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A generalized belief dissimilarity measure based on weighted conflict belief and distance metric and its application in multi-source data fusion

Mi Zhou [a,b,*], Ya-Jing Zhou [a,b], Jian-Bo Yang [c], Jian Wu [d]

[a] *School of Management, Hefei University of Technology, Hefei 230009, Anhui, China*
[b] *Key Laboratory of Process Optimization and Intelligent Decision-making, Ministry of Education, Hefei, Anhui 230009, China*
[c] *Alliance Manchester Business School, The University of Manchester, M15 6PB, UK*
[d] *School of Economics and Management, Shanghai Maritime University, Shanghai 201306, China*

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ABSTRACT

Dissimilarity measure between basic probability assignments (BPAs) in the Dempster-Shafer evidence structure is a vibrant research topic in artificial intelligence. However, there are flaws in the existing measurements. In particular, it is insufficient to characterize dissimilarity only from either evidential distance or conflict belief for a BPA. As such, we propose a new dissimilarity measure which takes into consideration both distance measure and conflict belief among betting commitments. These two factors complement with each other. Distance measure reflects diversity between the focal elements of two pieces of evidence. That is to say, the more intersections between the bodies of evidence (BOE) of two data sources, the more reliable it acts as a dissimilarity measure. Conversely, the conflict belief which is created based on the transformed Pignistic probability characterizes the product of singleton's belief from two pieces of evidence whose intersection is empty. It quantifies dissimilarity measure more efficiently when the focal elements of two pieces of evidence have small intersect. Theoretically, the new dissimilarity measure satisfies reflexivity, symmetry, nonnegativity, nondegeneracy and some other properties. Comparative analysis is provided with some cases to demonstrate the applicability and validity of the proposed dissimilarity measure. To determine the weight and reliability of evidence, the new dissimilarity measure among evidence and uncertainty of BPA are used. The dissimilarity metric is further applied for multi-source data fusion together with uncertainty measure of belief structure. The application of large-scale group decision making (LSGDM) problem is given to illustrate the effectiveness of the proposed multi-source data fusion process.

1. Introduction

Multi-source data fusion is very common in real word. It has been applied in many fields, such as sensor data fusion [1,2], target recognition [3,4], fault diagnosis [5], multiple attribute decision making [6,7] and group decision making [8–11]. It is important to investigate how to extract useful information from multiple sources of data and eliminate disturbing or useless information. Two main issues must be tackled in a proper way in multi-source data fusion. One is the determination of weights or reliabilities for different data

* Corresponding author
E-mail address: zhoumi@hfut.edu.cn (M. Zhou).

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sources or evidence before the fusion of information [12–14]. Inappropriate weight or reliability assignment will result in an ineffective data fusion process. The other one is conflict among different pieces of evidence to be dealt with in combination of data [3,13,15]. When conflict occurs among evidence, counter-intuitive results may be generated, which can lead to distorted fusion outcomes.

Dissimilarity/distance measure is an indispensable part of investigation in addressing the above two issues. How to measure dissimilarity and manage it in combination of evidence is an open question in muti-source data fusion. The weight of one data source can be generated by the average divergence between it and all other data sources [16,17]. The larger the divergence of a data source, the smaller weight it should be assigned. This is in accordance with the perspective of objective weight assignment methods [18–20]. As for the second issue, conflict between each pair of sources can also be computed by dissimilarity between them. Due to complex environments and dynamic decision-making situations, uncertainty is inevitably included in multi-source data fusion. This will lead to difficulty in measuring dissimilarity between pairs of data sources. Dempster-Shafer (D-S) evidence theory [21,22] can be applied to model uncertain information which is an extension of probability theory. It was developed from the probability theory and has been applied broadly in many fields, such as classification [23], belief rule-based reasoning [24,25], multi-sensor data fusion [18,26] and environmental impact assessment [27]. To date, some generalized or developed theories were also proposed including Dezert-Smarandache theory [28] and evidential reasoning rule [14]. For a long time, the conflict belief k which is the normalization factor in Dempster's rule [21,22] is considered to be the distance metric between two evidences. But it has been proved that k is not appropriate to quantify the dissimilarity, especially when the intersection of the cores of two BPAs contains a large proportion of elements. Because of this, some alternative measures have been produced, such as Tessem's distance [29], Tanimoto's similarity [30], Minkowski's distance [31], Cosine similarity [32], Jousselme's distance [33,34] correlation coefficient [35], PSD pignistic probability based distance [36], Liu's distance [13], Jiang's distance [37], Yu's distance [38], Xiao's distance [39] and so on. Different classifications of distance between BPAs have also been proposed by some researchers. Liu [31] classified some commonly used distances into two categories: Jousselme's distance & Bhattacharyya's distance and probabilistic-based distance. Jousselme made a more detailed classification in [33] which classified evidential distances into the composite distances [40], the Minkowski family [41], the inner product family [32], the fidelity family [42], the information-based distances [43] and two-dimensional distances [13,36,44]. The comparison of the above-mentioned dissimilarity measures is shown in Table 1.

Up to now, although a plethora of dissimilarity measures have been defined under the D-S structure [13,30-39,42,43], we are still confronted with the problem of selecting a rational dissimilarity metric because many of them more or less have defects. First, the property of reflexivity, symmetry, nondegeneracy and transitivity should be satisfied for a strict dissimilarity metric. All the listed distance or dissimilarity measures satisfy the property of 'symmetry' and 'nonnegativity'. But not all of them meet the conditions of 'reflexivity' and 'nondegeneracy'. For instance, the conflict belief and Jiang's distance don't satisfy 'reflexivity' unless evidence is a categorical belief function. The conflict belief and Tessem's distance don't satisfy 'nondegeneracy'. Even if two pieces of evidence are completely identical, the conflict belief between them may be positive provided that two or more focal elements are included. Since Tessem's distance is calculated based on the transformed Pignistic probability, dissimilarity between two pieces of evidence may be 0 although they are different. If the above four properties are not satisfied, it is not a strict dissimilarity metric. Second, due to the fact that some dissimilarity measures are derived from probability scheme, they're only applicable in the situation of Bayesian belief structure. In other words, these dissimilarity measures are valid when they're defined on the base set rather than the power set. Cosine

Table 1
Comparison of different distance/dissimilarity measures in the belief structure

Name	Symbol	Reflexivity	Nondegeneracy Definiteness (two BPAs are identical ⇐ distance is 0)	Separability (two BPAs are identical ⇒ distance is 0)	BOEs of two masses don't intersect ⇔ distance is 1	BOEs of two masses intersect ⇔ distance is less than 1	Characteristics
Conflict belief [21]	k	×	×	×	√	√	Reflects the non-mutual inclusion relationship of two BPAs
Minkowski's distance(t=1) [31]	$DisP_1$	√	×	√	√	√	Based on the transformed Pignistic probability and has information loss
Minkowski's distance(t>1) [31]	$DisP_t$	√	×	√	×	×	
Tessem's distance [29]	$difBetP$	√	×	√	√	√	
Tanimoto's dissimilarity [30]	$1 - simTa$	√	×	√	√	√	
Jousselme's distance [34]	d_{BPA}	√	√	√	×	×	Embodies BPAs and similarity between sets
Cosine-based dissimilarity [32]	$1 - \cos$	√	√	√	×	×	Applicable in Bayesian belief structure
Liu's distance [13]	d^L	√	×	√	×	√	Employs both conflict belief and Jousselme's distance
Jiang's distance [37]	d^J	×	√	×	×	√	

similarity is just one of them because it is effective only when the cardinality of each focal element is 1. Third, some developed dissimilarity measures [13,37] regard the conflict belief and Jousselme's distance measure as two basic metrics and combine them jointly. Theoretically, these two-dimensional measures ought to be more valid because they take the advantages of both the conflict belief and evidential distance. The characteristic of conflict belief lies in that it has higher ability to discriminate the divergence between two pieces of evidence when the intersection of focal elements for two BPAs decreases. Jousselme's distance measure is a popular distance metric, and the rationality of the measurement will be enhanced when the similarity of the cores for two pieces of evidence increases. Since either of the two basic metrics is satisfactory for complex circumstances, the measures in [13] and [36] also have their own defects although Jousselme's distance measure is a strict measure.

In this paper, the definition of conflict belief between betting commitments is first given. Then, a new dissimilarity measure under the D-S evidence structure is proposed which embodies both the conflict belief of transferable Pignistic probability and evidential distance. The former factor takes into consideration the non-intersection between the focal elements of two pieces of evidence, while the latter one quantifies the divergence between them. They are combined with discounting factors termed as dissimilarity/similarity coefficient derived from the betting commitments of two pieces of evidence. It is defined on the power set rather than the base set, and it utilizes sufficient information in the original BPAs. Conflict belief between betting commitments is used here to replace conflict belief between BPAs to overcome irrational results generated in specific situations which is illustrated in the paper. The proposed measure is especially applicable to muti-source data fusion under uncertain circumstance when the weights and reliabilities of evidence are unknown in advance. It satisfies the basic properties for a distance metric, i.e., reflexivity, symmetry, nondegeneracy, transitivity, whereas a large amount of existent dissimilarity measures do not meet. The advantage of the new measure lies in that it not only characterizes the contradiction between the mass of incompatible focal elements, but also reflects the discrepancy between the BPAs of compatible focal elements. The properties of the new dissimilarity measure are proved theoretically. It is finally combined with uncertainty measure of belief structure for the application in muti-source data fusion problem. The weight and reliability of evidence are automatically generated by the dissimilarity metric and uncertainty measure [12,45] of belief structure.

The rest of the paper is organized as follows: Section 2 briefly introduces some basic concepts in D-S evidence theory. In Section 3, some typical dissimilarity/distance measures are presented, and the flaws of them are discussed with some examples. Section 4 proposes a new dissimilarity measure considering both the conflict belief between betting commitments and evidential distance. The properties of the new dissimilarity measure are shown with theoretical proofs. In Section 5, comparative analysis of the proposed dissimilarity measure against some typical measures is conducted. Section 6 provides a multi-source data fusion method based on the new dissimilarity metric and the application in large-scale group decision making problem. This paper is concluded in Section 7.

2. Preliminary preparation and theoretical basis

2.1. D-S evidence theory

D-S evidence theory is well suited for dealing with uncertain and incomplete information. It was introduced by Dempster [21], and then refined by Shafer [22]. It belongs to the information fusion technique which has been widely applied in many fields.

Definition 1. (*The frame of discernment*) Let θ_n $(n=1,2,\cdots,N)$ be N distinct basic hypotheses (propositions). All the N hypotheses constitute the *frame of discernment* which is denoted as:

$$\Omega=\{\theta_1,\theta_2,\cdots,\theta_N\} \tag{1}$$

The N hypotheses in Ω are mutually exclusive and collectively exhaustive. The power set of Ω, represented by 2^Ω, is all the subsets of Ω indicated as:

$$2^\Omega=\{\varnothing,\{\theta_1\},\cdots,\{\theta_N\},\{\theta_1,\theta_2\},\cdots,\{\theta_1,\theta_2,\cdots,\theta_n\},\cdots,\Omega\} \tag{2}$$

Definition 2. (*Basic probability assignment*) Suppose there're L pieces of evidence $\boldsymbol{e}=\{e_1,e_2,\cdots,e_L\}$. *Basic probability assignment* (BPA) is the measurement of the extent to which the ith evidence e_i $(i=1,2,\dots,L)$ supports the subset A of Ω, formally signified by $m_i(A)(A\subseteq 2^\Omega)$. It is called a *mass function* denoted by m_i that is a mapping from 2^Ω to $[0,1]$.

$$m_i:\ 2^\Omega\rightarrow[0,1] \tag{3}$$

$m_i(A)$ is also coined as the degree of support for A from e_i or basic probability mass. A is called a *focal element* if it is given a positive mass such that $m_i(A)>0$, and the union of focal elements for e_i is termed as the *core* of Ω, represented by $\mathscr{C}_i=\bigcup\limits_{m_i(A)>0,A\subseteq 2^\Omega}A$. $m_i(A)$ quantifies the probability mass exactly distributed to A but not to any subset of A. Let $|A|$ be the cardinality of A. If $|A|\geq 2$, we call $m_i(A)$ the mass of ignorance or incompleteness. Specifically, when $2\leq|A|\leq N-1$, $m_i(A)$ is the measurement of local ignorance on e_i; the probability assigned to Ω on e_i dented by $m_i(\Omega)$ is called global ignorance.

Definition 3. [34] (*Body of evidence*). Let A be the focal element of 2^Ω such that $m_i(A)>0(A\subseteq 2^\Omega)$. The set of all the focal elements is called a *body of evidence* (BOE) as follows:

$$\mathfrak{B}_i=(\mathfrak{B},m_i)=\left\{A|m_i(A)\rangle 0,A\subseteq 2^\Omega\right\} \tag{4}$$

It is obvious that $\bigcup_{A\subseteq\mathfrak{B}_i} A = \mathscr{C}_i$. If $\forall A \subseteq \mathfrak{B}_i, |A| = 1$, then the mass function on e_i degenerates to the *Bayesian belief structure*. It is clear that a Bayesian belief structure doesn't contain ignorance although it may include uncertainty if $|\mathfrak{B}_i| \geq 2$.

Definition 4. [46] (*Categorical mass function*). Given a BPA m_i defined on 2^Ω, if it owns only a sole focal element such that $|\mathfrak{B}_i| = 1$, it is called a *categorical mass function*. Otherwise, it is called a *non-categorical mass function* ($|\mathfrak{B}_i| \geq 2$).

The definition of categorical mass function shows that it doesn't involve uncertainty although local or global ignorance may be embodied. A special case of categorical mass function is that the unique focal element is a singleton such that $|A| = 1 (A = \mathfrak{B}_i, |\mathfrak{B}_i| = 1)$, which means the information contains no uncertainty or incompleteness. On the contrary, a categorical mass function where the mass is attributed to the whole frame of discernment includes the largest global ignorance ($|A| = N(A = \mathfrak{B}_i, |\mathfrak{B}_i| = 1)$). In addition to the above two cases, a categorical mass function contains local ignorance when $2 \leq |A| \leq N-1 (A = \mathfrak{B}_i, |\mathfrak{B}_i| = 1)$.

Definition 5. [21] **(***Dempster's combination rule***)** Let m_1 and m_2 be two BPAs defined on 2^Ω. The *Dempster's combination rule* is an orthogonal sum formulated as:

$$m_{12}(A) = \begin{cases} 0 & A = \varnothing \\ \dfrac{1}{1-k_{12}} \sum\limits_{B\cap C=A} m_1(B)m_2(C) & A \neq \varnothing \end{cases} \tag{5}$$

where

$$k_{12} = \sum_{B\cap C=\varnothing} m_1(B)m_2(C) \tag{6}$$

k_{12} is called the *normalization factor* or *conflict belief* between m_1 and m_2. The *Dempster's* rule which can be used to combine different sources of evidence is the kernel of the D-S evidence theory. It assumes that information sources are independent, and satisfies commutativity and associativity of multiplication. So the combination results remain the same regardless of the order in which the multiple pieces of evidence are fused. Conflict belief reflects a non-mutual inclusion relationship between focal elements. Only two basic probabilities assigned to an empty set can be multiplied and accumulated to compute the conflict value, i.e. the belief assigned to empty intersections in the evidence fusion results.

There are two main reasons for the conflict of evidence. One is the incompleteness of the base set, or interpreted as the 'open-world assumption' from the perspective of Smets [47]. In this paper, the 'closed-world assumption' is employed because no potential object will be identified in our multi-source data fusion problem. The other one is the reliability of information source, which is especially applicable in the case such as target identification. By natural factors (e.g. atrocious weather) or human interference, a sensor's feedback may be inconsistent with another one.

In Dempster's rule, the mass to the non-intersections is evenly attributed to all focal elements to guarantee that the total mass sums to one. Counterintuitive results may be generated from aggregating high contradictory evidences as pointed out by Zadeh [48]. Nevertheless, the conflict belief is not effective to characterize the dissimilarity between BPAs, especially when two BPAs are identical [44] and are not categorical mass function. For example, $m_1(\theta_1) = 0.5$, $m_1(\theta_2) = 0.5$, $m_2(\theta_1) = 0.5$, $m_2(\theta_2) = 0.5$, then $k_{12} = 0.5$ which is irrational to assume that the dissimilarity between m_1 and m_2 is 0.5 because they're absolutely the same.

2.2. Typical dissimilarity/distance measures in evidence theory

In this subsection, the current dissimilarity/distance measures are categorized.

Definition 6. (*Distance measure function*). Suppose X and Y are two vectors of R^n, $d(\cdot): X \times Y \to [0,1]$ is called a distance metric if it satisfies the following conditions:

(d1) Reflexivity: $d(X,X) = 0$;
(d2) Symmetry: $d(X,Y) = d(Y,X)$;
(d3) Nondegeneracy: $d(X,Y) = 0 \Leftrightarrow X = Y$;
(d4) Nonnegativity: $d(X,Y) \geq 0$;
(d5) Transitivity: $Y = Z \Rightarrow d(X,Y) = d(X,Z)$;
(d6) Triangle inequality: $d(X,Y) \leq d(X,Z) + d(Z,Y)$.

Basically, a full distance metric between BPAs should satisfy all the six properties in Def. 6. Nondegeneracy (d3) is also called 'definiteness' in [33], and the inference that $d(X,Y) = 0 \Rightarrow X = Y$ is called 'separability'. If $d(X,Y) \leq 1$, then it is normed. The transitivity property (d5) is obvious, but the converse of it $d(X,Y) = d(X,Z) \Rightarrow Y = Z$ is not necessarily true. For instance, given the base set $\Omega = \{\theta_1, \theta_2, \theta_3\}$, three BPAs are $m_1(\{\theta_1, \theta_2\}) = 1$, $m_2(\{\theta_1, \theta_3\}) = 1$, $m_3(\{\theta_2, \theta_3\}) = 1$. Intuitively, $d(m_1, m_2)$ is equal to $d(m_1, m_3)$ because the number of elements contained in the intersection of m_1 and m_2 equals to that of m_1 and m_3 even though $m_2 \neq m_3$. The triangle inequality (d6) ensures that the direct path between two belief functions is always shorter than the path when a third belief function is involved. Jousselme [33] classified the dissimilarity metric into other five categories if only some of the properties in Def. 6 are satisfied, i.e., semi-metric, quasi-metric, pseudo-metric, semi-pseudo-metric and pre-metric. For instance, if a dissimilarity satisfies (d1) to (d5), it is a semi-metric; whereas a pre-metric only satisfies (d1) and (d4).

2.2.1. Probability-based distance

Definition 7. [49,50] (*Pignistic probability function*) Given the frame of discernment $\Omega = \{\theta_1, \theta_2, \cdots, \theta_N\}$, where θ_n is a singleton such that $|\theta_n| = 1$ and $\theta_n \cap \theta_m = \varnothing (1 \leq n,m \leq N)$. The BPA attributed to the subset A of Ω is denoted by m. The *Pignistic probability function* in the transferable belief model is defined as:

$$BetP_m(\theta_n) = \sum_{\theta_n \in A, A \subseteq \Omega} \frac{1}{|A|} \frac{m(A)}{1 - m(\varnothing)} \tag{7}$$

The Pignistic probability function can also be defined on 2^Ω as follows:

$$BetP_m(B) = \sum_{A \subseteq \Omega} m(A) \frac{|B \cap A|}{|A|} = \sum_{\theta_n \in B, B \subseteq \Omega} BetP_m(\theta_n) \tag{8}$$

The transformation from mass function to Pignistic probability function is called Pignistic transformation.

(1) Tessem's distance [29,44]

Suppose m_i and m_j are two independent BPAs on Ω, the associated Pignistic probability functions on them are $BetP_{m_i}$ and $BetP_{m_j}$ respectively. Then the distance between betting commitments of the two BPAs is defined as follows:

$$difBetP_{m_i}^{m_j} = \max_{A \subseteq \Omega} \left(\left| BetP_{m_i}(A) - BetP_{m_j}(A) \right| \right) \tag{9}$$

Eq.(9) reflects the maximum difference of the sum of BPAs on any union of singletons between two BPAs. Because the transformation changes the base set from the power set of atomic elements to singletons, the original information contained in the BPAs is altered, especially for the case where mass are mostly attributed to local or global ignorance.

(1) Tanimoto's similarity [30]

Bi [30] introduced the similarity measure based on the Pignistic probability function as follows:

$$Sim\left(m_i^{'}, m_j^{'}\right) = \frac{\sum_{n=1}^{N} m_i^{'}(\theta_n) \cdot m_j^{'}(\theta_n)}{\sum_{n=1}^{N} m_i^{'}(\theta_n)^2 + \sum_{n=1}^{N} m_j^{'}(\theta_n)^2 - \sum_{n=1}^{N} m_i^{'}(\theta_n) \cdot m_j^{'}(\theta_n)} \tag{10}$$

$$m_k^{'}(k = i, j) = \left(BetP_{m_k}(\theta_1), BetP_{m_k}(\theta_2), \cdots, BetP_{m_k}(\theta_N) \right) = \left(m_k^{'}(\theta_1), m_k^{'}(\theta_2), \cdots, m_k^{'}(\theta_N) \right) \tag{11}$$

Eq. (10) does not satisfy nondegeneracy (d3) because $1 - Sim(m_i^{'}, m_j^{'}) = 0$ cannot infer that the original two BPAs are identical. The reason for this deficiency lies in that the original information is lost when the mass function is transferred to Pignistic probability function.

(1) Minkowski's distance [31]

Given two masses m_i and m_j defined on the power set of Ω, the Minkowski's distance between them is defined as

$$DisP_t\left(m_i, m_j\right) = \left(\frac{1}{2} \sum_{\theta_n \in \Omega} \left| BetP_{m_i}(\theta_n) - BetP_{m_j}(\theta_n) \right|^t \right)^{\frac{1}{t}} \tag{12}$$

When $t = 1$, it characterizes the divergence by calculating the additive distance between the probabilities of the same singletons. If $m_i = m_j$, it can be easily inferred that $DisP_1(m_i, m_j) = 0$. But the converse of the statement is not true. In Example 1, although the BPAs of m_i and m_j are different, $DisP_t(m_i, m_j) = 0$ because the transformed Pignistic probability of the two evidences are identical.

2.2.2. Mass-based distance

(1) Cosine similarity [32]

Cosine similarity measure between m_i and m_j is defined as

$$Sim(m_i, m_j) = \cos(\theta) = \frac{m_i . m_j^T}{||m_i|| \cdot ||m_j||} \tag{13}$$

where $m_i (i = 1, 2)$ is a 2^{Ω} dimensional vector. The numerator represents the inner product of the two BPAs. $|| \cdot ||$ signifies the norm of vector such that $||m_i|| = \sqrt{\sum_{A \in 2^{\Omega}} (m_i(A))^2}$.

If the mass is a Bayesian belief structure, or in other words, the focal elements are confined to singletons, the generated Cosine similarity measure seems interpretable. However, when it is defined on the power set 2^{Ω}, the generated value will be unreasonable. So it is applicable only in the situation of Bayesian belief structure.

(1) Jousselme's distance [33,34]

Jousselme's distance employs two factors to generate a metric, i.e., the BPAs and the similarity between sets. It is defined as

$$d_{BPA_{ij}} = d_{BPA}(m_i, m_j) = \sqrt{\frac{1}{2}\left(\vec{m}_i - \vec{m}_j\right)^T \underbrace{D}\left(\vec{m}_i - \vec{m}_j\right)} \tag{14}$$

where $\vec{m}$ is a $2^{\Omega} \times 1$ dimensional matrix with $\{m(A) | A \subseteq 2^{\Omega}\}$ as its elements, and $(\vec{m}_i - \vec{m}_j)$ denotes the subtraction of the two vectors. $\underbrace{D} = (d_{ij})_{2^{\Omega} \times 2^{\Omega}}$ stands for a $2^{\Omega} \times 2^{\Omega}$ dimensional matrix where $d_{ij} = \frac{|A \cap B|}{|A \cup B|}$, and $\frac{|\varnothing \cap \varnothing|}{|\varnothing \cup \varnothing|} = 0$ (A and B are any subsets of 2^{Ω}). Another form of $d_{BPA}(m_i, m_j)$ is shown as

$$d_{BPA_{ij}} = d_{BPA}(m_i, m_j) = \sqrt{\frac{1}{2}\left(\vec{m}_i^2 + \vec{m}_j^2 - 2\vec{m}_i, \vec{m}_j\right)} \tag{15}$$

where $\vec{m}_i, \vec{m}_j$ is the product of each element in $\vec{m}_i$ with that in $\vec{m}_j$. And $\vec{m}_i^2$ represents the square norm of $\vec{m}_i$ such that $\vec{m}_i^2 = \vec{m}_i, \vec{m}_i$.

$$\vec{m}_i, \vec{m}_j = \sum_{A \subseteq 2^{\Omega}} \sum_{B \subseteq 2^{\Omega}} m_i(A) \cdot m_j(B) \cdot \frac{|A \cap B|}{|A \cup B|} \tag{16}$$

(1) Correlation coefficient based conflict [35]

In [35], Jiang defined a correlation coefficient between two pieces of evidence denoted by $r_{BPA}(m_i, m_j)$ as follows:

$$r_{BPA}(m_i, m_j) = \frac{c(m_i, m_j)}{\sqrt{c(m_i, m_i) \cdot c(m_j, m_j)}} \tag{17}$$

where $c(m_i, m_j)$ is calculated by

$$c(m_i, m_j) = \sum_{A \subseteq 2^{\Omega}} \sum_{B \subseteq 2^{\Omega}} m_i(A) \cdot m_j(B) \cdot \frac{|A \cap B|}{|A \cup B|} \tag{18}$$

Then, a conflict coefficient between m_i and m_j is measured by $k_r(m_i, m_j) = 1 - r_{BPA}(m_i, m_j)$. Here, we call it the C-C based dissimilarity measure.

(1) PSD Pignistic probability based distance [36]

In [36], the *power-set-distribution* (PSD) *Pignistic probability function PBetPm* : $2^{\Omega} \rightarrow [0, 1]$ of m is defined as

$$PBetPm(B) = \sum_{A \subseteq \Omega} m(A) \frac{2^{|B \cap A|} - 1}{2^{|A|} - 1} \tag{19}$$

where B denotes a subset of Ω, $A \neq \varnothing$.

Suppose $\mathrm{PBetPm_i}$ and $\mathrm{PBetPm_j}$ are the corresponding PSD Pignistic probability functions of m_i and m_j. Then the *distance between PSD betting commitments* is defined as

$$difPBetP_{\mathrm{m_i}}^{\mathrm{m_j}} = \max_{\mathrm{A} \subseteq \Omega}\left(\left|\mathrm{PBetPm_i(A)} - \mathrm{PBetPm_j(A)}\right|\right) \tag{20}$$

2.2.3. Distance considering two dimensions

(1) Liu's distance measure [13]

Suppose the conflict belief and distance measure are quantified by Eqs.(6) and (14) respectively. Then, the conflict coefficient between m_i and m_j is evaluated by the product of Jousselme's distance measure and conflict belief as follows:

$$d_{ij}^{L} = \sqrt{k_{ij} \cdot d_{BPA_{ij}}} \tag{21}$$

The drawback of this measure lies in that $d_{ij}^{L} = 0$ if the value of conflict belief equals to zero, especially under the non-Bayesian belief structure. In other words, if any pair of focal elements from two masses has intersection, the scalar of d_{ij}^{L} will be 0 although the BPAs of m_i and m_j may be quite different.

(1) **Jiang's distance measure** [37]

The average of Jousselme's distance measure and conflict belief is conducted to produce the distance between two masses as follows:

$$d_{ij}^{J} = \frac{1}{2}\left(k_{ij} + d_{BPA_{ij}}\right) \tag{22}$$

The deficiency of this distance measure is that the value of d_{ij}^{J} is large if and only if both k_{ij} and $d_{BPA_{ij}}$ are large. Let the base set be $\Theta = \{\theta_1, \theta_2, \cdots, \theta_{100}\}$, with the mass on m_i evenly attributed to $\theta_1, \theta_2, \cdots, \theta_{50}$ such that $m_i(\theta_n) = 0.02 (n = 1, 2, \cdots, 50)$, while m_j is assigned on $\theta_{51}, \theta_{52}, \cdots, \theta_{100}$ with equal probability such that $m_j(\theta_n) = 0.02 (n = 51, 52, \cdots, 100)$. The scalar of distance should be 1 because the BOEs of the two masses have no intersections. Unfortunately, the distance of $d_{BPA_{ij}}$ in this case is 0.141, which leads to the value of d_{ij}^{J} be 0.571 although $k_{ij} = 1$. Apparently, the value of d_{ij}^{J} is lowered by $d_{BPA_{ij}}$ which is affected by the dispersity of the mass attributed on the power set.

3. Illustrative examples

In this section, several examples are presented to illustrate the characteristics of the above-mentioned dissimilarity/distance measures. The advantages and disadvantages of these measures are then discussed.

3.1. Illustrative examples

Example 1.

Given the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$, the BPAs of two pieces of evidence are shown in Table 2.

The distance measures by different methods are shown in Table 3.

$k_{12} = 0$ because there is no pair of focal elements that doesn't intersect respect to m_1 and m_2. So the conflict belief is not appropriate to be taken as the dissimilarity measure between two evidences, especially in the case that the mass function of one piece of evidence is a non-Bayesian belief structure. Since the Tanimoto based similarity measure depends on the Pignistic probability function, if the transformed Pignistic probabilities of two pieces of evidence are identical, the distance calculated by Eq. (10) will be zero even though the two BPAs may be quite different. In Example 1, m_2 is a categorical belief function that the mass value is awarded to the whole base set; comparatively, the BPA is equally attributed to the three singletons in the frame for m_1. Clearly, m_2 contains the maximum ambiguity which is named as global ignorance, whereas m_1 only has uncertainty that no ignorance exists. The distance between betting commitments by Eq. (9) has the same drawback which leads to $difBetP_{m_1}^{m_2} = 0$. The Cosine similarity measure equals to 0 because there is no identical focal element between the two masses.

Example 2.

Table 2
The BPAs of two pieces of evidence

	θ_1	θ_2	θ_3	$\{\theta_1, \theta_2, \theta_3\}$
m_1	1/3	1/3	1/3	0
m_2	0	0	0	1

Table 3
The distance measures by different methods

	$1 - r_{BPA}(m_1, m_2)$	k_{12}	$1 - Sim(m_1', m_2')$	$1 - cos(m_1, m_2)$	$difBetP_{m_1}^{m_2}$	$d_{BPA_{12}}$	d_{12}^{L}	d_{12}^{J}
d_{12}	0.423	0	0	1	0	0.577	0	0.288

Table 4
The BPAs of two pieces of evidence

	$\{\theta_1, \theta_2\}$	$\{\theta_1, \theta_3\}$	$\{\theta_2, \theta_3\}$	$\{\theta_1, \theta_2, \theta_3\}$
m_1	1/3	1/3	1/3	0
m_2	0	0	0	1

Table 5
The distance measures by different methods

	$1 - r_{BPA}(m_1, m_2)$	k_{12}	$1 - Sim(m_1', m_2')$	$1 - cos(m_1, m_2)$	$difBetP_{m_1}^{m_2}$	$d_{BPA_{12}}$	d_{12}^L	d_{12}^J
d_{12}	0.1056	0	0	1	0	0.333	0	0167

The BPAs of two pieces of evidence are shown in Table 4.

The results of some distance measures are shown in Table 5.

Example 2 shows the same problem with Example 1 in that the Pignistic probability allocated to each singleton is identical for m_1 and m_2. Hence, either the distance measure based on Tanimoto metric or betting commitments is 0. The Cosine similarity measure still equals to 0 in Example 2 because there is no same focal element between the two BPAs. The difference between Example 1 and 2 lies in that the BPAs on m_1 contains local ignorance in Example 2, while the mass of m_1 is evenly attributed to the three singletons in Example 1, which means it does not embody either local or global ignorance. Intuitively, the distance between m_1 and m_2 decreases from Example 1 to 2 because either $\max\limits_{A\subseteq\mathfrak{B}_1, B\subseteq\mathfrak{B}_2} \frac{|A\cap B|}{|A\cup B|}$ or $\sum\limits_{A\subseteq\mathfrak{B}_1, B\subseteq\mathfrak{B}_2} \frac{|A\cap B|}{|A\cup B|}$ increases.

Example 3.

Given the BPAs of a hundred pieces of evidence as shown in Table 6. Obviously, $\bigcup\limits_{A\subseteq\mathfrak{B}_1} A \cap \bigcup\limits_{B\subseteq\mathfrak{B}_j} B = \varnothing (j = 2, 3, \cdots, 100)$ where A and B refer to the focal elements for m_1 and m_j ($j = 2,3,\cdots,100$). All the pieces of evidence contain uncertainty except m_2, and the uncertainty degree of mass increases from m_3 to m_{100}.

The distances calculated by some methods are shown in Table 7.

Intuitively, the distance between m_1 and any other evidence should be one because the intersection of focal elements for m_1 and m_j ($j = 2, 3, \cdots, 100$) is $\varnothing$. It can be seen that with the mass becoming more scattered from m_2 to m_{100}, the scalar of Jousselme's distance decreases gradually. Especially, $d_{BPA}(m_1, m_2) < 1$ although m_2 contains neither uncertainty nor ignorance. Thus, from this point of view, Jousselme's distance metric not only partially quantifies the distance between two pieces of evidence, but also reflects the uncertainty of individual evidence to some extent. $d_{BPA}(m_1, m_2)$ equals to 1 if and only if m_1 and m_2 definitely point to two subsets of 2^Ω with no intersection, i.e., $d_{BPA}(m_1, m_2) = 1$ iff $m_1(A) = 1, m_2(B) = 1, A \cap B = \varnothing, A, B \in 2^\Omega$. From another point of view, $d_{BPA}(m_1, m_2) < 1$ if the mass of either m_1 or m_2 is attributed to more than one subset of 2^Ω. So $d_{BPA}(m_1, m_2)$ does not satisfy Lemma 1 (see Section 4.1). The value of k_{ij} in this case still equals to 1. So if the focal elements of two pieces of evidence have no intersection, the conflict belief is definitely appropriate to measure the dissimilarity between them. With the increase of the number of elements included in the intersection between the cores of two pieces of evidence, the conflict belief becomes more invalid as a dissimilarity metric.

Example 4.

Table 6
The BPAs of a hundred pieces of evidence

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	…	θ_{101}	θ_{102}
m_1	0.2	0.8	0	0	0	0			
m_2	0	0	1	0	0	0			
m_3	0	0	1/2	1/2	0	0			
m_4	0	0	1/3	1/3	1/3	0			
m_5	0	0	1/4	1/4	1/4	1/4			
…									
m_{100}	0	0	1/100	1/100	1/100	1/100	…	1/100	1/100

Table 7
The distance measures by different methods

	k_{1j}	$1 - Sim(m_1', m_j')$	$1 - cos(m_1, m_2)$	$difBetP_{m_1}^{m_2}$	$d_{BPA_{12}}$
d_{12}	1	1	1	1	0.917
d_{13}	1	1	1	0.8	0.768
d_{14}	1	1	1	0.8	0.712
d_{15}	1	1	1	0.8	0.682
……					
$d_{1\ 100}$	1	1	1	0.8	0.587

Table 8
The BPAs of two pieces of evidence

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
m_1	0.2	0.2	0.2	0.2	0.2	0	0	0	0	0
m_2	0	0	0	0	0	0.2	0.2	0.2	0.2	0.2

Let $\Omega = \{\theta_1, \theta_2, \cdots, \theta_{10}\}$ be the frame of discernment, and there are eleven pairs of BPAs, and $Sim_{\mathscr{C}_{12}} = \frac{\left|\left(\underset{A\subseteq\mathfrak{B}_1}{\cup} A\right)\cap\left(\underset{B\subseteq\mathfrak{B}_2}{\cup} B\right)\right|}{\left|\left(\underset{A\subseteq\mathfrak{B}_1}{\cup} A\right)\cup\left(\underset{B\subseteq\mathfrak{B}_2}{\cup} B\right)\right|}$. The first pair is shown as follows:

For the 2$^{\text{nd}}$ to the 6$^{\text{th}}$ pair of BPAs, m_2 is assumed to be fixed, while m_1 is changed from the probability assignment evenly attributed to $\theta_1, \theta_2, \cdots, \theta_5$ to $\theta_1, \theta_2, \cdots, \theta_{10}$. So, in the 6$^{\text{th}}$ pair of BPAs, $m_1(\theta_n) = 0.1$ $(n = 1, 2, \ldots, 10)$. For the 7$^{\text{th}}$ to the 11$^{\text{th}}$ pair of BPAs, the mass of m_1 equals to the assignment in the 6$^{\text{th}}$ pair and fixed, while m_2 is changed from the probability assignment averagely allocated to θ_5, $\theta_6, \cdots, \theta_{10}$ to $\theta_1, \theta_2, \cdots, \theta_{10}$. Then, in the 11$^{\text{th}}$ pair, $m_2(\theta_n) = 0.1$ $(n = 1, 2, \ldots, 10)$. Intuitively, the dissimilarity between the first pair of BPAs should be one because they are definitely contradictory. Comparatively, the masses of the 11$^{\text{th}}$ pair of evidence are identical which will lead to the dissimilarity being zero. Fig. 1 shows the conflict belief and Jousselme's distance between m_1 and m_2 in the eleven pairs of BPAs. Here, the horizontal and vertical axes refer to the serial number of pair and conflict/distance measure. Obviously, the conflict belief k_{12} is valid for the first pair of evidence, whereas Jousselme's distance $d_{BPA_{12}}$ is effective for the last pair of evidence. Fig. 1 also shows the value of $1/2\sum_{n=1}^{10}|m_1(\theta_n) - m_2(\theta_n)|$ and $1 - Sim_{\mathscr{C}_{12}}$ for all the eleven pairs of evidence. Obviously, with the value of $1/2\sum_{n=1}^{10}|m_1(\theta_n) - m_2(\theta_n)|$ or $1 - Sim_{\mathscr{C}_{12}}$ decreasing, Jousselme's distance becomes more valid, especially when the BOEs of the two pieces of evidence are identical, i.e. the 11$^{\text{th}}$ pair. Conversely, with the value of $1/2\sum_{n=1}^{10}|m_1(\theta_n) - m_2(\theta_n)|$ or $1 - Sim_{\mathscr{C}_{12}}$ increasing, the discrepancy between the focal elements of the two pieces of evidence becomes larger, which leads to the conflict belief k_{12} being more effective to represent the dissimilarity measure. The 1$^{\text{st}}$ pair of evidence is just the case that the BOEs of the two pieces of evidence have no intersection.

3.2. Discussion

With regard to 'nondegeneracy' ($d(X, Y) = 0 \Leftrightarrow X = Y$), conflict belief k_{ij} equals to zero if any pair of focal elements from two pieces of evidence have intersection, so $k_{ij} = 0 \not=> m_i = m_j$. Conversely, when the BPAs of two pieces of evidence are identical, conflict belief is not necessarily equal to 0 if there are two focal elements that are mutually exclusive. For example, given that $m_i(\{\theta_1\}) = m_j(\{\theta_1\}) = 0.2$, $m_i(\{\theta_2, \theta_3\}) = m_j(\{\theta_2, \theta_3\}) = 0.8$, then $k_{ij} = 0.32$. Particularly, if two masses are Bayesian belief functions, $k_{ij} > 0$ occurs provided that one piece of evidence contains at least two focal elements. So one gets $m_i = m_j \not=> k_{ij} = 0$. Since k_{ij} is a component in Liu's and Jiang's distances, nor do they satisfy 'nondegeneracy'. Specifically, Jiang's distance doesn't satisfy $m_i = m_j \Rightarrow d_{ij} = 0$ because k_{ij} may be greater than zero even if $m_i = m_j$, while Liu's distance doesn't meet $d_{ij} = 0 \Rightarrow m_i = m_j$ since k_{ij} may equal to zero even though $m_i \neq m_j$. Moreover, Jiang's distance equals to zero if and only if the two pieces of evidence are identical and any pair of two focal elements does intersect. Hence, conflict belief and Jousselme's distance cannot be simply combined by either multiplication or addition. Otherwise, unreasonable results will be generated. Neither Tessem's distance nor Tanimoto's similarity satisfy

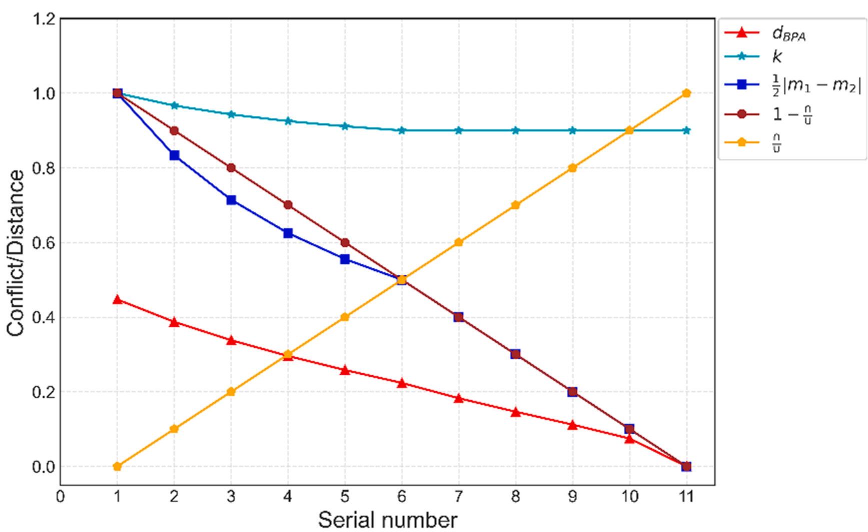

Fig. 1. Conflict or distance between two BPAs

'nondegeneracy' because they are both defined based on the transferable Pignisitic probability. So we have $difBetP_{m_i}^{m_j} = 0$ or $1 - Sim(m_i^{'}, m_j^{'}) = 0 \not=> m_i = m_j$ although $m_i = m_j$ can infer $difBetP_{m_i}^{m_j} = 0$ and $1 - Sim(m_i^{'}, m_j^{'}) = 0$. Examples 1 and 2 are just the cases used to explain it.

Cosine similarity satisfies the property that $1 - \cos(\theta) = 0 \Leftrightarrow m_i = m_j$. Nevertheless, it doesn't satisfy Lemma 1 (see Section 4.1) ($\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) = \varnothing \Leftrightarrow d_{ij} = 1$) in Section 4 because $1 - \cos(\theta) = 1$ occurs when there is no identical focal element between m_i and m_j even though $\bigcup_{A\subseteq\mathscr{F}_i} A$ and $\bigcup_{B\subseteq\mathscr{F}_j} B$ may have intersection. With regard to Jousselme's distance, although we have $d_{BPA}(m_i, m_j) = 1 \Rightarrow \underset{A\subseteq\mathfrak{B}_i}{\cup} A \cap \underset{B\subseteq\mathfrak{B}_j}{\cup} B = \varnothing$, the inverse proposition is invalid which is illustrated in Example 4. Nor do Liu's and Jiang's distances satisfy Lemma 1 (see Section 4.1) for the reason that Jousselme's distance is a component simply multiplied or added in their measures. They equal to one if and only if $k_{ij} = 1$ and $d_{BPA}(m_i, m_j) = 1$. It happens when the two masses each contain a sole focal element whose intersection is $\varnothing$. Otherwise, d_{ij}^L or d_{ij}^J is less than one. So we cannot infer $d_{ij}^L = 1$ or $d_{ij}^J = 1$ only from $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) = \varnothing$. Minkowski's distance ($t$>1) is less than one if one piece of evidence contains more than one focal element such that $\exists k \in \{i,j\}, |\mathfrak{B}_k| \geq 2$.

Jousselme's distance doesn't satisfy Lemma 3 (see Section 4.1) because $d_{BPA}(m_i, m_j) < 1$ cannot infer $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing$ although $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing \Rightarrow d_{BPA}(m_i, m_j) < 1$. Nor does Cosine similarity satisfy Lemma 3 (see Section 4.1) since $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing$ may infer $1 - \cos(\theta) = 1$ in some cases although $1 - \cos(\theta) < 1 \Rightarrow \left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing$. Example 1 is just the case of it. Comparatively, the following proposed dissimilarity measure satisfies all the listed properties.

4. New dissimilarity measure between two BPAs

The former section shows that terminologies such as distance, conflict, dissimilarity/similarity have all been used to represent the divergence or proximity between two bodies of evidence. In this paper, we employ the term 'dissimilarity' to denote the concept of distance or discrepancy between two BPAs. The key point for measuring the dissimilarity between two BPAs rests with the relevance of their focal elements. Generally speaking, if the intersection between the cores of two pieces of evidence consists of more elements, the dissimilarity will be smaller; and vice versa. This section is dedicated to discussing our proposed new dissimilarity measure with theoretical proofs.

4.1. Basic properties for a dissimilarity measure

Lemma 1. (Maximum dissimilarity between two BPAs) Let $\mathfrak{B}_i = \{A|m_i(A)\rangle 0)\}$ and $\mathfrak{B}_j = \{B|m_j(B)\rangle 0)\}$ be the set of focal elements in the ith and jth evidence with m_i and m_j as their corresponding BPAs respectively. If the intersection of focal elements for m_i and m_j is $\varnothing$ such that $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) = \varnothing$, then the dissimilarity between m_i and m_j ought to be 1, and vice versa.

Lemma 1 can be interpreted as a necessary and sufficient condition such that $\underset{A\subseteq\mathfrak{B}_i}{\cup} A \cap \underset{B\subseteq\mathfrak{B}_j}{\cup} B = \varnothing \Leftrightarrow d_{ij} = 1$. It indicates that the maximum contradiction occurs when two mass functions have no intersection between their focal elements, which will lead to the scalar of a rational dissimilarity measure between two BPAs being 1. Although this is a simple and basic property that a dissimilarity measure should satisfy, some metrics don't meet the condition.

Lemma 2. (Minimum dissimilarity between two BPAs) If BPAs m_i and m_j from two independent pieces of evidence e_i and e_j are identical such that $m_i = m_j$, then the dissimilarity between them should be zero. The converse of the statement should also be satisfied. That is to say, $m_i = m_j \Leftrightarrow Diss(m_i, m_j) = 0$.

Lemma 2 may have three situations. (1) The belief of a same singleton for m_i and m_j is 1; (2) The mass functions of m_i and m_j definitely point to a same subset A ($|A| \geq 2$) of Ω; (3) The BPAs of e_i and e_j are identical, and the BOEs of m_i or m_j contains at least two subsets of 2^{Ω}. m_i and m_j in situations (1) and (2) are both the case of categorical belief functions. The only difference lies in that the two pieces of evidence in situation (1) contains no uncertainty or ignorance, while situation (2) contains ignorance. Situation (3) at least involves uncertainty because the number of focal elements for any of the two pieces of evidence is more than one. In general, if the BPAs of two pieces of evidence are identical, the dissimilarity between them ought to be zero no matter how uncertain or ignorant the two pieces of evidence are. Although Lemma 2 is a comprehensible requirement to be met for a dissimilarity measure, some metrics don't satisfy, e.g., the conflict belief k_{ij}.

Lemma 3. Given two pieces of evidence with m_i and m_j being their mass functions respectively, the dissimilarity between m_i and m_j should be less than one if and only if the BOEs of the two masses have intersections. In terms of formula, $d_{ij} < 1$ if $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing$, and vise versa.

Table 9
The BPAs of four pieces of evidence

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
m_1	0.25	0.25	0.25	0.25	0	0	0	0
m_2	0	0	0	0.01	0.24	0.25	0.25	0.25
m_3	0.24	0.24	0.24	0.24	0.01	0.01	0.01	0.01
m_4	0.01	0.01	0.01	0.01	0.24	0.24	0.24	0.24

Lemma 3 is a fundamental property that a dissimilarity measure between BPAs under the Dempster-Shafer's evidence structure should satisfy. It can be easily interpreted because if two subsets from m_i and m_j with positive BPAs have at least one identical element, the two pieces of evidence may have similarity to some extent. In other words, the more belief assigned to compatible or same subsets from two sources of evidence, the less dissimilarity between them. Nevertheless, not all dissimilarity metrics meet the requirement of Lemma 3, e.g., the Cosine-based dissimilarity measure.

4.2. Dissimilarity coefficient between betting commitments

From the above analysis, it can be seen that with the increase of elements contained in the intersection of cores for two pieces of evidence, conflict belief is less valid to quantify dissimilarity between them, while Jousselme's distance seems more effective to measure divergence between pieces of evidences. Especially in the situation that two pieces of evidence are identical, conflict belief can no longer be a metric for dissimilarity because it just accumulates the incompatible belief of two BPAs. Conversely, when intersection between the cores of two pieces of evidence contains fewer elements, conflict belief will be more effective to characterize dissimilarity, whereas Jousselme's distance is less valid comparatively. Example 3 is just the case of the counter-intuitive results generated by Jousselme's distance when $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) = \varnothing$. So neither conflict belief nor Jousselme's distance can independently measure the dissimilarity between two BPAs in all possible scenarios. Nor does any other existing dissimilarity/distance measure defined on 2^Ω can tackle with all situations. Inspired by the previously mentioned two-dimensional dissimilarity measure for belief structure, we propose a two-tuple dissimilarity measure which takes into consideration both the conflict belief between betting commitments and Jousselme's distance. Since Jousselme's distance measure is more valid with the increase of elements included in $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)$, a weighting factor which is positively correlated with it should be computed to discount it.

Definition 8. (*Conflict belief between betting commitments*) Let $\Omega = \{\theta_1, \theta_2, \cdots, \theta_N\}$ be the frame of discernment. m_i and m_j are two BPAs defined on 2^Ω. $BetP_{m_l}(\theta_n)$ $(n = 1, 2, \ldots, N)$ is the Pignistic probability transformed from evidence $m_l(l = i, j)$. Then, the *conflict belief* between betting commitments of m_i and m_j is computed as follows:

$$k'_{ij} = \sum_{\theta_n \cap \theta'_n = \varnothing} BetP_{m_i}(\theta_n) \cdot BetP_{m_j}(\theta'_n) \tag{23}$$

k'_{ij} is different from k_{ij} in Eq. (6) for the reason that it is defined on the transformed Pignistic probability rather than the original mass function. Theoretically, we have $k_{ij} < k'_{ij}$.

Definition 9. (*Similarity coefficient between the cores of two* pieces of evidence) Suppose the intersection and union of the cores for m_i and m_j are denoted by $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)$ and $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cup \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)$ respectively. Then, the *similarity coefficient* between the cores of m_i and m_j is generated as follows:

$$Sim_{\mathscr{C}_{ij}} = \frac{\left|\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)\right|}{\left|\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cup \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)\right|} \tag{24}$$

So $1 - Sim_{\mathscr{C}_{ij}}$ represents the *dissimilarity coefficient* between the cores of m_i and m_j represented by $Diss_{\mathscr{C}_{ij}}$. If we only use $Sim_{\mathscr{C}_{ij}}$ to discount $d_{BPA_{ij}}$ and $\alpha_{k_{ij}} = 1 - Sim_{\mathscr{C}_{ij}}$ to be the weight of conflict belief, irrational results may be generated in some situations.

Example 5.
Suppose the frame of discernment is $\Omega = \{\theta_1, \theta_2, \cdots, \theta_8\}$, four BPAs are shown in Table 9.

In this example, $k'_{12} = 0.9975$, $d_{BPA_{12}} = 0.4951$. If we use $\alpha_{d_{BPA_{12}}} = \frac{\left|\left(\underset{A\subseteq\mathfrak{B}_1}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_2}{\cup} B\right)\right|}{\left|\left(\underset{A\subseteq\mathfrak{B}_1}{\cup} A\right) \cup \left(\underset{B\subseteq\mathfrak{B}_2}{\cup} B\right)\right|} = \frac{1}{8}$ as the weight of $d_{BPA_{12}}$, then $\alpha_{k'_{12}} = 1 - \alpha_{d_{BPA_{12}}} = \frac{7}{8}$, and we will have $Diss(m_1, m_2) = 0.9347$ (See Def. 12). It is obvious that m_1 and m_3 are very similar, and m_2 and m_4 are also alike. $k'_{34} = 0.9808$, $d_{BPA_{34}} = 0.46$. But for the latter two pieces of evidence, $\alpha_{d_{BPA_{34}}} = \frac{|\left(\underset{A\subseteq \mathfrak{B}_3}{\cup} A\right) \cap \left(\underset{B\subseteq \mathfrak{B}_4}{\cup} B\right)|}{|\left(\underset{A\subseteq \mathfrak{B}_3}{\cup} A\right) \cup \left(\underset{B\subseteq \mathfrak{B}_4}{\cup} B\right)|} = 1$ although m_3 and m_4 have taken minor changes from m_1 and m_2 respectively. If $\alpha_{d_{BPA_{34}}}$ is used to discount $d_{BPA_{34}}$ and $\alpha_{k'_{34}} = 1 - \alpha_{d_{BPA_{34}}} = 0$ is the weight allocated to k'_{34}, then $Diss(m_3, m_4) = 0.46$. It is irrational because the dissimilarity changes a lot although the two pieces of evidence have changed slightly.

Definition 10. (*Dissimilarity coefficient between the betting commitment of two* pieces of evidence) Given two pieces of evidence m_i and m_j defined on 2^Ω, then the *dissimilarity coefficient* between the betting commitments of m_i and m_j is generated as follows:

$$Diss_{BetP_{m_{ij}}} = \frac{1}{2}\sum_{n=1}^{N}\left|BetP_{m_i}(\theta_n) - BetP_{m_j}(\theta_n)\right| \tag{25}$$

Correspondingly, $Sim_{BetP_{m_{ij}}} = 1 - Diss_{BetP_{m_{ij}}}$ is called the *similarity coefficient* between the betting commitments of m_i and m_j. Def. 10 is a special case of Minkowski's distance when $t = 1$.

As we know, the conflict belief k_{ij} reveals the cumulative value of the product between incompatible focal elements associated with m_i and m_j. So the larger the value of $Diss_{BetP_{m_{ij}}}$, the more contradiction between two pieces of evidence which will lead to k_{ij} being more important in the dissimilarity measure. Especially, if $Diss_{BetP_{m_{ij}}} = 1$, then $Diss_{\mathscr{C}_{ij}} = 1$, the two pieces of evidence are absolutely incompatible. As such, $k_{ij} = 1$ completely reflect the dissimilarity of two pieces of evidence in this situation. Comparatively, Jousselme's distance reflects the difference of BPA on consistent focal elements between pieces of evidence. So the smaller the value of $Diss_{BetP_{m_{ij}}}$, the more consistent between two pieces of evidence. Thus, $d_{BPA_{ij}}$ is to be given more importance in the dissimilarity measure. Particularly, when $Diss_{BetP_{m_{ij}}} = 0$, then $Diss_{\mathscr{C}_{ij}} = 0$, and $d_{BPA_{ij}} = 0$. Since the two pieces of evidence are completely compatible in this situation, $d_{BPA_{ij}}$ can be employed to determine their dissimilarity with the weight of 1.

The dissimilarity (or similarity) coefficient between betting commitments and the cores of pieces of evidence have the following properties:

Properties:

(1) $0 \le Sim_{BetP_{m_{ij}}} \le 1$, $0 \le Diss_{BetP_{m_{ij}}} \le 1$;
(2) $Diss_{BetP_{m_{ij}}} = 0$ iff $BetP_{m_i} = BetP_{m_j}$;
(3) $Diss_{BetP_{m_{ij}}} = 1$ iff $\mathscr{C}_i \cap \mathscr{C}_j = \emptyset$;
(4) $Sim_{BetP_{m_{ij}}} + Diss_{BetP_{m_{ij}}} = 1$;
(5) $Diss_{BetP_{m_{ij}}} = 1$ iff $Diss_{\mathscr{C}_{ij}} = 1$;
(6) $Diss_{BetP_{m_{ij}}} = 0 \Rightarrow Diss_{\mathscr{C}_{ij}} = 0$.

Property (2) implies that the dissimilarity coefficient attains the minimum value when the Pignistic probability functions of two pieces of evidence are identical. Nevertheless, it doesn't mean the two BPAs must be the same because the betting commitments of two BPAs may be identical even though the two BPAs are different. Property (6) indicates that if $BetP_{m_i} = BetP_{m_j}$, the cores of m_i and m_j must be the same, and then $Diss_{\mathscr{C}_{ij}} = 0$. But the converse of Property (6) is not true.

4.3. New two-tuple dissimilarity measure between pieces of evidence

Based on the discussion in Section 4.2, a two-tuple dissimilarity measure in the D-S evidence structure is proposed. It is a strict dissimilarity metric because it satisfies the properties presented in Def. 6 and Lemmas 1 to 3.

Definition 11. **(***Two-tuple dissimilarity measure between pieces of evidence***)** Given two pieces of evidence m_i and m_j defined on 2^Ω. A two-tuple dissimilarity measure is represented as

$$Diss(m_i, m_j) = \langle k'_{ij}, d_{BP_{ij}} \rangle. \tag{26}$$

where k'_{ij} denotes the conflict belief derived from the Pignistic probability functions of m_i and m_j, $d_{BPA_{ij}}$ represents Jousselme's distance measure between m_i and m_j.

Definition 12. (*A new weighted dissimilarity measure*) Let Ω be the frame of discernment, and the mass of two pieces of evidence is denoted by m_i and m_j. The dissimilarity measure between the two pieces of evidence $Diss(m_i, m_j)$ is defined as:

$$Diss(m_i, m_j) = \alpha_{k'_{ij}} \cdot k'_{ij} + \alpha_{d_{BPA_{ij}}} \cdot d_{BPA_{ij}} \tag{27}$$

where $\alpha_{k'_{ij}}$ and $\alpha_{d_{BPA_{ij}}}$ are the discounting coefficient or weighting factor for k'_{ij} and $d_{BPA_{ij}}$. They are determined by

$$\alpha_{k'_{ij}} = Diss_{BetP_{m_{ij}}},\ \alpha_{d_{BPA_{ij}}} = Sim_{BetP_{m_{ij}}} = 1 - \alpha_{k'_{ij}} \tag{28}$$

The dissimilarity measure $Diss(m_i, m_j)$ in Def.12 is a weighted average of conflict belief between betting commitments and Jousselme's distance. As the value of similarity coefficient increases, the distance metric will occupy a larger proportion in the dissimilarity measure, and the effect of conflict belief will decrease gradually. On the contrary, when dissimilarity coefficient increases, the difference between betting commitments of m_i and m_j increases, which will lead to the dissimilarity measure being more dependent upon the conflict belief compared with the distance metric.

Properties:

(1) If$\alpha_{k'_{ij}} = 0, Diss(m_i, m_j) = d_{BPA_{ij}}$;
(2) If;$\alpha_{k'_{ij}} = 0.5, Diss(m_i, m_j) = \frac{1}{2}(k'_{ij} + d_{BPA_{ij}})$
(3) If $\alpha_{k'_{ij}} = 1, Diss(m_i, m_j) = k'_{ij}$.

Here, the conflict belief k'_{ij} is produced from the Pignistic probability function transferred from the mass function rather than the original two BPAs. If k_{ij} (the conflict measure generated by the product of the mass of subsets between two pieces of evidence that don't intersect with each other) is used to determine the conflict belief between two pieces of evidence, some irrational results may be generated.

Example 6.

Given the frame of discernment $\Omega = \{\theta_1, \theta_2, \cdots, \theta_{50}\}$, the BPAs of fifty pieces of evidence are shown as follows:

Each piece of evidence in this example is a categorical belief function which owns only one focal element. From m_1 to m_{50}, the hypotheses included in the focal element increases gradually such that $|\mathscr{C}_i| = i$. Since the intersection between the sole focal elements for any pair of evidence is a nonempty set, the conflict belief associated with m_i and m_j ($i, j = 1, 2, \ldots 50; i \neq j$) is zero such that $k_{ij} = 0$. Taking m_1, m_3 and m_{50} for example, if we use the conflict belief k_{ij} which is generated from the original BPAs to replace k'_{ij} in Eq. (27), then the result may be controversial.

$$Diss(m_1, m_3) = \alpha_{k_{13}} \cdot k_{13} + \alpha_{d_{BPA_{13}}} \cdot d_{BPA_{13}} = \frac{2}{3} \times 0 + \frac{1}{3} \times 0.8165 = 0.2722$$

$$Diss(m_1, m_{50}) = \alpha_{k_{1,50}} \cdot k_{1,50} + \alpha_{d_{BPA_{1,50}}} \cdot d_{BPA_{1,50}} = \frac{49}{50} \times 0 + \frac{1}{50} \times 0.9899 = 0.0198$$

Intuitively, the dissimilarity between m_1 and m_3 ought to be less than that between m_1 and m_{50} because m_{50} is more uncertain than m_3. From another aspect, the similarity coefficient associated with m_1 and m_3 ($\alpha_{d_{BPA_{13}}}$) is much larger than that of m_1 and m_{50} ($\alpha_{d_{BPA_{1,50}}}$). So the dissimilarity generated above is abnormal. Moreover, the above two dissimilarity values are less than the normal level because Jousselme's distance is discounted by the similarity coefficient, while the first part in the equation doesn't contribute to the final dissimilarity because of the misuse of k_{ij}. When k'_{ij} is utilized to calculate the conflict belief between m_i and m_j, we have:

$$Diss(m_1, m_3) = \alpha_{k'_{13}} \cdot k'_{13} + \alpha_{d_{BPA_{13}}} \cdot d_{BPA_{13}} = \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times 0.8165 = 0.7166$$

$$Diss(m_1, m_{50}) = \alpha_{k'_{1,50}} \cdot k'_{1,50} + \alpha_{d_{BPA_{1,50}}} \cdot d_{BPA_{1,50}} = \frac{49}{50} \times \frac{49}{50} + \frac{1}{50} \times 0.9899 = 0.9802$$

It is rational because $Diss(m_1, m_{50}) > Diss(m_1, m_3)$, and the dissimilarity incorporates both the conflict belief associated with Pignistic probability and evidential distance between two masses. If we only consider Jousselme's distance, the difference between $Diss(m_1, m_3)$ and $Diss(m_1, m_{50})$ is not obvious because the total ignorance is only restricted to θ_1, θ_2 and θ_3 for m_3, while it is attributed to fifty elements for m_{50}. So the dissimilarity between m_1 and m_{50} should be much larger than that between m_1 and m_3 compared with the value of $|d_{BPA_{1,50}} - d_{BPA_{13}}|$. The new dissimilarity measure just consists with our intuition.

Theorem 1. Suppose m_i and m_j are two BPAs defined on the same frame of discernment Ω, $Diss(m_i, m_j)$ is the proposed dissimilarity measure in Def.12. Then properties (d1) - (d5) in Def. 6 are satisfied. Additionally, Lemma 1 ($Diss(m_i, m_j) = 1$ iff $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) = \varnothing$) to Lemma 3 ($0 < Diss(m_i, m_j) < 1$ iff $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right) \cap \left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right) \neq \varnothing$ and $m_i \neq m_j$) are also satisfied.

Property (d3) and Lemma 1 together are called 'Extreme consistency' in [15]. It means that the dissimilarity reaches the minimum value iff the two BPAs are absolutely non-conflicting, while attains the maximum value iff the two BPAs are totally conflicting.

Proof of Theorem 1.

(d1) *Reflexivity*: $Diss(m_i, m_i) = 0$

From Eq. (25), we have

$$Diss_{BetP_{m_{ii}}} = 0$$

Then from Eq. (28), we have

$$\alpha_{k'_{ii}} = Diss_{BetP_{m_{ii}}} = 0, \alpha_{d_{BPA_{ii}}} = 1 - \alpha_{k'_{ii}} = 1$$

From Eq. (27), we have

$$Diss(m_i, m_i) = \alpha_{k'_{ii}} \cdot k'_{ii} + \alpha_{d_{BPA_{ii}}} \cdot d_{BPA_{ii}} = 0 \cdot k'_{ii} + 1 \cdot d_{BPA_{ii}}$$

Since $d_{BPA_{ii}} = 0$ can be inferred from Eq. (14), so

$$Diss(m_i, m_i) = 0$$

(d2) Symmetry: $\boldsymbol{Diss(m_i, m_j) = Diss(m_j, m_i)}$

From Eq. (27), we have

$$Diss(m_i, m_j) = \alpha_{k'_{ij}} \cdot k'_{ij} + \alpha_{d_{BPA_{ij}}} \cdot d_{BPA_{ij}}$$

$$Diss(m_j, m_i) = \alpha_{k'_{ji}} \cdot k'_{ji} + \alpha_{d_{BPA_{ji}}} \cdot d_{BPA_{ji}}$$

According to Eqs. (14), (23) and (28), we can infer that

$$d_{BPA_{ij}} = d_{BPA_{ji}}, k'_{ji} = k'_{ji}, \alpha_{d_{BPA_{ij}}} = \alpha_{d_{BPA_{ji}}}, \alpha_{k'_{ji}} = \alpha_{k'_{ji}}$$

So

$$Diss(m_i, m_j) = Diss(m_j, m_i)$$

(d3) Nondegeneracy: $\boldsymbol{Diss(m_i, m_j) = 0}$ iff $\boldsymbol{m_i = m_j}$

Proof of Sufficiency:

If $m_i = m_j$, from Property (1) in Theorem 1, we have

$$Diss(m_i, m_j) = 0$$

So

$$m_i = m_j \Rightarrow Diss(m_i, m_j)$$

Proof of Necessity:

If $Diss(m_i, m_j) = 0$, from Def. 12, we have $\alpha_{k'_{ij}} \cdot k'_{ij} = 0$ and $\alpha_{d_{BPA_{ij}}} \cdot d_{BPA_{ij}} = 0$ because $\alpha_{k'_{ij}} \geq 0$, $k'_{ij} \geq 0$, $\alpha_{d_{BPA_{ij}}} \geq 0$, $d_{BPA_{ij}} \geq 0$.

a) If $\alpha_{k'_{ij}} = 0$, then $\alpha_{d_{BPA_{ij}}} = 1$, $d_{BPA_{ij}}$ must be equal to 0.

   In this case, $m_i = m_j$ because $d_{BPA_{ij}} = 0 \Rightarrow m_i = m_j$.

a) If $0 < \alpha_{k'_{ij}} < 1$, then $0 < \alpha_{d_{BPA_{ij}}} < 1$, k'_{ij} and $d_{BPA_{ij}}$ must be zero to guarantee $Diss(m_i, m_j)$ equals to 0. This will not occur because if $\alpha_{d_{BPA_{ij}}} < 1$, then there exist some discrepancies between the focal elements of m_i and m_j, which deduces the result that $d_{BPA_{ij}} > 0$. So, we have $Diss(m_i, m_j) > 0$ in this situation.

b) If $\alpha_{k'_{ij}} = 1$, then $\alpha_{d_{BPA_{ij}}} = 0, k'_{ij}$ equals to 1 because the focal elements of the two pieces of evidence have no intersection. In this case, $Diss(m_i, m_j) = 1$.

In summary, we can infer that $Diss(m_i, m_j) = 0$ iff $\alpha_{d_{BPA_{ij}}} = 1$ and $d_{BPA_{ij}} = 0$ which occurs when $m_i = m_j$.

Therefore, the proposed dissimilarity measure has the minimum value if and only if the two pieces of evidence are completely consistent. Hence, $Diss(m_i, m_j) = 0 \Rightarrow m_i = m_j$.

(d4) Nonnegativity: $\boldsymbol{Diss(m_i, m_j) \geq 0}$;

Because $\alpha_{k'_{ij}}$, k'_{ij}, $\alpha_{d_{BPA_{ij}}}$ and $d_{BPA_{ij}}$ are all non-negative and bounded within [0,1], it is easy to infer that

$$Diss(m_i, m_j) = \alpha_{k'_{ij}} \cdot k'_{ij} + \alpha_{d_{BPA_{ij}}} \cdot d_{BPA_{ij}} \geq 0$$

Lemma 1: $Diss(m_i, m_j) = 1$ iff $\left(\underset{A \subseteq \mathfrak{B}_i}{\cup} A \right) \cap \left(\underset{B \subseteq \mathfrak{B}_j}{\cup} B \right) = \varnothing$

If $\left(\underset{A\subseteq\mathfrak{B}_i}{\cup} A\right)\cap\left(\underset{B\subseteq\mathfrak{B}_j}{\cup} B\right)=\varnothing$, then

$$Diss_{\mathscr{C}_{ij}} = Diss_{BetP_{m_{ij}}} = 1$$

From Def. 12, we have

$$\alpha_{k'_{ij}} = 1,\ \alpha_{d_{BPA_{ij}}} = 0$$

From Def. 8, we get

$$k'_{ij} = 1$$

As such, $Diss(m_i,m_j) = \alpha_{k'_{ij}} \cdot k'_{ij} + \alpha_{d_{BPA_{ij}}} \cdot d_{BPA_{ij}} = k'_{ij} = 1$.

(d5) Transitivity: $\boldsymbol{m_i = m_i \Rightarrow Diss(m_k,m_i) = Diss(m_k,m_j)}$**;**

From Eq. (27), we have

$$Diss(m_g,m_i) = \alpha_{k'_{gi}} \cdot k'_{gi} + \alpha_{d_{BPA_{gi}}} \cdot d_{BPA_{gi}}$$

$$Diss(m_g,m_j) = \alpha_{k'_{gj}} \cdot k'_{gj} + \alpha_{d_{BPA_{gj}}} \cdot d_{BPA_{gj}}$$

Because $m_i = m_j$, according to Eqs. (14), (23) and (28), we can infer that

$$d_{BPA_{gi}} = d_{BPA_{gj}},\ k'_{gi} = k'_{gi}, \alpha_{k'_{gi}} = \alpha k'_{gi}, \alpha_{d_{BPA_{gi}}} = \alpha_{d_{BPA_{gj}}}$$

So, we have $Diss(m_g,m_i) = Diss(m_g,m_j)$.

To illustrate (d6), simulation is conducted in Supplementary Material. Obviously, the proposed dissimilarity measure satisfies Theorem 1. Based on this, a similarity measure can be defined below.

Definition 13. (*Similarity measure between pieces of evidence*) Given the frame of discernment Ω, the mass of two pieces of evidence is denoted by m_i and m_j. The dissimilarity measure between m_i and m_j is computed by Eqs. (27), (28). Then, the *similarity measure* between m_i and m_j profiled by $Sim(m_i,m_j)$ is defined as:

$$Sim(m_i,m_j) = 1 - Diss(m_i,m_j) = 1 - k'_{ij} + \alpha_{d_{BPA_{ij}}} \cdot (k'_{ij} - d_{BPA_{ij}}) \tag{29}$$

5. A multi-source data fusion approach based on uncertainty measure and the new dissimilarity metric

In data fusion procedure, the weight and reliability of each piece of evidence may be different. As mentioned by Smarandache et al. [21], reliability reveals the capability to provide the correct information of the given problem which ought to be determined from statistical data or other ways. Weight can be interpreted as the relative importance of a piece of evidence compared with other evidence. Due to the fact that reliability and weight don't have the same explanation, they should be discriminated in data fusion process and tackled indifferently. Since Dempster's rule doesn't differentiate the reliability and weight of evidence clearly and assumes both of them to be 1, counter-intuitive combination results will be generated provided that pieces of evidence are contradictory. Yang's ER rule [14] distinguished the concept of reliability and weight of evidence and combined them comprehensively in the aggregation of evidence. But how to generate the values of these two parameters still needs to be discussed.

5.1. Weight assignment based on dissimilarity among evidence

The main purpose of this subsection is to assign rational weighting factors to evidence to make a valid combination procedure. As mentioned by Deng [52], if a piece of evidence is supported by a majority of evidence greatly, it should be automatically given more importance in data fusion. Otherwise, the evidence should play less important role in the aggregation provided that its cumulative dissimilarity is large. That is to say, the truth is generally held by the majority.

Suppose there are L pieces of evidence denoted as $\boldsymbol{e} = \{e_1,e_2,\ldots,e_L\}$. Let the dissimilarity between two pieces of evidence e_i and e_j $(i,j=1,2,\ldots,L)$ be generated by Def. 12, and the similarity measure is calculated by Def. 13. When the similarity measures between all pairs of evidence are computed, an L dimensional similarity measure matrix is constructed as follows:

$$\left[Sim_{ij}\right]_{L\times L}=\begin{array}{c} \\ e_1 \\ \cdots \\ e_i \\ \cdots \\ e_L \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & \cdots & e_j & \cdots & e_L \end{array} \\ \left[\begin{array}{ccccc} 1 & \cdots & Sim_{1j} & \cdots & Sim_{1L} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Sim_{i1} & \cdots & Sim_{ij} & \cdots & Sim_{iL} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Sim_{L1} & \cdots & Sim_{Lj} & \cdots & 1 \end{array}\right]_{L\times L} \end{array} \tag{30}$$

where Sim_{ij} is the abbreviation of $Sim(m_i,m_j)$. As such, the *cumulative similarity measure* for e_i is $\sum_{j=1\,j\neq i}^{L} Sim_{ij}$. The degree of support for e_i can then be calculated by the *average similarity measure* between e_i and other L-1 pieces of evidence as follows:

$$Sim_i=\frac{1}{L-1}\sum_{j=1\,j\neq i}^{L} Sim_{ij} \tag{31}$$

After the normalization, the weight of e_i can be calculated as:

$$w_i=\frac{Sim_i}{\sum_{i=1}^{L} Sim_i} \tag{32}$$

where $\sum_{i=1}^{T} w_i=1$. Then the weighting vector of evidence is generated as $w=\{w_1,\ w_2,\ldots,w_L\}$.

5.2. Determination of evidence reliability

Since reliability reflects the ability to provide correct information, it should be computed objectively by digging out the information contained in data source other than the comparison among different pieces of evidence.

Remark 1. The more uncertainty contained in the BPA of a piece of evidence, the less reliability should be allocated to the evidence in the combination of multi-source data, and vice versa.

The uncertainty mentioned in Remark 1 consists of two implications. One is the dispersity of a BPA, which means how scattered belief degrees are allocated to the power set of discernment framework. It reflects the randomness of a BPA. For example, two BPAs are given as $m_1(\{\theta_1\})=0.2$, $m_1(\{\theta_2\})=0.8$, $m_2(\{\theta_1\})=0.2$, $m_2(\{\theta_2\})=0.6$, $m_2(\{\theta_3\})=0.2$. Obviously, m_2 is more uncertain than m_1 because m_2 has three focal elements. Hence, the reliability of m_1 is assumed to be larger than m_2. The other component is the uncertainty measure of nonspecificity [12]. It refers to the uncertainty that the basic probability of singleton element is not specified provided that the cardinality of focal element is larger than 1. Suppose two BPAs are given as $m_1(\{\theta_1\})=0.2$, $m_1(\{\theta_2\})=0.8$, $m_2(\{\theta_1\})=0.2$, $m_2(\{\theta_1,\theta_2\})=0.8$. Since $|\theta_1,\theta_2|>|\theta_2|$, the uncertainty of m_2 is larger than m_1. $|\theta_1,\theta_2|$ is also interpreted as local ignorance if $|\Omega|>2$, or global ignorance when $\Omega=\{\theta_1,\theta_2\}$. The above two components jointly constitute a total uncertainty measure which should reflect both the randomness and non-specificity of a BPA. Here, we use the uncertainty measure proposed by Zhou et al. [12] because it satisfies probabilistic consistency, non-negativity, monotonicity and other basic properties that an uncertainty measure should satisfy.

Let $\Omega=\{\theta_1,\theta_2,\cdots,\theta_N\}$ be the frame of discernment. The BPA of evidence e_i is represented as $m_i=\{(A,m_i(A))|m_i(A)\rangle 0,A\subseteq\Omega\}$, and the BOE is denoted by $\mathfrak{B}_i=\{A|m_i(A)\rangle 0,A\subseteq\Omega\}$. Then the total uncertainty measure of m_i is computed as:

$$\widetilde{E}(m_i)=\begin{cases}\dfrac{1}{2(|\mathfrak{B}_i|-1)}\displaystyle\sum_{A_l\subseteq\Omega}\sum_{A_h\subseteq\Omega} SC_{A_l,A_h}\cdot\left[-m(A_l)\log_2\frac{m(A_l)}{2^{|A_l|}-1}-m(A_h)\log_2\frac{m(A_h)}{2^{|A_h|}-1}\right], & |\mathfrak{B}_i|\geq 2\\ -m(A)\log_2\dfrac{m(A)}{2^{|A|}-1}, & |\mathfrak{B}_i|=1\end{cases} \tag{33}$$

$$(A_l,A_h\in\mathfrak{B}_i;A\in\mathfrak{B}_i)$$

A_l and A_h $(l,h=1,2,\ldots,|\mathfrak{B}_i|;\ l\neq h)$ refer to the lth and hth focal elements in $\mathfrak{B}_i$.

$\log_2\sum_l(2^{|A_l|}-1)$ is the maximum value of Deng's uncertainty measure [45] and Zhou et al's uncertainty measure is no larger than that of Deng's. Therefore, we use the following equation to quantify the reliability of evidence e_i.

$$r_i=1-\frac{\widetilde{E}(m_i)}{\log_2\sum_l(2^{|A_l|}-1)} \tag{34}$$

Remark 2. r_i is bounded within [0,1]. When $\widetilde{E}(m_i)=0$, e_i expresses determinate information and it is assumed to be completely reliable. This situation occurs when there is only one focal element whose cardinality is 1. On the contrary, when $\widetilde{E}(m_i)=\log_2\sum_l(2^{|A_l|}-$

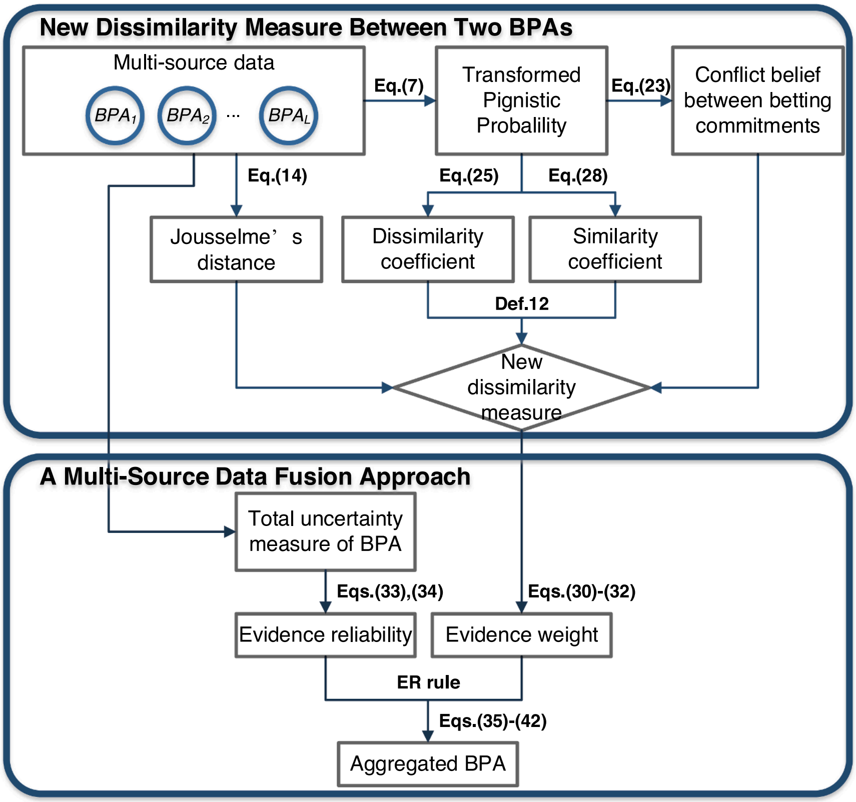

Fig. 2. The architecture of data fusion process based on the proposed dissimilarity measure

1), it attains the maximum uncertainty value. This case means that the data source expresses worthless information.

5.3. *Multi-source data fusion based on uncertainty measure and the new dissimilarity metric*

In a multi-source data fusion problem including L pieces of evidence, let m_i be the BPA of evidence e_i $(i=1,2,\cdots,L)$ such that $m_i = \{(A,m_i(A))|m_i(A)\rangle 0, A\subseteq 2^{\Omega}\}$, $\sum_{A\subseteq\Omega} m_i(A) = 1$. Based on the weights and reliabilities generated by Eqs. (31)-(34), the ER rule can then be used to generate the aggregated BPA.

The discounted BPA for e_i defined in the ER rule with evidence weight and reliability [14] is profiled as:

$$\widetilde{m}_i(A) = \begin{cases} \widetilde{w}_i \cdot m_i(A)\ A \subseteq 2^{\Omega},\ A \neq \varnothing \\ 0\ A = \varnothing \\ 1 - \widetilde{w}_i\ A = P(\Omega) \end{cases} \tag{35}$$

where $\widetilde{w}_i = \frac{w_i}{1+w_i-r_i}$. $1-\widetilde{w}_i$ signifies the residual support for e_i from w_i and r_i. A weighted BPA is then constructed as follows:

$$\widetilde{m}_i = \left\{(A,\widetilde{m}_i(A))|\widetilde{m}_i(A)\rangle 0, A\subseteq 2^{\Omega}; \left(P(\Omega), m_{P(\Omega),i}\right)\right\} \tag{36}$$

The combination of multiple pieces of evidence is conducted as follows:

$$\widehat{m}_{A,e(i)} = \left[(1-\widetilde{w}_i)\widetilde{m}_{A,e(i-1)} + \widetilde{m}_{P(\Omega),e(i-1)}\widetilde{m}_{A,i}\right] + \sum_{B\cap C=A} \widetilde{m}_{B,e(i-1)}\widetilde{m}_{C,i},\ \forall A\subseteq 2^{\Omega} \tag{37}$$

$$\widehat{m}_{P(\Omega),e(i)} = (1-\widetilde{w}_i)\widetilde{m}_{P(\Omega),e(i-1)} \tag{38}$$

$$\widetilde{m}_{A,e(i)} = k\cdot\widehat{m}_{A,e(i)} = \frac{\widehat{m}_{A,e(i)}}{\sum_{D\subseteq\Omega}\widehat{m}_{D,e(i)} + \widehat{m}_{P(\Omega),e(i)}}, \forall A\subseteq 2^{\Omega} \tag{39}$$

Table 10
The BPAs of two pieces of evidence

	θ_1	$\{\theta_1, \theta_2\}$	$\{\theta_1, \theta_2, \theta_3\}$	…	$\{\theta_1, \theta_2, \ldots, \theta_{50}\}$
m_1	1	0	0		0
m_2	0	1	0		0
m_3	0	0	1		0
…					
m_{50}	0	0	0		1

$$\widetilde{m}_{P(\Omega),e(i)} = k \cdot \widehat{m}_{P(\Omega),e(i)} = \frac{\widehat{m}_{P(\Omega),e(i)}}{\sum\limits_{D\subseteq\Omega} \widehat{m}_{D,e(i)} + \widehat{m}_{P(\Omega),e(i)}} \tag{40}$$

$1 - \widetilde{w}_i$ is allocated to the power set of the discernment framework instead of any single subset. So $1 - \widetilde{w}_i$ is attached to $P(\Omega)$ that allows it to be redistributed to all propositions in the power set of the frame of discernment because $A \cap P(\Omega) = A.e(i)$ represents the combination result of the first i pieces of evidence.

After $L-1$ times of iteration, the combined probability mass can be obtained as $\widetilde{m}_{A,e(L)}(A \subseteq 2^{\Omega})$, and the combined residual support for $e(L)$ is generated as $\widetilde{m}_{P(\Omega),e(L)}$. The final combined BPA for A is then generated as:

$$m_{A,e(L)} = \frac{\widetilde{m}_{A,e(L)}}{1 - \widetilde{m}_{P(\Omega),e(L)}} = \frac{\widehat{m}_{A,e(L)}}{\sum\limits_{D\subseteq\Omega} \widehat{m}_{D,e(i)}} \quad \left(A \subseteq 2^{\Omega}\right) \tag{41}$$

So the distribution after the fusion of L data sources is obtained as:

$$m = \left\{(A, m(A)) | m(A) \rangle 0, A \subseteq 2^{\Omega}\right\} \tag{42}$$

The generalized dissimilarity measure and its application in muti-source data fusion is depicted in Fig. 2.

5.4. Illustrative example

In this subsection, we use the numerical example in [52] to verify the rationality and validity of the proposed multi-source data fusion method based on uncertainty measure and the new dissimilarity metric. The detailed calculation steps of the proposed method are described as follows.

Step 1. Obtain multisource data.

There are five different sensors used in a multisensor-based automatic target recognition system, and suppose the real target is A. The frame of discernment is $\Theta = \{A, B, C\}$. Five bodies of evidence have been collected as follows:

$$S_1: \ m_1(A) = 0.5, m_1(B) = 0.2, m_1(C) = 0.3$$

$$S_2: \ m_2(A) = 0, \ m_2(B) = 0.9, \ m_2(C) = 0.1$$

$$S_3: \ m_3(A) = 0.55, m_3(B) = 0.1, m_3(A, C) = 0.35$$

$$S_4: \ m_4(A) = 0.55, m_4(B) = 0.1, \ m_4(A, C) = 0.35$$

$$S_5: \ m_5(A) = 0.6, \ m_5(B) = 0.1, \ m_5(A, C) = 0.3$$

Step 2. Determination of evidence weight.

Step 2.1: Transforming the original mass function to Pignistic probability function by applying Eq.(7);
Step 2.2: Calculating the dissimilarity coefficient between the betting commitments $Diss_{BetP_{m_{ij}}}$ by using Eq.(25);
Step 2.3: Computing Jousselme's distance $d_{BPA_{ij}}$ by Eq.(14) and its discounting coefficient $\alpha_{d_{BPA_{ij}}}$ by Eq.(28);
Step 2.4: Computing conflict belief between betting commitments k'_{ij} by Eq.(23) and its discounting coefficient $\alpha_{k'_{ij}}$ by Eq.(28);
Step 2.5: Obtaining the final dissimilarity measure between the two pieces of evidence $Diss(m_i, m_j)$ by Def.12.
Step 2.6: By applying Eqs.(29)-(32), the weight w_i of evidence e_i can be calculated. The results of weight vector are calculated as $\boldsymbol{w} = \{0.2226, 0.0653, 0.2381, 0.2381, 0.2359\}$.

Step 3. Determination of evidence reliability.

Step 3.1: Computing the total uncertainty $\widetilde{E}(m_i)$ of evidence e_i by using Eq.(33);
Step 3.2: Using Eq.(34) to quantify the reliability r_i of evidence e_i. The result of reliability vector is calculated as $\boldsymbol{r} = \{0.5314, 0.8828, 0.7529, 0.7529, 0.7676\}$.

Table 11
Results of different data fusion methods

	m_1, m_2	m_1, m_2, m_3	m_1, m_2, m_3, m_4	m_1, m_2, m_3, m_4, m_5
Dempster- Shafer's combination rule	$m(A)$=0	$m(A)$=0	$m(A)$=0	$m(A)$=0
	$m(B)$=0.8571	$m(B)$=0.6316	$m(B)$=0.3288	$m(B)$=0.1228
	$m(C)$=0.1429	$m(C)$=0.3684	$m(C)$=0.6712	$m(C)$=0.8772
Murphy's combination rule	$m(A)$=0.1543	$m(A)$=0.3500	$m(A)$=0.6027	$m(A)$=0.7958
	$m(B)$=0.7469	$m(B)$=0.5224	$m(B)$=0.2627	$m(B)$=0.0932
	$m(C)$=0.0988	$m(C)$=0.1276	$m(C)$=0.1346	$m(C)$=0.1110
Deng's combination rule	$m(A)$=0.1543	$m(A)$=0.4861	$m(A)$=0.7773	$m(A)$=0.8909
	$m(B)$=0.7469	$m(B)$=0.3481	$m(B)$=0.0628	$m(B)$=0.0086
	$m(C)$=0.0988	$m(C)$=0.1657	$m(C)$=0.1600	$m(C)$=0.1005
Proposed	**$m(A)$=0.3040**	**$m(A)$=0.4278**	**$m(A)$=0.5965**	**$m(A)$=0.6925**
	$m(B)$=0.4729	**$m(B)$=0.2794**	**$m(B)$=0.1467**	**$m(B)$=0.0920**
	$m(C)$=0.2231	**$m(C)$=0.1048**	**$m(C)$=0.0448**	**$m(C)$=0.0230**
		$m(A,C)$=0.1880	**$m(A,C)$=0.2120**	**$m(A,C)$=0.1925**

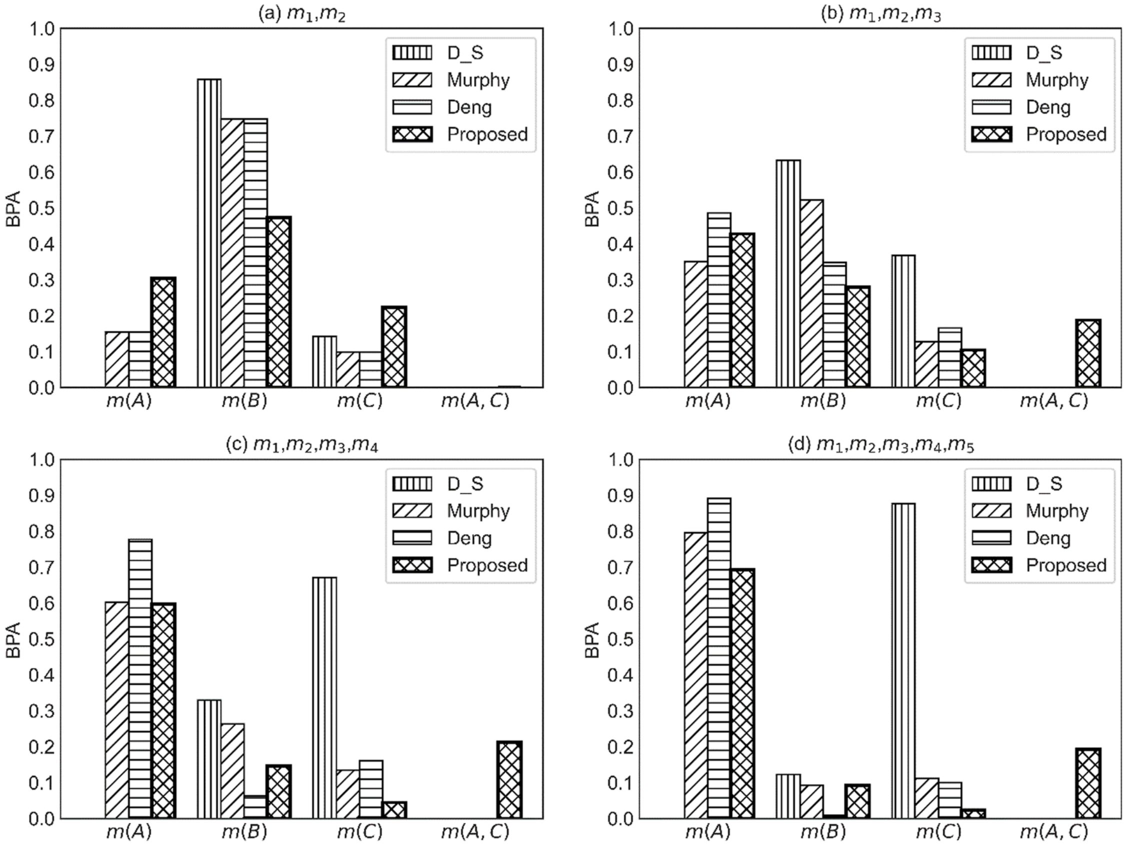

Fig. 3. Comparison of different data fusion methods

Step 4. Multi-source data fusion based on uncertainty measure and the new dissimilarity metric.

After implementing Steps 1-3, the weight and reliability of evidence e_i are both generated. And ER rule can then be used to generate the aggregated BPA $m = \{(A, m(A)) | m(A) > 0, A \subseteq 2^{\Omega}\}$.

The results by applying the proposed data fusion methods and other methods in [52] are shown in Table 11 and Fig. 3. Next, the calculation results of different methods will be analyzed from the following two aspects.

One is in terms of their ability to correctly identify the real target A. From Table 11 and Fig. 3, it can be intuitively seen that the D-S combination rule is counter-intuitive regardless of the amount of evidence, which is caused by the conflict between S_2 and other sensors. Besides, Murphy's method, Deng's method and the proposed method can effectively avoid the irrationality of D-S combination rule, especially as the evidence grows. However, from Fig. 3 (b) we can figure that Murphy's method cannot recognize the correct target when there are only three sensors S_1,S_2,S_3. Furthermore, from Fig. 3 (b)-(d), it can be seen that except for our proposed method, none of the other methods assign probability to $\{A, C\}$ which represents local ignorance. From the point of view in practical application, it is important to provide basis for failure mode and effect analysis (FMEA), fault diagnosis and so on.

The other is in terms of their evidence aggregation methods. By further analysis, we can infer that different ways for the aggregation of evidence lead to different results from Deng's method. On the one hand, from the original data, it is obvious that the second sensor S_2 has the biggest dissimilarity from other sensors, which is reflected in the smallest weight such that $\boldsymbol{w}=\{0.2226$, **0.0653**, 0.2381, 0.2381, 0.2359}. On the other hand, if evidence has greater ability to provide determinate information, it should have larger reliability such that $\boldsymbol{r}=\{0.5314$, **0.8828**, 0.7529, 0.7529, 0.7676}. By considering the weight and reliability simultaneously, the pieces of evidence can be aggregated by applying ER rule in the proposed method. Comparatively, D-S rule and Murphy's rule don't consider weight and reliability simultaneously. Specifically, D-S rule assumes the reliability of evidence equals to weight with the value of 1. This is irrational because the two concepts really have different meanings. In [52], the reliability of evidence is calculated by the similarity between the evidence and other evidences. It also doesn't distinguish weight and reliability. Therefore, the proposed method can not only recognize the correct target, but also has higher practicability and rationality in tackling with real-world problems.

5.5. Application in large-scale group decision making (LSGDM) process

In this study, a multi-source data fusion approach based on uncertainty measure and the new dissimilarity metric has been proposed. To demonstrate the practicality of the proposed method in real-world scenarios, we applied our multi-source data fusion method to a LSGDM scenario. In the context of GDM, the integration of multiple data sources is essential for attribute and expert aggregation. Typically, multi-source data in LSGDM is derived from a range of channels, including surveys, interviews and social media platforms, to capture diverse perspectives and opinions within the group. In this paper, we examine the multi-source data gathered from diverse decision makers (DMs). The proposed approach enables the integration of these disparate sources by simultaneously considering their relative weights and reliabilities.

In this section, we implement the proposed methodology to address the issue of selecting a Software-as-a-Service (SaaS) suite, as presented in [53], which is useful for assisting electric vehicle (EV) companies in making strategic decisions. There are 21 experts or managers denoted as $=\{e_1,\ldots,e_{21}\}$, tasked with selecting the optimal carbon footprint SaaS provider from alternative set $A=\{a_1,a_2,a_3,a_4\}$. Due to differences in expertise and knowledge background among DMs, each DM has their own evaluation criteria and ultimately provides a complete decision-making process report based on evaluation grades $\Omega=\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}=$ {absolutely worse, worse, slightly worse, indifference, slightly better, better, absolutely better}. The score values of seven evaluation grades are set as $S=\{s(H_1),s(H_2),s(H_3),s(H_4),s(H_5),s(H_6),s(H_7)\}=\{-1,-0.7,-0.3,0,0.3,0.7,1\}$. By following the step-by-step approach proposed in Section 5, we are able to aggregate 21 decision matrices into a coactive one and compare it with the results presented in [53], as shown in Table 12.

The application of the proposed method in this scenario shows its ability to simultaneously consider the relative importance and reliability of decision makers (information sources). This approach enhanced the scientific rigor and persuasiveness of the decision-making process in real-world applications. Notably, our method is not only applicable to traditional belief function assignments (BPA), but also to preference relations based on pairwise comparisons, making it versatile and applicable to various decision-making contexts.

The results obtained from the application of our method in the SaaS selection scenario demonstrated its effectiveness and universality. The approach provided a comprehensive and robust framework for integrating and analyzing multiple data sources, facilitating informed decision-making in complex and dynamic situations. Another strength of the proposed method is its scalability. It is designed to handle both small-scale and large-scale data, ensuring its applicability in various scenarios and accommodating the diverse needs of decision-making processes.

In summary, the application of the proposed method in real-world scenario shows its effectiveness in facilitating decision-making processes. By simultaneously considering the importance and reliability of decision-makers, our method measures these two parameters automatically by the given assessment information. Furthermore, its applicability to different data scales and its versatility in accommodating various analysis approaches make it a valuable tool in decision support systems.

6. Comparative analysis

Here, three cases are selected to analyze the validity and rationality of the proposed dissimilarity measure. Among them, Case 2 is a common example used for comparative analysis, such as in [31,35,36] and [51]. Then the comparison of some typical distance or dissimilarity measures with the proposed measure is conducted.

Case 1.

Suppose the frame of discernment is $\Omega=\{\theta_1,\theta_2,\theta_3\}$. Three BPAs are constructed as shown in Table 13. The dissimilarity measures between m_1 and $m_j (j=1,2)$ are presented in Table 14.

From the correlation coefficient defined in [35], we can get that $1-r_{BPA}(m_1,m_2)=1-r_{BPA}(m_1,m_3)=0.4227$. The result is irrational because m_1 and m_2 are more contradictory with each other than m_1 and m_3. Specifically, m_2 is a piece of certain evidence that definitely points to θ_1, while m_3 is assigned with the largest ignorance. So the dissimilarity between m_1 and m_2 should be undoubtedly different with that between m_1 and m_3. $d_{BPA_{12}}=d_{BPA_{13}}=0.5774$ is also unreasonable for the same reason. The value of $difBetP_{m_1}^{m_j}$ and $1-Sim(m_1^{'},m_j^{'})$ decreases excessively from 0.6667 to 0. Since m_1 and m_3 are two different BPAs, the dissimilarity between them should be positive. The PSD pignistic probability-based dissimilarity also decreases too much.

Table 12
Comparison analysis in the LSGDM application

Comparisons	The proposed method					Model in [53]				
The aggregation result		$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$		$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$
	$\boldsymbol{a}_1$	-	$\{(H_1, 0.05), (H_2, 0.03), (H_3, 0.04), (H_4, 0.32), (H_5, 0.04), (H_6, 0.03), (\Omega, 0.49)\}$	$\{(H_1, 0.02), (H_2, 0.07), (H_3, 0.05), (H_4, 0.22), (H_5, 0.11), (H_6, 0.04), (\Omega, 0.48)\}$	$\{(H_2, 0.08), (H_3, 0.10), (H_4, 0.16), (H_5, 0.04), (H_6, 0.11), (\Omega, 0.57)\}$	$\boldsymbol{a}_1$	-	$\{(H_1, 0.05), (H_2, 0.01), (H_3, 0.04), (H_4, 0.03), (H_5, 0.19), (H_6, 0.09), (H_7, 0.16), (\Omega, 0.44)\}$	$\{(H_1, 0.01), (H_2, 0.01), (H_3, 0.03), (H_4, 0.04), (H_5, 0.06), (H_6, 0.11), (H_7, 0.32), (\Omega, 0.42)\}$	$\{(H_1, 0.01), (H_2, 0.05), (H_3, 0.10), (H_4, 0.16), (H_5, 0.04), (H_6, 0.11), (H_7, 0.05), (\Omega, 0.49)\}$
	$\boldsymbol{a}_2$	$\{(H_2, 0.03), (H_3, 0.04), (H_4, 0.32), (H_5, 0.04), (H_6, 0.03), (H_7, 0.05), (\Omega, 0.49)\}$	-	$\{(H_1, 0.02), (H_2, 0.03), (H_3, 0.08), (H_4, 0.25), (H_5, 0.09), (H_6, 0.07), (H_7, 0.03), (\Omega, 0.42)\}$	$\{(H_1, 0.03), (H_2, 0.01), (H_3, 0.07), (H_4, 0.14), (H_5, 0.12), (H_6, 0.03), (H_7, 0.03), (\Omega, 0.57\}$	$\boldsymbol{a}_2$	$\{(H_1, 0.15), (H_2, 0.09), (H_3, 0.19), (H_4, 0.03), (H_5, 0.04), (H_6, 0.01), (H_7, 0.05), (\Omega, 0.44)\}$	-	$\{(H_1, 0.04), (H_2, 0.02), (H_3, 0.13), (H_4, 0.16), (H_5, 0.08), (H_6, 0.10), (H_7, 0.03), (\Omega, 0.44)\}$	$\{(H_1, 0.01), (H_2, 0.03), (H_3, 0.10), (H_4, 0.19), (H_5, 0.07), (H_6, 0.08), (H_7, 0.05), (\Omega, 0.48)\}$
	$\boldsymbol{a}_3$	$\{(H_2, 0.03), (H_3, 0.11), (H_4, 0.21), (H_5, 0.05), (H_6, 0.07), (H_7, 0.02), (\Omega, 0.50)\}$	$\{(H_1, 0.04), (H_2, 0.06), (H_3, 0.09), (H_4, 0.25), (H_5, 0.08), (H_6, 0.03), (H_7, 0.02), (\Omega, 0.44)\}$	-	$\{(H_1, 0.01), (H_2, 0.06), (H_3, 0.12), (H_4, 0.15), (H_5, 0.04), (H_6, 0.04), (\Omega, 0.60)\}$	$\boldsymbol{a}_3$	$\{(H_1, 0.32), (H_2, 0.11), (H_3, 0.06), (H_4, 0.04), (H_5, 0.03), (H_6, 0.01), (H_7, 0.01), (\Omega, 0.42)\}$	$\{(H_1, 0.03), (H_2, 0.10), (H_3, 0.08), (H_4, 0.16), (H_5, 0.13), (H_6, 0.02), (H_7, 0.04), (\Omega, 0.44)\}$	-	$\{(H_3, 0.03), (H_4, 0.07), (H_5, 0.05), (H_6, 0.05), (H_7, 0.33), (\Omega, 0.46)\}$
	$\boldsymbol{a}_4$	$\{(H_2, 0.14), (H_3, 0.08), (H_4, 0.08), (H_5, 0.12), (H_6, 0.09), (\Omega, 0.49)\}$	$\{(H_1, 0.04, (H_2, 0.03), (H_3, 0.14), (H_4, 0.16), (H_5, 0.08), (H_6, 0.02), (H_7, 0.04), (\Omega, 0.49)\}$	$\{(H_2, 0.04), (H_3, 0.04), (H_4, 0.18), (H_5, 0.15), (H_6, 0.07), (H_7, 0.01), (\Omega, 0.50)\}$		$\boldsymbol{a}_4$	$\{(H_1, 0.05), (H_2, 0.11), (H_3, 0.04), (H_4, 0.16), (H_5, 0.10), (H_6, 0.05), (H_7, 0.01), (\Omega, 0.49)\}$	$\{(H_1, 0.05, (H_2, 0.08), (H_3, 0.07), (H_4, 0.19), (H_5, 0.10), (H_6, 0.03), (H_7, 0.01), (\Omega, 0.48)\}$	$\{(H_1, 0.33), (H_2, 0.05), (H_3, 0.05), (H_4, 0.07), (H_5, 0.03), (\Omega, 0.46)\}$	
The comparison result	$a_2 \succ a_4 \succ a_3 \succ a_1$					$a_1 \succ a_2 \succ a_3 \succ a_4$				

Table 13
The BPAs of three pieces of evidence

	θ_1	θ_2	θ_3	$\{\theta_1, \theta_2, \theta_3\}$
m_1	1/3	1/3	1/3	0
m_2	1	0	0	0
m_3	0	0	0	1

Table 14
The dissimilarity computed by different methods

	$1-r_{BPA}(m_1,m_j)$	$difBetP_{m_1}^{m_j}$	1-cos	$1-Sim(m_1^{'},m_j^{'})$	$d_{BPA_{1j}}$	$difPBetP_{m_1}^{m_j}$	Proposed
d_{12}	0.4227	0.6667	0.4227	0.6667	0.5774	0.6667	0.6369
d_{13}	0.4227	0	1	0	0.5774	0.2381	0.5774

Case 2.

Suppose the frame of discernment is given as $\Omega = \{\theta_1,\theta_2,\cdots,\theta_{20}\}$. Two pieces of evidence are constructed below:

Suppose set A is changing as $\{\theta_1\}$, $\{\theta_1,\theta_2\}$, $\{\theta_1,\theta_2,\ \theta_3\}$, …, $\{\theta_1,\theta_2,\cdots,\theta_{20}\}$ respectively. Then there will be 20 pairs of BPAs. Intuitively, the dissimilarity between m_1 and m_2 attains to the minimum value when $A = \{\theta_1,\theta_2,\cdots,\theta_5\}$, and the greater the discrepancy between A and $\{\theta_1,\theta_2,\cdots,\theta_5\}$, the larger the dissimilarity. The comparisons of different dissimilarity measures between m_1 and m_2 are shown in Table 16 and Fig. 4.

It can be seen that the core of m_2 is $\{\theta_1,\theta_2,\cdots,\theta_5\}$ for all the twenty groups of evidence. The Cosine-based dissimilarity (symbol '1-cos' in Fig. 4) always equals to 1 because there is no identical focal element between the two pieces of evidence although the intersection of the cores for m_1 and m_2 is nonempty. The conflict belief k_{12} (symbol 'K' in Fig. 4) equals to 0.05 because only $m_1(\{\theta_7\})$ and $m_2(\{\theta_1,\theta_2,\cdots,\theta_5\})$ do not intersect with each other from the focal elements of the two pieces of evidence. The value of correlation coefficient [35] (symbol '1-r_{BPA}' in Fig. 4) is 0.0094 when A is $\{\theta_1,\theta_2,\cdots,\theta_5\}$. It is less than our intuition because the mass on A is only 0.8, while the rest 0.2 is attributed to other three focal elements. It is the same problem for Tanimoto's similarity [30] (symbol '1-simTa' in Fig. 4). So the dissimilarity which is approaching to 0 in this situation is irrational. The curves of Liu's distance measure [13] (symbol '$\sqrt{k\cdot d_{BPA}}$' in Fig. 4) and Jiang's distance measure [37] (symbol '$\frac{1}{2}(k\cdot d_{BPA})$' in Fig. 4) are also controversial because the twentieth pair of evidence does conflict with each other to a great extent. In this case, the proposed dissimilarity, Jousselme's distance [34] (symbol 'd_{BPA}' in Fig. 4) and Tessem's distance [29] (symbol '*difBetP*' in Fig. 4) are relatively valid.

Case 3.

Table 15
BPAs of two pieces of evidence

	$\{\theta_2, \theta_3, \theta_4\}$	$\{\theta_7\}$	Ω	A	$\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$
m_1	0.05	0.05	0.1	0.8	0
m_2	0	0	0	0	1

Table 16
Comparisons between typical dissimilarity measures and the proposed one in Case 2

	d_{BPA}	k	k'	1-*simTa*	1-cos	1-r_{BPA}	*difBetP*	$(k*d_{BPA})^{0.5}$	$0.5(k+d_{BPA})$	Proposed
1	0.7858	0.05	0.825	0.7418	1	0.7348	0.605	0.1982	0.4179	0.8144
2	0.6867	0.05	0.825	0.5285	1	0.5483	0.5285	0.1853	0.3683	0.763
3	0.5705	0.05	0.825	0.3493	1	0.369	0.2483	0.1689	0.3103	0.6655
4	0.4237	0.05	0.825	0.1966	1	0.1964	0.125	0.1455	0.2368	0.5019
5	0.1323	0.05	0.825	0.0376	1	0.0094	0.125	0.0813	0.0911	0.2189
6	0.3884	0.05	0.8517	0.196	1	0.1639	0.2583	0.1394	0.2192	0.5081
7	0.5029	0.05	0.8707	0.3465	1	0.2808	0.3536	0.1586	0.2765	0.633
8	0.5705	0.05	0.885	0.4187	1	0.3637	0.425	0.1689	0.3103	0.7042
9	0.6187	0.05	0.8961	0.4749	1	0.4288	0.4806	0.1759	0.3344	0.752
10	0.6554	0.05	0.905	0.5198	1	0.477	0.525	0.181	0.3527	0.7864
11	0.6844	0.05	0.9123	0.5566	1	0.5202	0.5614	0.185	0.3672	0.8123
12	0.7082	0.05	0.9183	0.5872	1	0.5565	0.5917	0.1882	0.3791	0.8325
13	0.7281	0.05	0.9235	0.6131	1	0.5872	0.6173	0.1908	0.389	0.8487
14	0.7451	0.05	0.9279	0.6353	1	0.6137	0.6393	0.193	0.3976	0.8619
15	0.7599	0.05	0.9317	0.6546	1	0.6367	0.6583	0.1949	0.405	0.873
16	0.773	0.05	0.935	0.6714	1	0.6569	0.675	0.1966	0.4115	0.8823
17	0.7846	0.05	0.9379	0.6863	1	0.6748	0.6897	0.1981	0.4173	0.8904
18	0.7951	0.05	0.9406	0.6995	1	0.6907	0.7028	0.1994	0.4225	0.8973
19	0.8046	0.05	0.9429	0.7113	1	0.705	0.7145	0.2006	0.4273	0.9034
20	0.8133	0.05	0.945	0.722	1	0.7178	0.725	0.2017	0.4317	0.9088

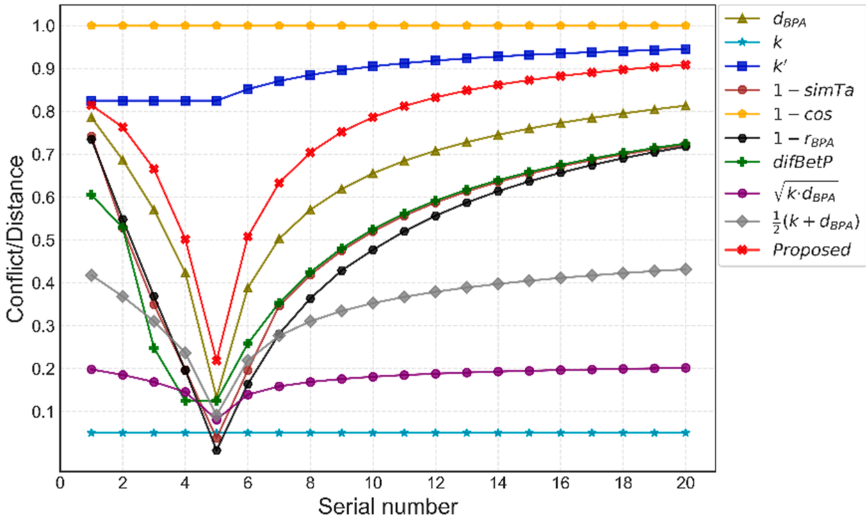

Fig. 4. Comparisons between different dissimilarity measures and the proposed one in Case 2

Table 17
Comparisons of the seven dissimilarity measures in Case 3

	d_{BPA}	*difBetP*	1-*cos*	1-simTa	1-r_{BPA}	*difP BetP*	Proposed
1	0.6172	0.6667	0.622	0.6667	0.4908	0.7415	0.6502
2	0.6236	0.6667	0.6449	0.6667	0.5	0.7476	0.6523
3	0.6305	0.6667	0.6725	0.6667	0.5095	0.7537	0.6546
4	0.638	0.6667	0.7031	0.6667	0.5193	0.7598	0.6571
5	0.646	0.6667	0.7348	0.6667	0.5292	0.7658	0.6598
6	0.6546	0.6667	0.7659	0.6667	0.5393	0.7719	0.6626
7	0.6636	0.6667	0.7954	0.6667	0.5494	0.778	0.6656
8	0.673	0.6667	0.8228	0.6667	0.5595	0.7841	0.6688
9	0.683	0.6667	0.8477	0.6667	0.5695	0.7902	0.6721
10	0.6933	0.6667	0.8701	0.6667	0.5794	0.7963	0.6755
11	0.7041	0.6667	0.8903	0.6667	0.5891	0.8024	0.6791
12	0.7152	0.6667	0.9083	0.6667	0.5986	0.8084	0.6829
13	0.7267	0.6667	0.9244	0.6667	0.608	0.8145	0.6867
14	0.7386	0.6667	0.9388	0.6667	0.6171	0.8206	0.6907
15	0.7509	0.6667	0.9518	0.6667	0.626	0.8267	0.6947
16	0.7634	0.6667	0.9635	0.6667	0.6346	0.8328	0.6989
17	0.7762	0.6667	0.974	0.6667	0.643	0.8389	0.7032
18	0.7894	0.6667	0.9835	0.6667	0.6511	0.845	0.7076
19	0.8028	0.6667	0.9921	0.6667	0.659	0.8511	0.712
20	0.8165	0.6667	1	0.6667	0.6667	0.8571	0.7166

We take another example in [36] to illustrate the validity of our proposed method. Given two BBAs m_1 and m_2 be defined on the frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$. $m_1\{\theta_3\}$ is set to be 1 and kept stable. m_2 is constantly changing from situation 1 to 20. In situation 1, m_2 is evenly attributed to the seven elements in the power set of Ω such that $m_2\{A\} = \frac{1}{7}(\forall A \in 2^\Omega, A \notin \Omega)$. From situation 2 to 20, each step has an increase of $\frac{6}{133}$ for $m_2(\{\theta_1, \theta_2, \theta_3\})$ together with a decrease of $\frac{1}{133}$ for other 6 elements. So in situation 20, $m_2(\{\theta_1, \theta_2, \theta_3\}) = 1$ and $m_2\{A\} = 0(\forall A \in 2^\Omega, A \notin \Omega)$. The comparison of d_{BPA}, *difBetP*, $1-cos$, $1-$simTa, $1-r_{BPA}$, *difP BetP* and the proposed measure for all the 20 situations are presented in Table 17 and depicted in Fig. 5.

It can be seen that both the proposed dissimilarity and *difP BetP* increase gradually from situation 1 to 20. Although the value of 1 $-$cos increases from 0.622 to 1, it is irrational because the last couple of evidence is not in complete conflict.

The reason that $1-simTa$ and *difBetP* remain the same value lies in that the transformed Pignistic probabilities for all the 20 groups of evidence are identical even though the original BPA changes gradually. And these two measures are only derived from the transformed Pignistic probability.

7. Conclusions

This paper proposes a novel dissimilarity measure defined on the Dempster-Shafer's belief structure which employs both the conflict belief on transformed Pignistic probability and Jousselme's distance measure. The concept of dissimilarity coefficient between betting commitments is defined. The first advantage of the new dissimilarity measure lies in that two dimensions are jointly

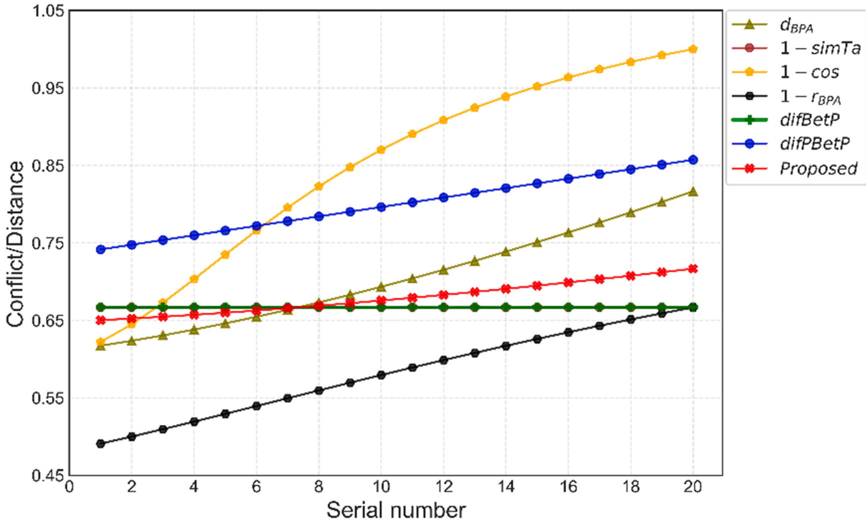

Fig. 5. Different dissimilarities between m_1 and m_2 in Case 3

considered. One is the contradiction between incompatible focal elements, the other one is the discrepancy of BPAs between compatible portions. So it can not only tackle with the situation that two pieces of evidence have small intersections, but also deal with the case where two BPAs intersect much. The proposed dissimilarity measure satisfies the basic properties of a distance metric. The second advantage of the new dissimilarity measure is that it is not originated from Pignistic probability. So it will not generate counter-intuitive results by some probability-based distance. A method of determining evidence reliabilities and weights is also presented based on uncertainty measure and the new dissimilarity metric when no prior knowledge is acquired. The new dissimilarity metric can be effectively used for multi-source data fusion under uncertain and complex environment, such as large-scale group decision making problem. The case presented in this paper illustrates the applicability of the proposed measure. Future research may be focused on how to fuse conflict data sources under complex environments based on the new dissimilarity measure. And the effectiveness of generating evidence weights and reliabilities also needs to be validated with the increasing number of evidence.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.fss.2023.108719.

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