the feedback interconnection of two nonlinear systems. One of these results will be obtained using graph separation methods. For completeness, we state an IQC version of Corollary 5.1 in (Teel, 1996) as follows:

Theorem 6.1 (Teel (1996))**.** *Let Δ_1 and Δ_2 be two causal and bounded systems. Let Ψ be a stable linear system. Assume that:*

(1) the feedback interconnection of G and Δ is well-posed;
(2) the time-domain IQC

$$\int_0^T \left(\Psi \begin{bmatrix} u \\ \Delta_1 u \end{bmatrix}\right)^\top M \left(\Psi \begin{bmatrix} u \\ \Delta_1 u \end{bmatrix}\right) dt \geq 0, \tag{27}$$

is satisfied for any $T > 0$ and $u \in \mathscr{L}_{2e}[0,\infty)$;

[3] See (Partington, 2004) for a definition.

(3) *the time-domain inverse-graph IQC*

$$\int_0^T \left(\Psi \begin{bmatrix} \Delta_2 u \\ u \end{bmatrix}\right)^\top M \left(\Psi \begin{bmatrix} \Delta_2 u \\ u \end{bmatrix}\right) dt < -\varepsilon \left\| \begin{bmatrix} \Delta_2 u \\ u \end{bmatrix} \right\|_T^2, \tag{28}$$

is satisfied for any $T > 0$ and $u \in \mathscr{L}_{2e}[0,\infty)$.

Then the feedback interconnection between Δ_1 and Δ_2 is $\mathscr{L}_2$-stable.

In the spirit of Jönsson (2011), we can establish the following corollary for the interconnection of two nonlinear systems:

Corollary 6.2 (Corollary of Theorem 2.2)**.** *Let $\Delta_1 : \mathscr{L}_{2e}^l[0,\infty) \to \mathscr{L}_{2e}^m[0,\infty)$ and $\Delta_2 : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$ be bounded causal operators, and let $\Pi \in \mathbf{RL}_\infty^{(m+l)\times(m+l)}$. Assume that:*

(I) *for every $\tau \in [0,1]$, the feedback interconnection of $\tau\Delta_1$ and $\tau\Delta_2$ is well-posed;*
(II) *for every $\tau \in [0,1]$, $\tau\Delta_1$ satisfies the IQC defined by Π;*
(III) *for every $\tau \in [0,1]$, $\tau\Delta_2$ strictly satisfies the inverse-graph IQC defined by Π.*

Then, the feedback interconnection of Δ_1 and Δ_2 is stable.

Remark 1. The current state-of-the-art in the use of the IQC theorem requires one of the system to be LTI. In (Megretski and Rantzer, 1997), it is used that $G\tau\Delta(v) = \tau G\Delta(v)$ in the last equation of the first step of the proof, hence the linearity of G is explicitly used in the proof. To the best of authors' knowledge, the above Corollary is the least conservative use of the IQC theorem for two nonlinear systems.

Proof. The result follows from the application of the IQC theorem using

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \text{ and } G = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \tag{29}$$

and the following augmented multiplier:

$$\Pi_a = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & -\Pi_{22} - \varepsilon I & 0 & -\Pi_{12}^* \\ \Pi_{12}^* & 0 & \Pi_{22} & 0 \\ 0 & -\Pi_{12} & 0 & -\Pi_{11}^* - \varepsilon I \end{bmatrix} \tag{30}$$

Some straightforward algebra is required to show that the conditions in Theorem 2.2 are satisfied. □

Using Theorem 4.5, then it is possible to remove the homotopy condition in the above result if the matrix Π is positive-negative. Formally we can state the following result:

Corollary 6.3 (Corollary of Theorem 6.1)**.** *Let $\Delta_1 : \mathscr{L}_{2e}^l[0,\infty) \to \mathscr{L}_{2e}^m[0,\infty)$ and $\Delta_2 : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$ be bounded causal operators, and let $\Pi \in \mathbf{RL}_\infty^{(m+l)\times(m+l)}$. Assume that:*

(i) *the feedback interconnection of Δ_1 and Δ_2 is well-posed;*
(ii) *Δ_1 satisfies the IQC defined by Π;*
(iii) *Δ_2 strictly satisfies the inverse-graph IQC defined by Π;*

(iv) Π *is a positive-negative multiplier.*

Then, the feedback interconnection of Δ_1 *and* Δ_2 *is stable.*

Proof. If Π is a positive-negative multiplier, then there exists a factorization (Ψ, M) such that Ψ and Ψ^{-1} are both stable (Seiler, 2015). Therefore the factorization (Ψ, M) is doubly-hard as it satisfies the conditions in Theorem 4.5. The frequency-domain conditions (ii) and (iii) can be transformed into truncated time-domain conditions by using the factorization (Ψ, M). As a result, Theorem 6.1 can be used to establish the stability of feedback interconnection between Δ_1 and Δ_2. □

Remark 2. It would not be possible to prove Corollary 2 by using triangular factorizations as it fails to guarantee that condition (iii) is equivalent to the truncated time-domain condition (28).

6.2. *Discussion*

A naïve comparison of the results would suggest that condition (iv) in Corollary 6.3 an extra condition over the conditions of Corollary 6.2. It is well known that the homotopy condition in (II) is satisfied if Π_{11} is positive. Similarly, the homotopy condition in (III) is satisfied if Π_{22} is negative. Hence one can think of a superiority of Corollary 6.2 over Corollary 6.3.

However, if Δ_1 and Δ_2 are both nonlinear, the IQC theorem requires homotopy conditions for both systems. If condition (II) holds, the requirement of the condition to be true when $\tau = 0$ implies $\Pi_{11}(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Similarly, if condition (III) holds, the same argument when $\tau = 0$ implies $\Pi_{22}(j\omega) \leq -\varepsilon I$ for some $\varepsilon > 0$.

A perturbation argument as in (Carrasco et al., 2012; Seiler, 2015) in conjunction with a substitution argument (Carrasco, Heath, and Lanzon, 2013) is required here; although $\Pi_{11}(j\omega) \geq 0$ does not guarantee the existence of a factorization, the following Lemma ensures the existence of a new $\bar{\Pi}$ with $\bar{\Pi}_{11}(j\omega) \geq \delta I$ for some $\delta > 0$, hence $\bar{\Pi}$ can be factorised:

Lemma 6.4. *Let* $G \in \mathbf{RH}_\infty^{m\times l}$*, let* $\Delta : \mathscr{L}_{2e}^m[0,\infty) \to \mathscr{L}_{2e}^l[0,\infty)$ *be a bounded causal operator. If conditions (2) and (3) in Theorem 2.2 are satisfied for some* Π*, then there exists some* $\delta > 0$ *such that conditions (2) and (3) are satisfied for*

$$\bar{\Pi} = \begin{bmatrix} \Pi_{11} + \delta I_m & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}.$$

Proof. See Appendix. □

Remark 3. The counterpart result for Corollary 6.2 is trivially obtained as the only required condition is the boundedness of Δ_2.

As a result, we can consider without loss of generality that Corollary 6.2 can only be satisfied if Π is positive-negative. In conclusion, the IQC theorem may only provide better results over the graph separation theory when (a) Δ_2 is linear and (b) Π_{22} is non-negative. Otherwise, graph separation and IQC theories lead to the same stability result for rational multipliers.

Finally, it must be highlighted that this paper has used the IQC theorem following the proof by Megretski and Rantzer (1997). It remains open if an alternative proof of the IQC theorem without the explicit use of the linearity of *G* may provide some theoretical advantages as the homotopy condition in condition (III) would no longer be required.

7. Conclusion

The aim of this paper is to complete the classification of IQC-factorizations. It concludes previous work presented in (Carrasco and Seiler, 2015; Seiler, 2015), establishing a novel connection between IQC and graph separation theories. Here we propose the term doubly-hard factorizations, where both frequency conditions can be transformed into truncated time-domain conditions. We show that the standard triangular factorization is hard factorization but fails to be a doubly-hard. Then it cannot be used to establish an equivalence between the IQC theorem and separation results in the truncated time-domain. We have shown that (Ψ, M) is a doubly-hard factorization if Ψ and Ψ^{-1} are both stable.

The new results allow us to compare both theories for the feedback interconnection of two nonlinear systems. As a result we conclude that the straightforward application of the IQC theorem in Megretski and Rantzer (1997) for two nonlinear systems given by Corollary 6.2 does not provide any significant advantage over its counterpart result derived using graph separation tools. However, the IQC theorem may provide some advantages when one of the systems is linear and the term Π_{22} is non-negative. We must highlight that an alternative proof of the IQC theorem may render in less conservative conditions, but it remains as an open question.

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Appendix A. Proof of Lemma 6.4

If condition (2) is satisfied for Π, then it is trivial that it is also satisfied for $\bar{\Pi}$ since

$$\begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta u}(j\omega) \end{bmatrix}^* \begin{bmatrix} \delta I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta u}(j\omega) \end{bmatrix} = \delta |\widehat{u}(j\omega)|^2 \geq 0 \tag{A1}$$

for all $\omega \in \mathbb{R}$. If condition (3) is satisfied for Π, then there exists $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\varepsilon I \qquad \forall \omega \in \mathbb{R}. \tag{A2}$$

Moreover, for any $\delta > 0$, it follows

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \delta I_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \delta (G(j\omega)^* G(j\omega)) \leq \delta \|G\|_\infty^2 I, \tag{A3}$$

for all ω. As a result, taking $\delta = \frac{\varepsilon}{2\|G\|_\infty^2}$,

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \bar{\Pi}(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -(\varepsilon - \frac{\varepsilon}{2}) I = -\frac{\varepsilon}{2} I \qquad \forall \omega \in \mathbb{R}, \tag{A4}$$

Appendix B. Proof of Lemma 4.4

For any $T \geq 0$, the frequency-domain inequality (20) can be converted into time-domain (by Parserval's theorem) and re-arranged as

$$\int_0^T z(t)^\top Mz(t)dt \leq -\varepsilon \int_0^\infty z(t)^\top z(t)dt - \int_T^\infty z(t)^\top Mz(t)dt \tag{B1}$$

Note that $-\varepsilon \int_T^\infty z(t)^\top z(t)dt \leq 0$ and hence the following bound is also valid:

$$\int_0^T z(t)^\top Mz(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt - \int_T^\infty z(t)^\top Mz(t)dt \tag{B2}$$

Next let $\tilde{w} \in \mathscr{L}_2[0,\infty)$ be any signal satisfying $\tilde{w}_T = w_T$. Define $\tilde{v} = \Delta_2 \tilde{w}$ and let $\tilde{z} = \Psi \left[\begin{smallmatrix} \tilde{v} \\ \tilde{w} \end{smallmatrix}\right]$ be the response of Ψ with null initial condition. By causality of Δ_2 and Ψ, $w_T = \tilde{w}_T$ implies $v_T = \tilde{v}_T$ and $z_T = \tilde{z}_T$. Hence for all $\tilde{w}$, it holds

$$\int_0^T z(t)^\top Mz(t)dt = \int_0^T \tilde{z}(t)^\top M\tilde{z}(t)dt.$$

Moreover, the IQC holds for any input/output pairs of Δ_2. In particular, Equation B2 holds with z replaced by $\tilde{z}$. As a result, any $\tilde{w} \in \mathscr{L}_2$ satisfying $\tilde{w}_T = w_T$ can be used to upper bound the integral $\int_0^\infty z(t)^\top Mz(t)dt$ obtained with w:

$$\int_0^T z(t)^\top Mz(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt - \int_T^\infty \tilde{z}(t)^\top M\tilde{z}(t)dt \tag{B3}$$

Minimizing over all feasible $\tilde{w}$ yields the upper bound

$$\int_0^T z(t)^\top Mz(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt + \inf_{\tilde{w} \in \mathscr{L}_2, \tilde{w}_T = w_T} \left(- \int_T^\infty \tilde{z}(t)^\top M\tilde{z}(t)dt \right), \tag{B4}$$

The suitable set of signals $\tilde{w}$ can be rewritten as

$$\tilde{w}(t) = \begin{cases} w(t) & \text{if } t \leq T \\ w_f(t) & \text{if } t > T \end{cases}$$

for any $w_f \in \mathscr{L}_2[T,\infty)$. We can rewrite the minimisation as

$$\int_0^T z(t)^\top Mz(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt + \inf_{w_f \in \mathscr{L}_2[T,\infty)} \left(- \int_T^\infty \tilde{z}(t)^\top M\tilde{z}(t)dt \right), \tag{B5}$$

such that $\tilde{v} = \Delta_2 \tilde{w}$ and $\tilde{z} = \Psi \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$. The dependence on Δ_2 can be removed following similar arguments to those given in (Seiler, 2015). Partition $\tilde{v} = \Delta_2 \tilde{w}$ as:

$$\tilde{v}(t) = \begin{cases} \Delta_2 w(t) & \text{if } t \leq T \\ v_f(t) & \text{if } t > T \end{cases} \tag{B6}$$

The bound in Equation B5 only involves $\tilde{z}$ defined on $[T,\infty)$. This signal can be computed from the state of Ψ at time T, i.e. x_T, as well as the signals w_f and v_f. Note that $x(T) = x_T$ is the same for any feasible choice of $\tilde{w}$ because $\tilde{w}_T = w_T$ and $\tilde{v}_T = v_T$. The dependence on Δ_2 is removed, with some conservatism, by simply maximizing over all possible future signals v_f defined on $[T,\infty)$ instead of using $\tilde{v} = \Delta_2 w$. In other words,

$$\int_0^T z(t)^\top M z(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt + \inf_{w_f \in \mathscr{L}_2[T,\infty)} \sup_{v_f \in \mathscr{L}_2[T,\infty)} \left(-\int_T^\infty \tilde{z}(t)^\top M \tilde{z}(t)dt \right), \quad \text{(B7)}$$

This is subject to constraint $x(T) = x_T$. This can be rewritten using the cost function J as:

$$\int_0^T z(t)^\top M z(t)dt \leq -\varepsilon \int_0^T z(t)^\top z(t)dt - \underline{J}(x_T), \quad \text{(B8)}$$