Evidential Reasoning-Based Nonlinear Programming Model for MCDA Under Fuzzy Weights and Utilities

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In a multiple-criteria decision analysis (MCDA) problem, qualitative information with subjective judgments of ambiguity is often provided by people, together with quantitative data that may also be imprecise or incomplete. There are several uncertainties that may be considered in an MCDA problem, such as fuzziness and ambiguity. The evidential reasoning (ER) approach is well suited for dealing with such MCDA problems and can generate comprehensive distributed assessments for different alternatives. Many researches in dealing with imprecise or uncertain belief structures have been conducted on the ER approach. In this paper, both triangular fuzzy weights of criteria and fuzzy utilities assigned to evaluation grades are introduced to the ER approach, which may be incurred in several circumstances such as group decision-making situation. The Hadamard multiplicative combination of judgment matrix is extended for the aggregation of triangular fuzzy judgment matrices, the result of which is applied as the fuzzy weights used in the fuzzy ER approach. The consistency of the aggregated triangular fuzzy judgment matrix is also proved. Several pairs of ER-based programming models are designed to generate the total fuzzy belief degrees and the overall expected fuzzy utilities for the comparison of alternatives. A numerical example is conducted to show the effectiveness of the proposed approach.

1. INTRODUCTION

In an MCDA problem, various types of criteria need to be taken into account, which may be quantitative, measured by numerical values with certain units, or

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qualitative, assessed using subjective judgments with uncertainties. The evidential reasoning (ER) approach, which was introduced in 1990s[1–4] on the basis of Dempster–Shafer (D–S) theory[5,6] and decision-making theory, is well suited to address such complex MCDA problems. The unique features of the ER approach include its ability to represent incomplete and fuzzy subjective judgments and its convenience for the combination of attributes.

In recent years, the ER approach has been greatly improved in both theory and applications. First, in the theory aspect, Yang has made great efforts on the improvement of the ER approach.[1–4,7–9] A basic framework of the ER approach is proposed in Ref. 1 and extended to structure of multiple hierarchies in Ref. 2. In Ref. 3, an improved method for the ER approach is proposed for the transformation of different sets of linguistic evaluation grades associated with qualitative criteria and certain or random numbers associated with quantitative criteria to one set of evaluation grades. In Ref. 4, the original ER approach is analyzed for the explanation of its irrationality and an improved ER approach is introduced that could satisfy independency axiom, consensus axiom, completeness axiom, and incompleteness axiom simultaneously. Second, on dispelling the absurdities of D–S rule of combination, many researches have been conducted, for example, Yager,[10] Sun,[11] Liang,[12] etc. Third, on the combination for dependent evidences, Xiao,[13] Sun,[14] Voobraak,[15] and Hummel[16] have conducted a great deal of research. Fourth, the ER approach has been applied to many real-world decision-making issues, for instance, the oil reserve forecast,[17] motorcycle evaluation,[1–4] strategic R&D projects assessment,[8,9] expert system,[18] new product development,[19] knowledge reduction,[20] risk analysis,[21,22] securities market forecasting,[23] radar fault diagnosis,[24] prequalifying construction contractors,[25] etc. Finally, a window-based and graphically designed decision support software package called intelligent decision system is introduced on the basis of the ER methodology.[26] It provides a flexible and easy-to-use interface for modeling and decision-making analysis. It is capable of solving complex MCDA problems with deterministic belief degrees and certain weights associated with the attributes.

In MCDA problem circumstances, the subjective judgments are often provided by a group of assessors because an individual may be incapable of providing reliable judgments due to a lack of information or experiences. In this situation, the deterministic value associated with the weights of attributes could not always reflect the perspective of all the experts or departments. Moreover, although the relative weight of each criterion is considered in either the recursive[1–4] or analytical[7] ER approach, the relative importance of each criterion is not always provided precisely by the experts because of a lack of information or their limited knowledge. The decision-making process is significantly different from the natural science because many fuzzy or qualitative factors that would lead to the uncertain result or judgment are involved in. In the study of ER approach under fuzziness and uncertainties, Xu has investigated the ER approach for MCDA under interval uncertainties that is caused by interval assessment grades.[27] Yang also studied the ER approach for MCDA under both probabilistic and fuzzy uncertainties in which precise data, ignorance, and fuzziness are all modeled under the unified framework of a distributed fuzzy belief structure.[28]

In both the recursive and analytical ER approaches, the weight of each criterion is associated with an accurate value. This would lead to the failure or distortion of the final results on the combined belief degrees through the ER approach. Guo[29] extended the ER approach to cope with the weights of interval values. In this paper, a triangular fuzzy number is introduced for the representation of weights and the aggregation of total belief degrees is then conducted through several nonlinear programming models.

In the ER approach, the aggregated belief degree on an alternative is an $N + 1$ dimensional vector, which provides a visualized distributed presentation for the assessment of an alternative. It is useful to transfer it into a definite value through utility function.[1–4,7,8] Different utility functions may be estimated by decision-making units (DMUs) depending on their different preferences that would be derived from their discrepancy on backgrounds or value judgments. It may also be changed with the time variable. In this paper, to capture the diversity of preferences from different kind of DMUs, the utility of evaluation grades is extended from accurate values to interval ones and programming models are constructed for the computing of the final assessment value on an alternative through the aggregated $N + 1$ dimensional vector.

2. DEMPSTER–SHAFER'S THEORY OF EVIDENCE

Dempster–Shafer's evidence theory is well suited for handling incomplete information. It is introduced by Dempster[5] in 1967 and refined by Shafer[6] in the publication in 1976, so the theory of evidence is called the D–S theory of evidence. The D–S theory is a realistic approach to handle problems with fuzziness and uncertainty and arouses the interest of experts in the field of artificial intelligence.

In the D–S theory of evidence, a basic hypothesis is denoted by H_n. All hypotheses together constitute a set $H = \{H_1, H_2, \ldots, H_N\}$, which is called the frame of discernment. The hypotheses in H are collectively exhaustive and mutually exclusive and the elements in H could be enumerated by 2^H, which is the power set of H, consisting of all the subsets of H.

Let $m(A)$ denote the basic probability assignment (mass) to the subset A, which measures the extent to which the evidence supports A. It is a number between [0,1], satisfying the following two equations:

$$\sum_{A \subseteq H} m(A) = 1 \quad 0 \le m(A) \le 1 \quad \forall A \subset H \tag{1}$$

$$m(\phi) = 0 \tag{2}$$

where A is called a focal element if it satisfies $m(A) > 0$, $A \subseteq H$, and all the focal elements together are the core of H. $m(A)$ expresses the probability mass exactly assigned to A but not to any subset of A. The assigned probability to H, which is denoted by $m(H)$, is the measurement of the degree of ignorance. It is assumed to be the negation of the hypothesis A if A is the only focal element.

Belief function is another important concept associated with the evidence theory, which is defined as

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad (\forall A \subset H). \tag{3}$$

It reflects the exact support to the hypothesis A and is a function $Bel : 2^H \rightarrow [0, 1]$. $Bel(A)$ is the probability assigned to A considering all the premises of A. There are several other functions associated with the evidence theory, such as the plausibility function, commonality function, etc. Each of them reflects the probability based on the basic probability assignment from different points of view under the frame of discernment.

The kernel of the D–S theory of evidence is the combination rule that could be used for the aggregation of different sources of evidence. Suppose there are N pieces of evidence in H, and each provides a basic probability assignment to a subset A of H, that is, $m_1, m_2, \ldots, m_N$. The evidence combination rule is defined as follows:

$$\begin{aligned} K &= \left(1 - \sum_{\substack{A_1, A_2, \ldots, A_N \subset H \\ A_1 \cap A_2 \cap \cdots \cap A_N = \phi}} m_1(A_1) m_2(A_2) \ldots m_N(A_N)\right)^{-1} \\ &= \left(\sum_{\substack{A_1, A_2, \cdots, A_N \subset H \\ A_1 \cap A_2 \cap \cdots \cap A_N \neq \phi}} m_1(A_1) m_2(A_2) \ldots m_N(A_N)\right)^{-1} \end{aligned} \tag{4}$$

$$m(A) = \begin{cases} 0, & A = \phi \\ K \sum_{\substack{A_1, \cdots, A_N \subset H \\ A_1 \cap \ldots \cap A_N = A}} m_1(A_1) m_2(A_2) \ldots m_N(A_N), & A \neq \phi \end{cases}. \tag{5}$$

where K is called the normalization factor and $K > 1$. It is the degree of conflict that reflects the conflict between N pieces of evidence. In the combination rule, it is assumed that the information sources are independent and the process of calculation is called the orthogonal sum.

Dempster's rule of combination satisfies commutativity and associativity of multiplication. As such, it ensures that the combination results remain the same regardless of the order in which the N pieces of evidence are aggregated.

3. THEORY OF TRIANGULAR FUZZY NUMBER

In this section, the theory of triangular fuzzy number is briefly described.

DEFINITION 1. *Suppose R as a set of real number, M, is defined as a triangular fuzzy number in the bound of R if the subjection degree function of M could be represented*

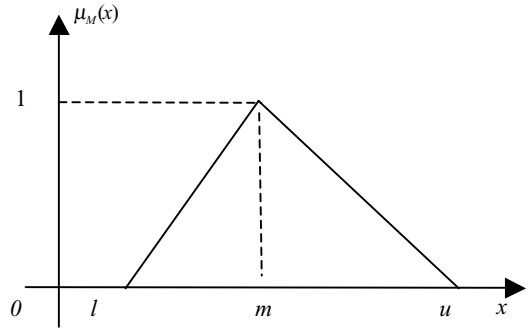

Figure 1. Geometrical explanation of triangular fuzzy number (l, m, u).

as:

$$\mu_M(x) = \begin{cases} 0 & x \in (-\infty, l] \cup [u, +\infty) \\ \dfrac{1}{m-l}x - \dfrac{l}{m-l} & x \in [l, m] \\ \dfrac{1}{m-u}x - \dfrac{u}{m-u} & x \in [m, u] \end{cases} \tag{6}$$

where $l \leq m \leq u$; l and u are the lower and upper bound of fuzzy number M, respectively; and m is the middle value of M, with the degree of 1 subjected to R. The triangular fuzzy number is denoted by (l, m, u). The geometrical explanation of triangular fuzzy number is shown in Figure 1.

The operational rules of triangular fuzzy number could be found in Ref. 30.

In this paper, the relative importance of criterion e_i is denoted by $\omega_i = (\omega_i^l, \omega_i^m, \omega_i^u)$. It denotes that the relative degree of importance of the ith factor is ω_i^m, whereas ω_i^l and ω_i^u are the right and left extensions of ω_i, respectively. They denote the most conservative assessment and the most optimistic assessment of experts, respectively. The distance of $|\omega_i^u - \omega_i^l|$ denotes the fuzzy degree of the relative importance, that is, the larger the absolute value, the fuzzier the judgment. When $|\omega_i^u - \omega_i^l| = 0$, the judgment is not fuzzy.

4. EVIDENTIAL REASONING APPROACH

The ER approach is one of MCDA methods such as AHP,[31,32] TOPSIS, ELECTRE, and PROMETHEE. Different from other MCDA methods, the ER approach utilized the belief decision matrix (BDM), in which each element of the matrix is a vector compared with the single value in other MCDA methods. In the BDM, a set of linguistic evaluation grades for the assessment of a factor on an alternative is defined as follows:

$$H = \{H_1, H_2, \ldots, H_N\} \tag{7}$$

It is equivalent to the frame of discernment described in the D–S evidence theory in Section 2, and $H_n (n = 1, 2, \ldots, N)$ each denotes an evaluation grade. They are collectively exhaustive and mutually exclusive.[2–4] H_1 represents the least preferred evaluation grade and H_N represents the most preferred evaluation grade. H_{n+1} is assumed to be preferred over H_n. The symbols and expressions in the ER approach are described as follows:

e_i the ith criterion;
a_l the lth alternative;
$E_{I(i)}$ set of the first i criteria associated with their upper-level criterion, $E_{I(i)} = \{e_1, \ldots, e_i\}$;
ω_i the relative importance (weight) of the ith criterion e_i, and $\sum_{i=1}^{L} \omega_i = 1$;
$\beta_{n,i}(a_l)$ the intensity to which the state of the ith criterion e_i at a_l is confirmed to a linguistic evaluation degree H_n;
$\beta_{H,i}(a_l)$ the degree of uncertainty that the lth alternative a_l is assessed on the ith criterion e_i;
$m_{n,i}(a_l)$ the basic probability assignment (mass) of a_l being assessed to the assessment grade H_n on the basic criterion e_i;
$m_{H,i}(a_l)$ the remaining probability mass unassigned to any individual grade after all the N grades have been considered for assessing the general attribute as far as e_i is concerned; and
$S(e_i(a_l))$ the state (vector of belief degree) of a basic criterion e_i to be evaluated on a_l to grade H_n, $S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)), n = 1, 2, \ldots, N; H, \beta_{H,i}(a_l)\}$.

$$(i = 1, 2, \ldots, L; n = 1, 2, \ldots, N; l = 1, 2, \ldots, S)$$

In the statement, $0 \le \beta_{n,i}(a_l) \le 1$ and $\sum_{n=1}^{N} \beta_{n,i}(a_l) \le 1$ are two basic conditions for $\beta_{n,i}(a_l)$. If $\beta_{n,i}(a_l)$ $(n = 1, 2, \ldots, N)$ sums to unity, the information provided by a DMU is said to be complete. Otherwise, if $\sum_{n=1}^{N} \beta_{n,i}(a_l) < 1$, then the judgment is incomplete.

On the basis of these symbols and expressions, a BDM may be modeled to represent the belief degrees that are to be assigned to all the alternatives on all criteria as follows:

$$\text{BDM} = (S(e_i(a_l)))_{L \times S}. \tag{8}$$

where L is the total number of basic attributes and S is the number of alternatives been assessed.

The ER approach is applied to deal with MCDA problem for the aggregation of multiple criteria based on the BDM and the D–S combination theory. In the ER approach, each criterion is assigned with belief degrees on several linguistic evaluation grades, as denoted in (7) for the consideration to assessing the subjective uncertainties and ambiguities of both quantitative and qualitative criteria simultaneously. The ER algorithms provide two systematic processes for the aggregation of assessment information. They are the recursive ER algorithm[1–4] and the analytical

ER algorithm,[7] as mentioned in Section 1. The latter one is an improved ER arithmetic to the former one. In the analytical ER approach, the evidence combination process is conducted only once instead of $L-1$ times in the recursive ER approach. Thus, the complexity of the ER model is simplified and the efficiency of evidence combination is enhanced greatly. The recursive ER algorithm is shown as follows:

$$m_{n,i}(a_l) = \omega_i \beta_{n,i}(a_l), \quad i = 1, 2, \ldots, L, \quad n = 1, 2, \ldots, N, l = 1, 2, \ldots, S \tag{9}$$

$$m_{H,i}(a_l) = 1 - \sum_{n=1}^{N} m_{n,i}(a_l) = 1 - \omega_i \sum_{n=i}^{N} \beta_{n,i}(a_l). \tag{10}$$

$$\beta_{H,i}(a_l) = 1 - \sum_{n=1}^{N} \beta_{n,i}(a_l). \tag{11}$$

$$\begin{aligned} m_{n,I(i+1)}(a_l) = {} & \frac{1}{k_{I(i+1)}}[m_{n,I(i)}(a_l)m_{n,i+1}(a_l) + m_{n,I(i)}(a_l)m_{H,i+1}(a_l) \\ & + m_{H,I(i)}(a_l)m_{n,i+1}(a_l)] \end{aligned} \tag{12}$$

$$m_{H,I(i+1)}(a_l) = \frac{1}{k_{I(i+1)}}[m_{H,I(i)}(a_l)m_{H,i+1}(a_l)] \tag{13}$$

$$k_{I(i+1)} = 1 - \sum_{t=1}^{N} \sum_{\substack{j=1 \\ j \neq t}}^{N} m_{t,I(i)}(a_l)m_{j,i+1}(a_l) \tag{14}$$

where $m_{n,I(i+1)}(a_l)$ is the combined probability assignment to H_n ($n = 1, 2, \ldots, N$, H) on a_l generated by assessing $E_{I(i+1)}$. Suppose $\beta_n(a_l)$ is the total belief degree that the lth alternative a_l be assessed on linguistic evaluation grade H_n and $\beta_H(a_l)$ is the degree of belief unassigned to any individual evaluation grade after all the L basic attributes have been assessed. Then, we have

$$\beta_n(a_l) = \delta m_{n,I(L)}(a_l), \quad \delta = \frac{1 - \beta_H(a_l)}{1 - m_{H,I(L)}(a_l)}. \quad n = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, S \tag{15}$$

$$\beta_H(a_l) = \sum_{i=1}^{L} \omega_i \beta_{H,i}(a_l). \quad l = 1, 2, \ldots, S \tag{16}$$

where

$$\beta_H(a_l) + \sum_{n=1}^{N} \beta_n(a_l) = 1. \tag{17}$$

From the ER approach, the aggregated belief degrees that are to be assigned to a_l could then be represented by a $N+1$ dimensional vector as follows:

$$S(y(a_l)) = \{(H_n, \beta_n(a_l)), n = 1, 2, \ldots, N; H, \beta_H(a_l)\}. \quad (l = 1, 2, \ldots, S) \quad (18)$$

5. THE HADAMARD MULTIPLICATIVE COMBINATION OF JUDGMENT MATRIX FOR COMPUTING TRIANGULAR FUZZY WEIGHTS

Considering the complexity of handling multiple criteria simultaneously, it is important to get a group of experts involved for assigning criteria weights. However, experts are always achieving different degree of consistency because they have different backgrounds or expertise and may represent conflicting interests. The Hadamard multiplicative combination of judgment matrix[33–35] is an ideal method for the aggregation of multiple matrices that are constructed by different experts. It could minimize the impact of inconsistency among individual expert's judgment and opinion. The theory of the Hadamard multiplicative combination of judgment matrix is shown in the Appendix.

Because of the complexity of decision-making environment, the comparison of every two criteria is not always definitely fixed to a certain value in the hierarchy of Saaty's 1–9 rating, the triangular fuzzy judgment matrix is then proposed by Laarhoven,[36] in which each element of the matrix is a triangular fuzzy number. Here, the Hadamard multiplicative combination of judgment matrix is extended for the combination of triangular fuzzy judgment matrices.

DEFINITION 2. *$A = (a_{ij})_{n\times n}$ $(i, j = 1, 2, \ldots, n)$ is a $n \times n$ triangular fuzzy judgment matrix if the elements a_{ij} are represented as triangular fuzzy number as follows:*

$$a_{ij} = (l_{ij}, m_{ij}, u_{ij}). \quad (19)$$

DEFINITION 3. *Suppose $A_1, A_2, \ldots, A_m$ as a set of triangular fuzzy judgment matrices given by m experts, $A_t = (a_{ij}^{(t)})_{n\times n} = (l_{ij}^{(t)}, m_{ij}^{(t)}, u_{ij}^{(t)})_{n\times n}$ $(t = 1, 2, \ldots, m)$, $\overline{A}$ is defined as a Hadamard multiplicative combination from $A_1, A_2, \ldots, A_m$. Let us assume*

$$\overline{A} = (\bar{a}_{ij})_{n\times n} = A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m}. \quad (20)$$

where $\lambda_t \in [0, 1] \quad (t = 1, 2, \ldots, m)$, $\sum_{t=1}^{m} \lambda_t = 1$,

$$\bar{a}_{ij} = (a_{ij}^{(1)})^{\lambda_1}(a_{ij}^{(2)})^{\lambda_2} \ldots (a_{ij}^{(m)})^{\lambda_m} = \prod_{t=1}^{m} (l_{ij}^{(t)}, m_{ij}^{(t)}, u_{ij}^{(t)})^{\lambda_t}$$

$$= \left(\prod_{t=1}^{m} l_{ij}^{(t)^{\lambda_t}}, \prod_{t=1}^{m} m_{ij}^{(t)^{\lambda_t}}, \prod_{t=1}^{m} u_{ij}^{(t)^{\lambda_t}} \right) \quad (i, j = 1, 2, \ldots, n) \tag{21}$$

In the real decision-making problem, n represents the number of criteria and $\lambda_t (t = 1, 2, \ldots, m)$ is the relative importance of each expert. If there are m experts involved in the decision-making problem for the generation of criteria weights, then there will be m judgment matrices constructed by each of them to be aggregated.

THEOREM 1. *Assume $A_1, A_2, \ldots, A_m$ as a set of triangular fuzzy judgment matrices given by m experts for the same problem and satisfy the requirement of consistency, then the Hadamard multiplicative combination $\overline{A} = A_1^{\lambda_1} A_2^{\lambda_2} \cdots A_m^{\lambda_m}$ meets the requirement of consistency as well.*

Theorem 1 is called consistency theorem. It shows that if the judgment presented by each expert is consistent, then the aggregated judgment calculated by the Hadamard multiplicative combination is consistent too. The proof is shown in the Appendix.

After the combination of m triangular fuzzy judgment matrices constructed by m experts, an aggregated fuzzy judgment matrix could then be generated. Fuzzy analytic hierarchy process (FAHP) approach[36] is then used to calculate the triangular fuzzy weight of each criterion. The calculated weight is also in the form of triangular fuzzy number.

6. ER-BASED NONLINEAR PROGRAMMING MODELS FOR COMPUTING BELIEF DEGREES UNDER FUZZY WEIGHTS

After the generation of combined triangular fuzzy weights, the total fuzzy belief degrees are calculated on the basis of the ER approach. In the field of fuzzy ER approach, some researches have been conducted. Denoeux[37] has systematically explored the combination and normalization of interval belief degrees. Yager[38] studied the combination and normalization of interval evidence too. Lee[39] also studied the combination of interval evidence. But their approaches for dealing with interval-valued belief structures are not really satisfied, which is illustrated by Wang.[40,41] Wang[41] studied the ER approach under interval belief degrees in which the combination and normalization process are optimized simultaneously, and Guo[29] further extended the ER approach in dealing with both interval belief degrees and interval weights.

6.1. Programming Models for Computing Belief Degrees under Fuzzy Weights

On the basis of the ER approach and the calculated combined triangular fuzzy weights described in Section 5, nonlinear programming models for ER under fuzzy weights could be constructed according to Guo's programming models[29] as follows. To calculate the maximum and minimum values of $\beta_n(a_l)$, the following model is constructed:

$$\max / \min \ \beta_n(a_l) = \delta m_{n,I(L)}(a_l) \quad (n = 1, 2, \ldots, N;\ l = 1, 2, \ldots, S) \tag{22}$$

s.t.

$$\begin{cases} \text{Equations 9–14, and 16} \\ \delta = \dfrac{1 - \beta_H(a_l)}{1 - m_{H,I(L)}(a_l)} & (23) \\ \omega_i^l \le \omega_i \le \omega_i^u, \quad i = 1, 2, \ldots, L & (24) \\ \displaystyle\sum_{i=1}^{L} \omega_i = 1 & (25) \end{cases}$$

This programming model contains L variables, which could be computed by MATLAB software or by Microsoft Office Excel. Suppose $\beta_n^u(a_l)$ and $\beta_n^l(a_l)$ $(n = 1, 2, \ldots, N)$ are the optimized values of the objective function in this ER-based nonlinear programming model, respectively. Then, the aggregated fuzzy belief degree of a_l $(l = 1, 2, \ldots, S)$ that is assigned to $H_n (n = 1, 2, \ldots, N)$ will be obtained, which could be denoted by $\beta_n(a_l) \in [\beta_n^l(a_l), \beta_n^u(a_l)] (n = 1, 2, \ldots, N)$.

The nonlinear programming model for the belief degree of total uncertainty $\beta_H(a_l)$ could also be constructed as follows:

$$\max / \min \beta_H(a_l) = \sum_{i=1}^{L} \omega_i \left(1 - \sum_{n=1}^{N} \beta_{n,i}(a_l)\right) \quad (l = 1, 2, \ldots, S) \tag{26}$$

s.t. Equations 9–14, 24, and 25.

Suppose $\beta_H^u(a_l)$ and $\beta_H^l(a_l)$ $(n = 1, 2, \ldots, N)$ are the optimized values of this ER-based nonlinear programming model and $\beta_H(a_l) \in [\beta_H^l(a_l), \beta_H^u(a_l)]$. On the basis of this ER-based optimization processes, the aggregated belief degrees of the lth alternative a_l could then be represented by an $N + 1$ dimensional vector as follows:

$$\begin{aligned} S(a_l) = \{&(H_n, \beta_n \in [\beta_n^l(a_l), \beta_n^u(a_l)]), \\ &n = 1, 2, \ldots, N;\ H, \beta_H \in [\beta_H^l(a_l), \beta_H^u(a_l)]\} \end{aligned} \tag{27}$$

where

$$\beta_H(a_l) + \sum_{n=1}^{N} \beta_n(a_l) = 1. \tag{28}$$

Equation 27 is an improved concept of Equation 18, in which each element of the $N + 1$ dimensional vector is extended to fuzzy value. It provides a distributed view to the assessment of a_l after combining all the criteria associated with the evaluation framework. From this description, it is clear that there are $(N + 1) \times 2$ nonlinear programming models for the calculation of the aggregated belief degrees of each alternative.

If the belief degree of one basic criterion is incomplete or fuzzy, then from the ER algorithm, we could see that the aggregated belief degrees on the general level are incomplete too. It could also be shown as:

$$\text{if } \exists\, i \in \{1, 2, \ldots, L\}, \quad \beta_{H,i}(a_l) = 1 - \sum_{n=1}^{N} \beta_{n,i}(a_l) > 0,$$

$$\text{then, } \beta_H(a_l) = 1 - \sum_{n=1}^{N} \beta_n(a_l) > 0. \quad l = 1, 2, \ldots, S$$

From Equation 27, it is clear that the total combined belief degree is still a vector of $N + 1$ dimensions, so the comparison and analysis with several collected alternatives $(a_l, (l = 1, 2, \ldots, S))$ from the vectors $(S(a_l), (l = 1, 2, \ldots, S))$ seems impossible and unreliable.

6.2. Center of Gravity Method

According to the operation features of triangular fuzzy value and the uncertainty, center of gravity method (CGM) could be used to transfer the triangular fuzzy weights into nonfuzzy value CGM. The equation is as follows:

$$\bar{\omega}_i = \left[\left(\omega_i^u - \omega_i^l\right) + \left(\omega_i^m - \omega_i^l\right)\right]/3 + \omega_i^l \quad (i = 1, 2, \ldots, L) \tag{29}$$

This equation calculated by CGM should be standardized as follows:

$$\bar{\omega}_i' = \bar{\omega}_i \Big/ \sum_{i=1}^{L} \bar{\omega}_i \quad (i = 1, 2, \ldots, L) \tag{30}$$

From the deterministic values of weights calculated by this equation and the ER algorithm (Equations 9–16), the combined belief degree could also be obtained for a selected alternative. The result is not fuzzy because the weights calculated by Equations 29 and 30 are accurate numbers. So, it is necessary for the comparison

between the aggregated belief degrees using the weights calculated by Equations 29–30 and the results using nonlinear programming models for ER methodology under fuzzy weights. By combining the fuzzy belief degrees (Equation 27) calculated by the $(N+1) \times 2$ ER-based nonlinear programming models and the certain belief degrees (Equation 18) computed by the recursive ER algorithm, a triangular fuzzy distributed assignment could then be constructed as follows:

$$S(a_l) = \{(H_n, \beta_n \in [\beta_n^l(a_l), \beta_n^m(a_l), \beta_n^u(a_l)]), \\ n = 1, 2, \ldots, N;\ H, \beta_H \in [\beta_H^l(a_l), \beta_H^m(a_l), \beta_H^u(a_l)]\} \tag{31}$$

where $\beta_n^m(a_l)$ and $\beta_H^m(a_l)$ are calculated by the recursive ER algorithm under the weights computed by Equations 29–30.

7. ER-BASED NONLINEAR PROGRAMMING MODELS FOR UTILITY FUNCTION UNDER FUZZY WEIGHTS AND UTILITIES

7.1. Expected Utility Theory

Equation 27 provides the decision maker with a panoramic view about the assessment of the alternative; however, it is not straightforward to use the distributed assessments for ranking alternatives. For the convenience of the ranking of several options, it is necessary to calculate the expected utility of each of them. In the ER approach under certain weights and utilities, the maximum utility, minimum utility, and average utility in both the recursive and analytical ER approaches are calculated for the transformation of the $N+1$ dimensional vector to a total utility value.[1–4,7–9] The expected utility of the aggregated assessment $S(a_l)$ is defined as follows:

$$u(y(a_l)) = \sum_{n=1}^{N} \beta_n(a_l) u(H_n) \tag{32}$$

To calculate this equation, a function must be defined for the assessment grades $H_n (n = 1, 2, \ldots, N)$, which are mutually exclusive and collectively exhaustive. They may be defined precisely or roughly, which depends on the need of real application. In Refs. 1 and 2, the concept of preference degree ($p(H_n)$) is introduced and a utility function ($u(H_n)$) is used in Refs. 3, 4, 7–9, 25. $p(H_n)$ takes the value in the close interval [−1,1], whereas $u(H_n)$ is estimated from 0 to 1. There are several utility estimation methods. For example, Winston[42] introduced an alternative technique "the probability assignment approach," and three approaches to estimate utilities on a quantitative attribute are discussed in Ref 3.

This illustrated process is also called utility function decision-making (UFDM) approach, in which the value of expected utility is considered as the evaluation rule of decision making. The lager the expected value is, the better an alternative is considered and will possibly be collected. Utility represents the degree of preference that a DMU considers the value of an option. So, different types of utility function

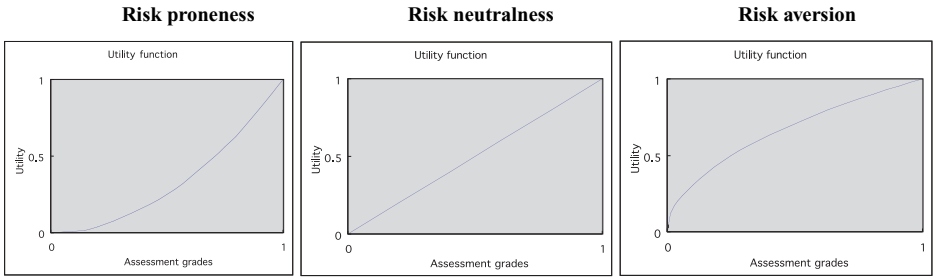

Figure 2. Three different types of utility function depending on the preference of DMU towards risk: risk proneness, risk neutralness, and risk aversion.

may be constructed to show the attitude of different DMUs toward risk. Three types of utility functions could be constructed; they are risk proneness, risk neutralness, and risk aversion, respectively, which are shown in Figure 2.

The characteristic of UFDM is that the subjective judgment of a DMU is involved in the utility value. There are several stochastic and uncertain factors in most MCDA problems. In the process of alternative selection, a DMU is always embedded with risk preference and the utility function is quite different among different DMUs, time variables, and places. In other words, the utility of the same evaluation grade to different experts is different, as well as in different time variables and places. So, in the real assessment problem under group decision-making circumstance, the utility of an evaluation grade being estimated by a group of experts is not an accurate value anymore. In this paper, utility of each assessment grade is estimated as a fuzzy number, whose value is between the interval $[u_n^l, u_n^u]$ or is represented by

$$u_n^l \leq u(H_n) \leq u_n^u \tag{33}$$

7.2. ER-Based Nonlinear Programming Model for Utility Function

Considering that the weights are often presented as fuzzy numbers and evaluation-grade utility estimation associated with each expert may be uncertain or ambiguity due to the complexity of real decision making problems, then these defined expected utility measures are no longer certain or accurate. Under this circumstance, it is more realistic and reliable to calculate the overall maximum and minimum utilities of each alternative on a holistic view instead of the upper and lower bounds of $u(y(a_l))$ calculated after the fuzzy belief degrees of each alternative have been obtained.

On the basis of the ER approach in the situation of fuzzy weights and utilities, the ER-based nonlinear programming model for utility function is constructed. Suppose the utility of linguistic evaluation grade H_n is $u(H_n)$ $(n = 1, 2, \ldots, N)$ and $u(H_{n+1}) > u(H_n)$.

$$\max/\min u'(y(a_l)) = u(y(a_l)) + \beta_H(a_l)u'(H)(l = 1, 2, \ldots, S) \tag{34}$$

s.t.

$$\begin{cases} \text{Equations 9–16, 23–25.} \\ u(y(a_l)) = \sum_{n=1}^{N} \beta_n(a_l) u(H_n) \\ u_n^l \le u(H_n) \le u_n^u \quad n = 1, 2, \ldots, N \end{cases}$$

where $u'(H) = u\left(H_{\frac{N+1}{2}}\right)$ when N is an odd number and $u'(H) = \frac{1}{2}\left[u\left(H_{\frac{N}{2}}\right) + u\left(H_{\frac{N}{2}+1}\right)\right]$ when N is an even number. In this optimization model, $L + N$ variables are contained. From the ER-based nonlinear programming model for computing expected utilities of alternative a_l, the overall maximum and minimum expected utilities could then be obtained, which are respectively presented by $U^u(a_l)$ and $U^l(a_l)$. Here, we assume that the minimum and maximum values of utility are 0 and 100, respectively. So, the total fuzzy utility of each alternative can be represented as follows:

$$U(a_l) = [U^l(a_l), U^m(a_l), U^u(a_l)] \tag{35}$$

where $U^m(a_l)$ is calculated with the belief degrees under the weights calculated by standardized *CGM* and the middle value of each utility interval in formula (33).

8. APPLICATION

In the vendor management inventory (VMI) alliance, it is important for a core enterprise to construct a set of scientific and rational criteria structures for the general assessment and selection of alliance enterprises. The rationality of assessment hierarchical structure directly affects the effectiveness of performance of VMI alliance enterprises. Thus, the strategic decision-making process of the core enterprise and the construction of supply chain management are influenced next.

In this section, the fuzzy evidential reasoning (FER) approach is applied to analyze the performance of VMI alliance. The research was conducted in close collaboration with the leaders and members in most departments of several accessory vendors associated with a core enterprise. A ranking order could be obtained on the general performance of several alliance corporations in the period of assessment through the investigation to their real management. The suppliers that have satisfactory performance could then be authorized to have cooperation qualifications by the core enterprise, whereas the suppliers that ranked low should be supervised to improve their management strategies for the purpose to achieve the requirement of the core enterprise and VMI alliance. Four steps are mentioned as follows:

Step 1: The Construction of the Criteria Framework for Assessment

In this paper, the balanced scorecard (BSC)[43–45] is introduced as the evaluation criteria to assess the performance of several vendors. Apart from the BSC assessment framework, many questionnaires of this evaluation system were sent to the faculties

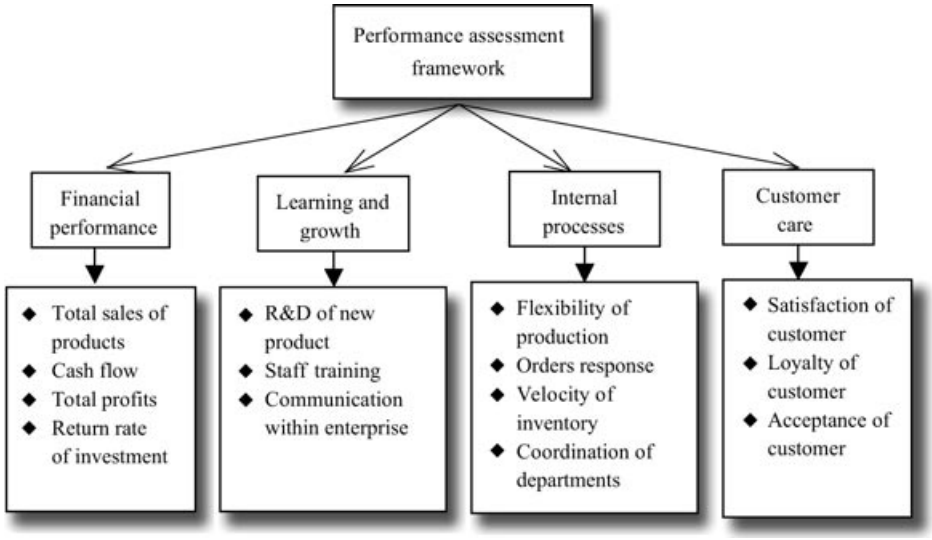

Figure 3. Evaluation hierarchy for the performance of corporation in the VMI alliance.

of these suppliers to check its efficiency and reliability. The improved framework of BSC and its three-level attributes are shown in Figure 3.

The balances of attributes in the BSC are represented as the following two aspects:

First, short-term and long-term balances. To obtain the maximum profit, continuous growing and profit must be depended on. An enterprise should manage the relation of short-term and long-term behaviors efficiently for the continuous development.

Second, financial and nonfinancial balances. Besides the financial attributes, many nonfinancial attributes such as process control, learning and innovation, and customer care should also be considered simultaneously in the assessment for the development and improvement of performance.

Step 2: Acquiring Belief Degrees and the Fuzzy Weights of Criteria

Suppose a supplier could be evaluated by the following six linguistic evaluation grades: *Bad*, *Poor*, *Average*, *Good*, *Very good*, and *Excellent*, which form the set of frame of discernment as follows:

$$\begin{aligned} H &= \{H_1, H_2, H_3, H_4, H_5, H_6\} \\ &= \{\text{Bad, Poor, Average, Good, Very good, Excellent}\} \end{aligned} \tag{36}$$

Three alternatives that represent three suppliers associated with a core enterprise are involved in our assessment process. They form the set of alternatives as follows:

$$A = \{a_1, a_2, a_3\} \tag{37}$$

Table I. Generalized decision matrix for VMI assessment and belief degrees of each criterion.

General attributes	Subcriteria	Belief degrees		
		a_1	a_2	a_3
Financial performance (E_1)	Total sales (e_{11})	(H_1, 0.1; H_2, 0.2; H_3, 0.7)	(H_3, 0.4; H_4, 0.5; H_5, 0.1)	(H_4, 0.3; H_5, 0.6)
	Cash flow (e_{12})	(H_3, 0.6; H_4, 0.4)	(H_5, 0.3; H_6, 0.7)	(H_2, 0.1; H_3, 0.9)
	Total profits (e_{13})	(H_4, 0.7; H_5, 0.2)	(H_2, 0.9)	(H_4, 1.0)
	Return rate (e_{14})	(H_3, 0.5; H_4, 0.5)	(H_3, 0.5; H_4, 0.4; H_5, 0.1)	(H_5, 0.7; H_6, 0.3)
Learning and growth (E_2)	R&D (e_{21})	(H_5, 0.4; H_6, 0.5)	(H_2, 0.5; H_3, 0.4)	(H_3, 0.2; H_4, 0.2; H_5, 0.5; H_6, 0.1)
	Staff training (e_{22})	(H_5, 1.0)	(H_4, 0.6; H_5, 0.3; H_6, 0.1)	(H_5, 0.9)
	Communication (e_{23})	(H_4, 0.2; H_5, 0.8)	(H_4, 0.8; H_5, 0.2)	(H_1, 0.2, H_2, 0.3, H_3, 0.5)
Internal processes (E_3)	Flexibility of production (e_{31})	(H_2, 0.1; H_3, 0.2; H_4, 0.5; H_5, 0.2)	(H_3, 0.5; H_4, 0.5)	(H_2, 0.6, H_3, 0.4)
	Orders response (e_{32})	(H_3, 0.2; H_4, 0.2; H_5, 0.6)	(H_2, 0.8)	(H_4, 0.8; H_5, 0.2)
	Velocity of Inventory (e_{33})	(H_3, 1.0)	(H_3, 0.7; H_4, 0.3)	(H_3, 0.9)
	Coordination of departments (e_{34})	(H_4, 0.2; H_5, 0.4; H_6, 0.4)	(H_5, 0.9)	(H_5, 0.6; H_6, 0.4)
Customer care (E_4)	Satisfaction of customer (e_{41})	(H_2, 0.5; H_3, 0.4; H_4, 0.1)	(H_3, 0.3; H_4, 0.6)	(H_4, 0.5; H_5, 0.5)
	Loyalty of customer (e_{42})	(H_3, 0.6; H_4, 0.3)	(H_4, 0.9; H_5, 0.1)	(H_2, 0.1; H_3, 0.3; H_4, 0.4; H_5, 0.2)
	Acceptance of customer (e_{43})	(H_4, 0.6; H_5, 0.2)	(H_4, 0.3; H_5, 0.7)	(H_3, 0.1; H_4, 0.9)

After some investigations, the belief degrees assigned to these three suppliers (a_i, ($i = 1, 2, 3$)) on each criterion were captured and shown in Table I.

From Table I, a distributed view of the belief degree assignment on each supplier can be seen clearly. For example, the "total profits" for a_1 is evaluated to be good with a belief degree of 0.7, and to be very good with belief degree of 0.2." This statement could be represented by the following expectation:

$$S(\text{Total profits}) = \{(\text{Good}, 0.7), (\text{Very good}, 0{,}2)\}. \tag{38}$$

In the statement, the total belief degree for the statement of "total profits" does not sum to 1, which means the information provided by experts is not complete or is partly ignorant. In other words, the incompleteness of the assessment to "total profits" is 0.1, which may have been resulted from the uncertainty of the decision environment.

Considering the complexity of handling multiple criteria simultaneously, it is important to get a group of experts involved who may have different background or

expertise and may represent conflicting interests for assigning the weights of criteria. For such a group of experts, a question appears as how to achieve an aggregated group judgment from individual expert's estimation of criteria weights.

At first, criteria that have great relevance to the assessment should be selected. Here, the BSC is considered as the assessment structure. Then, pairwise comparisons between every two criteria are provided by each expert in the form of triangular fuzzy number to construct his or her judgment matrix that is then used to generate a ranking vector by each expert.[30,36] The triangular fuzzy judgment matrix of each expert can then be aggregated to form an overall triangular fuzzy judgment matrix. There are a few approaches, such as the ideal synthesis matrix in a group context, additive convex combination, to combine the subjective judgment matrix. In this paper, the Hadamard multiplicative combination of judgment matrix is to be extended to combine triangular fuzzy judgment matrices and then implemented to construct an overall triangular fuzzy pairwise matrix. This approach could not only eliminate the impact of subjectivity but also maintain and improve the consistency of judgment matrix (see the Appendix). Then, the FAHP approach is applied to calculate the weight of each criterion. For the consideration of the length of this paper, the detail of the FAHP to compute fuzzy weights is not mentioned here. The calculated fuzzy weights of criteria are shown in Table II.

In Table II, the weight of each criterion is in the form of a triangular fuzzy number. For instance, the relative importance of "total profits" associated with its upper-level criterion "financial performance" is between interval [0.237, 0.396] and is considered to be 0.310 to the largest extent.

Table II. The values about fuzzy weights of criteria.

General attributes	Fuzzy weight (ω_i)	Basic attributes	Fuzzy weight (ω_{ij})		
			ω_{ij}^l	ω_{ij}^m	ω_{ij}^u
Financial performance (E_1)	(0.277, 0.333, 0.427)	Total sales (e_{11})	0.193	0.287	0.384
		Cash flow (e_{12})	0.157	0.235	0.325
		Total profits (e_{13})	0.237	0.31	0.396
		Return rate (e_{14})	0.122	0.168	0.208
Learning and growth (E_2)	(0.174, 0.239, 0.345)	R&D (e_{21})	0.351	0.413	0.593
		Staff training (e_{22})	0.258	0.327	0.513
		Communication (e_{23})	0.187	0.26	0.341
Internal processes (E_3)	(0.226, 0.284, 0.342)	Flexibility of production (e_{31})	0.193	0.269	0.355
		Orders response (e_{32})	0.228	0.338	0.47
		Velocity of inventory (e_{33})	0.164	0.219	0.273
		Coordination of departments (e_{34})	0.113	0.174	0.229
Customer care (E_4)	(0.106, 0.144, 0.204)	Satisfaction of customer (e_{41})	0.398	0.46	0.547
		Loyalty of customer (e_{42})	0.257	0.319	0.409
		Acceptance of customer (e_{43})	0.162	0.221	0.306

Step 3: The Calculation Process of the Aggregated Fuzzy Belief Degrees

The fuzzy weights in Table I are still of relative importance for the basic criteria compared with their upper-level attributes. To apply the ER-based nonlinear programming model, they need to be transferred to abstract fuzzy weights on the general level as follows:

$$\omega'_{ij} = \omega_i \cdot \omega_{ij}^T = \left(\omega_i^l, \omega_i^m, \omega_i^u\right) \cdot \left(\omega_{ij}^l, \omega_{ij}^m, \omega_{ij}^u\right) = \left(\omega_i^l \cdot \omega_{ij}^l, \omega_i^m \cdot \omega_{ij}^m, \omega_i^u \cdot \omega_{ij}^u\right) \quad (39)$$

For example, to calculate the abstract fuzzy weights of "total profits" (e_{13}), we have

$$\omega'_{13} = \omega_1 \cdot \omega_{13}^T = (0.277, 0.333, 0.427) \cdot (0.237, 0.310, 0.396)^T$$

$$= (0.065649, 0.103230, 0.169092).$$

The abstract triangular fuzzy weights of all the attributes are shown in the following text.

After the calculation of abstract weights, the ER-based nonlinear programming algorithm for computing belief degrees could then be applied for the calculation of the total fuzzy belief degrees. In this application, there are 14 variables altogether contained in the optimization model for the calculation of total belief degrees. The combined fuzzy belief degrees of the three assessed suppliers are shown in Table IV as follows.

Each of the assessed suppliers could also be evaluated on the basis of the deterministic weights calculated by the standardized CGM in Table III, and the aggregated belief degrees of each alternative are shown in Table V.

Table III. The abstract weights of basic attributes.

Fuzzy weights	$\omega_{ij}^{1'}$	$\omega_{ij}^{m'}$	$\omega_{ij}^{u'}$	CGM ($\bar{\omega}_{ij}$)	Standardized CGM ($\bar{\omega}'_{ij}$)
Total sales (e_{11})	0.053461	0.095571	0.163968	0.104333	0.093422
Cash flow (e_{12})	0.043489	0.078255	0.138775	0.08684	0.077758
Total profits (e_{13})	0.065649	0.10323	0.169092	0.112657	0.100875
Return Rate (e_{14})	0.033794	0.055944	0.088816	0.059518	0.053293
R&D (e_{21})	0.061074	0.098707	0.204585	0.121455	0.108753
Staff training (e_{22})	0.044892	0.078153	0.176985	0.10001	0.089551
Communication (e_{23})	0.032538	0.06214	0.117645	0.070774	0.063373
Flexibility of Production (e_{31})	0.043618	0.076396	0.12141	0.080475	0.072058
Orders response (e_{32})	0.051528	0.095992	0.16074	0.102753	0.092007
Velocity of inventory (e_{33})	0.037064	0.062196	0.093366	0.064209	0.057494
Coordination of departments (e_{34})	0.025538	0.049416	0.078318	0.051091	0.045747
Satisfaction of customer (e_{41})	0.042188	0.06624	0.111588	0.073339	0.065669
Loyalty of customer (e_{42})	0.027242	0.045936	0.083436	0.052205	0.046745
Acceptance of customer (e_{43})	0.017172	0.031824	0.062424	0.03714	0.033256

Table IV. The total aggregated fuzzy belief degrees of the three assessed suppliers.

	a_1		a_2		a_3	
	β_n^l	β_n^u	β_n^l	β_n^u	β_n^l	β_n^u
H_1	0.004458	0.016252	0	0	0.00514	0.022354
H_2	0.030317	0.098559	0.113428	0.422913	0.034664	0.128283
H_3	0.149168	0.483645	0.109334	0.341148	0.108115	0.394243
H_4	0.129952	0.387484	0.16506	0.506615	0.169402	0.531029
H_5	0.143606	0.517532	0.067826	0.250429	0.16461	0.517851
H_6	0.039407	0.159482	0.028893	0.113984	0.021186	0.074303
H	0.012724	0.037738	0.029751	0.082747	0.013542	0.043432

Table V. The total aggregated belief degrees on the weights calculated by standardized CGM.

$\beta_n^m (n = 1, \ldots, N, H)$	H_1	H_2	H_3	H_4	H_5	H_6	H
a_1	0.008522	0.053488	0.287124	0.244145	0.307745	0.077564	0.021413
a_2	0	0.21721	0.202222	0.331955	0.13983	0.058277	0.050506
a_3	0.011155	0.067692	0.215534	0.31426	0.326894	0.040419	0.024047

From Tables IV and V, a clear distributed presentation of the total assessment on each evaluation grade is obtained. We combine Tables IV and V, and the aggregated triangular fuzzy belief degrees of the three assessed suppliers could then be constructed according to formula (31). The following three figures are the presentation of the global triangular fuzzy belief degrees of a_1, a_2, and a_3 (Figures 4–6).

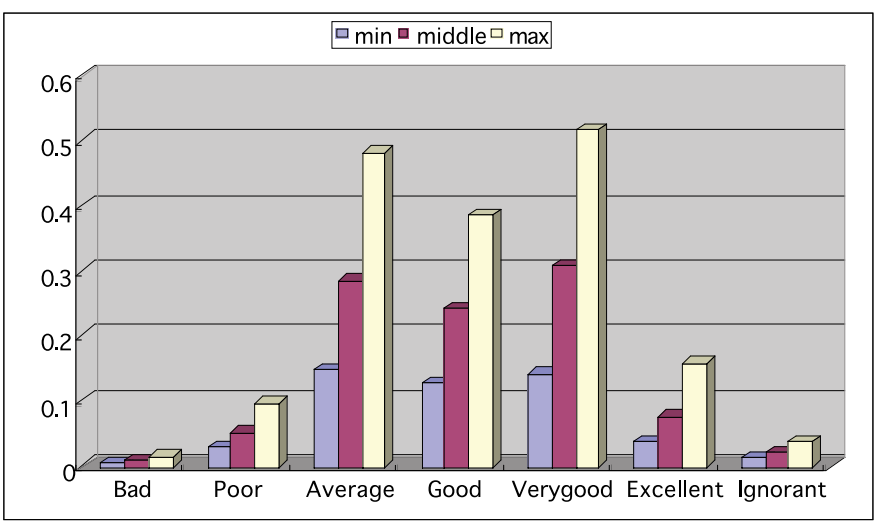

Figure 4. Triangular fuzzy belief degrees of a_1.

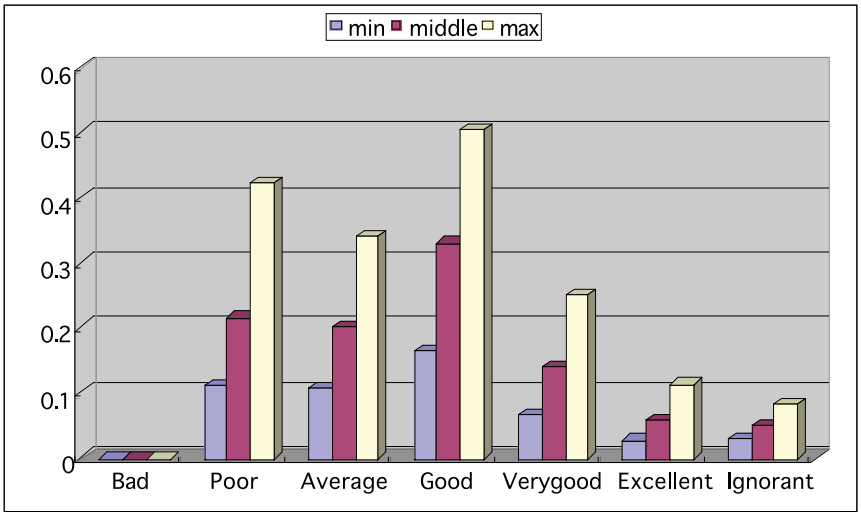

Figure 5. Triangular fuzzy belief degrees of a_2.

Step 4: The Calculation Process to Compute the Total Fuzzy Utility of Performance

These fuzzy distributed assessments provide a panoramic view of the overall performance of each of these different alternatives individually. However, it is not easy to rank the three alternatives just on the basis of fuzzy distributed assessments.

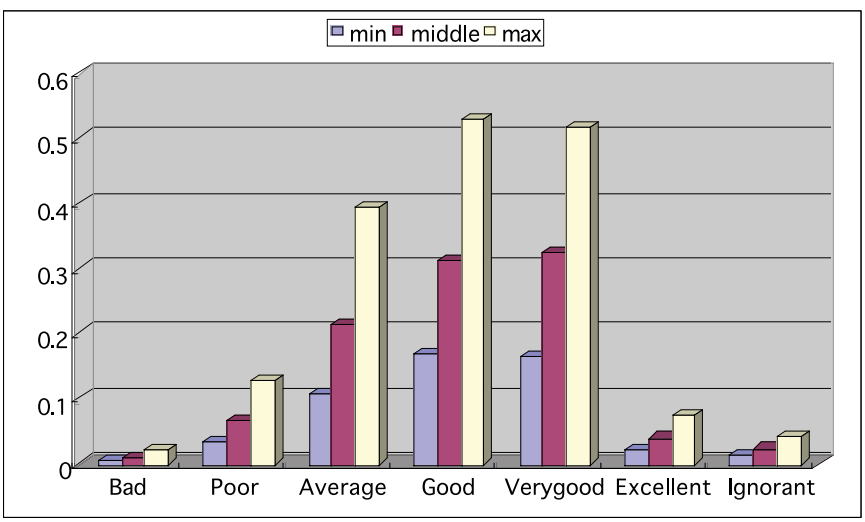

Figure 6. Triangular fuzzy belief degrees of a_3.

From the ER-based nonlinear programming models for utility function shown in the following, we could compute the fuzzy utility of each assessed suppliers.

$$\max / \min u'(y(a_l)) = u(y(a_l)) + \beta_H(a_l)u'(H) \quad (l = 1, 2, 3)$$

$$\text{s.t.} \quad \begin{cases} \text{Equations 9–16, 23–25.} \\ u(y(a_l)) = \sum_{n=1}^{6} \beta_n(a_l)u(H_n) \\ u'(H) = \frac{1}{2}[u(H_3) + u(H_4)] \\ 0 \le u(H_1) \le 5 \\ 15 \le u(H_2) \le 21 \\ 32 \le u(H_3) \le 40 \\ 55 \le u(H_4) \le 61 \\ 71 \le u(H_5) \le 78 \\ 94 \le u(H_6) \le 100 \end{cases}$$

Twenty variables are contained in this optimization model. By using "Solver" in Microsoft Office Excel, we could obtain the total fuzzy utility of each assessed suppliers in Table VI and Figure 7 in the following text.

After the calculation of expected utility, the ranking and selection of these three options should be conducted next. Suppose the degree that a_s prefers to a_t is denoted by $P(a_s \succ a_t)$, then we have:

$$P(a_s \succ a_t) = \max\left\{1 - \max\left[\frac{U^u(a_t) - U^l(a_s)}{[U^u(a_t) - U^l(a_t)] + [U^u(a_s) - U^l(a_s)]}, 0\right], 0\right\} \quad (40)$$

From formula (40), the relative comparison degree of a_1, a_2, and a_3 is calculated as follows:

$$P(a_1 \succ a_2) = 65.512\%, \quad P(a_1 \succ a_3) = 52.836\%, \quad P(a_2 \succ a_3) = 36.376\%$$

So, the relative importance of the three assessed suppliers is $a_1 \underset{52.836\%}{\succ} a_3 \underset{63.624\%}{\succ} a_2$, where the symbol "$\succ$" means "better than."

The total utility of each assessed supplier could also be calculated from the three risk preferences associated with different DMUs as shown in Figure 2. Suppose the preference of the DMU is risk neutralness and his or her utility values for each

Table VI. Fuzzy utility of the three assessed suppliers.

	a_1			a_2			a_3		
	U^l	U^m	U^u	U^l	U^m	U^u	U^l	U^m	U^u
U	43.23383	56.93804	70.54988	34.52186	48.88714	62.20069	43.32657	56.63707	67.53443

linguistic evaluation grade are given as follows:

$$u(H_1) = 0, u(H_2) = 20,\ u(H_3) = 40, u(H_4) = 60, u(H_5) = 80, u(H_6) = 100$$

Then, the expected utility of each assessed supplier could be computed using the optimization model (Equations 32 and 34) together with the belief degrees under the weights calculated by standardized *CGM*.

Suppose the DMU is risk proneness and provides the utility values as follows:

$$u(H_1) = 0,\ u(H_2) = 17,\ u(H_3) = 33,\ u(H_4) = 55,\ u(H_5) = 78,\ u(H_6) = 100$$

Also, if the DMU is risk aversion, he or she will provide the following utility values:

$$u(H_1) = 0,\ u(H_2) = 24,\ u(H_3) = 47,\ u(H_4) = 65,\ u(H_5) = 83,\ u(H_6) = 100$$

The calculated expected utilities under these three risk preferences are shown in Table VII.

The final selected alternative still needs further consideration on economic capability, but it is beyond the purpose of our paper and is not analyzed here.

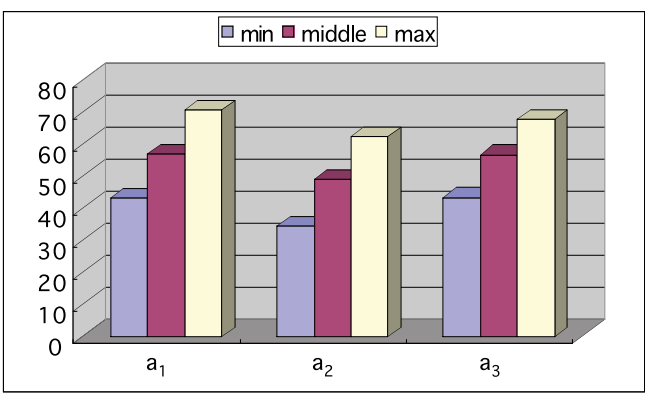

Figure 7. Fuzzy utility of the three assessed suppliers.

Table VII. The expected utilities of each assessed supplier under three different DMUs' preference information.

	Risk neutralness	Risk proneness	Risk aversion
a_1	60.65002	56.515	65.14627
a_2	51.88977	47.58011	56.55647
a_3	60.22654	56.14536	64.70231

9. CONCLUDING REMARKS

The ER approach is a suitable and rational method for dealing with MCDA problem. In the MCDA problem, the complexity and ambiguity of the real decision-making environment are always existent, which may result from the incomplete human judgments, inconsistency of different DMUs' opinion, and different time period. In this paper, the ER approach is extended to apply fuzzy information for MCDA with uncertainty. The Hadamard multiplicative combination of judgment matrix is extended to aggregate triangular fuzzy number pairwise matrices, from which the FAHP is used to compute the combined fuzzy weights of the attributes. Then, both fuzzy weights and fuzzy utilities are considered together to construct several programming models, from which the aggregated distributed fuzzy belief degrees and fuzzy utility values of each assessed alternative are generated. A case study to select the suppliers in the VMI alliance circumstance is introduced for the implementation of the ER approach under fuzzy weights and fuzzy utilities. From the application of the FER approach, it could be seen that the FER approach can provide a rational, reliable, and transparent way for decision-making analysis with group of experts and many uncertainties.

Acknowledgments

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APPENDIX

Here, the Hadamard multiplicative combination of judgment matrix is to be presented and then the proof of the consistency theorem is to be shown for the Hadamard multiplication in combining triangular fuzzy judgment matrices.

A.1. The Hadamard Multiplicative Combination of Judgment Matrix[33–35]

DEFINITION A-1. *The Hadamard multiplication algorithm of matrix* $A = (a_{ij})_{n\times n}$ *and matrix* $B = (b_{ij})_{n\times n}$ *is defined as follows, represented by* $C = (c_{ij})_{n\times n}$*:*

$$c_{ij} = a_{ij} b_{ij}. \tag{A.1}$$

which is denoted by $C = AB$.

DEFINITION A-2. *Suppose* $\lambda \in R$. *The exponential algorithm of matrix* $A = (a_{ij})_{n\times n}$ *is defined as:*

$$A^{\lambda} = \left(a_{ij}^{\lambda}\right)_{n\times n}. \tag{A.2}$$

DEFINITION A-3. *If* $a_{ij} = a_{ji}^{-1}$, $i \neq j; i, j = 1, 2, \ldots, n$, *then matrix* $A = (a_{ij})_{n\times n}$ *is called reflexive matrix.*

DEFINITION A-4. *Suppose $A_1, A_2, \ldots, A_m$ are m judgment matrices for the same problem. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ exist, $\lambda_t (t = 1, 2, \ldots, m)$ satisfy $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{m} \lambda_i = 1$, and $\bar{A} = A_1^{\lambda_1} A_2^{\lambda_2} \ldots A_m^{\lambda_m}$, then $\bar{A} = (\bar{a}_{ij})_{n\times n}$ is called the Hadamard multiplicative combination from $A_1, A_2, \ldots, A_m$, where*

$$\bar{a}_{ij} = \left(a_{ij}^{(1)}\right)^{\lambda_1} \left(a_{ij}^{(2)}\right)^{\lambda_2} \ldots \left(a_{ij}^{(m)}\right)^{\lambda_m}, \quad (i, j = 1, 2, \ldots, n) \tag{A.3}$$

DEFINITION A-5. *Suppose $A = (a_{ij})_{n\times n} \quad (i, j = 1, 2, \ldots, n)$ is a judgment matrix of a problem, A^* is the consistent judgment matrix of the problem, assume $A = A^*\varepsilon$, and*

$$\varepsilon = \begin{pmatrix} \varepsilon_{11} & \cdots & \varepsilon_{1n} \\ \cdots & \cdots & \cdots \\ \varepsilon_{n1} & \cdots & \varepsilon_{nn} \end{pmatrix}. \tag{A.4}$$

where $\varepsilon_{ij} = \varepsilon_{ij}^{-1} \quad (i, j = 1, 2, \ldots, n)$. If $\frac{1}{n(n-1)} \sum_{1\le i<j\le n} |\log \varepsilon_{ij}| \le R$ ($R > 0$ is a number with possible small value), then A is called a satisfied consistent judgment matrix.

THEOREM A-1. *Assume that $A_1, A_2, \ldots, A_m$ are m judgment matrices for the same problem, $R > 0$ is a number with possible small value, $\frac{1}{n(n-1)} \sum_{1\le i<j\le n} |\log \varepsilon_{ij}^{(t)}| \le R \quad (t = 1, 2, \ldots, m)$, and $\bar{A} = (\overline{a_{ij}})_{n\times n}$ is the Hadamard multiplicative combination from $A_1, A_2, \ldots, A_m$, where $\bar{A} = A^*\bar{\varepsilon}$, $\bar{\varepsilon} = (\overline{\varepsilon_{ij}})_{n\times n}$, $\overline{\varepsilon_{ij}} = (\varepsilon_{ij}^{(1)})^{\lambda_1} \cdot (\varepsilon_{ij}^{(2)})^{\lambda_2} \cdot \cdots \cdot (\varepsilon_{ij}^{(m)})^{\lambda_m}$. Then, we will have $\frac{1}{n(n-1)} \sum_{1\le i<j\le n} |\log \bar{\varepsilon}_{ij}| \le R$.*

A.2. Proof of Consistency Theorem in the Extension of Hadamard Multiplication

Proof of Theorem 1. From Definitions A-4 and A-5 and Definition 2 in Section 5, it is clear that $A_t = A^*\varepsilon_t \quad (t = 1, 2, \ldots, m)$ and $\varepsilon_t = (\varepsilon_{ij}^{(t)})_{n\times n}$, $\varepsilon_{ij}^{(t)} = (\varepsilon_{ij}^{(t)l}, \varepsilon_{ij}^{(t)m}, \varepsilon_{ij}^{(t)u})$, and the Hadamard multiplicative combination of $A_1, A_2, \ldots, A_m$ could be represented by $\bar{A} = A^*\bar{\varepsilon}$, where $\bar{\varepsilon} = (\overline{\varepsilon_{ij}})_{n\times n}$, $\overline{\varepsilon_{ij}} = (\overline{\varepsilon_{ij}^{l}}, \overline{\varepsilon_{ij}^{m}}, \overline{\varepsilon_{ij}^{u}})$, $\overline{\varepsilon_{ij}} = (\varepsilon_{ij}^{(1)})^{\lambda_1} (\varepsilon_{ij}^{(2)})^{\lambda_2} \ldots (\varepsilon_{ij}^{(m)})^{\lambda_m}$. A^* is the consistent triangular fuzzy judgment matrix. Since A_t are satisfied consistent judgment matrices, so we will have

$$\frac{1}{n(n-1)} \sum_{1\le i<j\le n} \left|\log \varepsilon_{ij}^{(t)}\right| \le R. \quad (t = 1, 2, \ldots, m). \tag{A.5}$$

Here, we prove only the following inequation:

$$\frac{1}{n(n-1)} \sum_{1 \le i < j \le n} |\log \bar{\varepsilon}_{ij}| \le R. \tag{A.6}$$

where $R = (R^l, R^m, R^u)$ $(R^l > 0, R^m > 0, R^u > 0)$ is the triangular fuzzy number with small enough value to check the degree of consistency.

$$\begin{aligned}
&\frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \left|\log \bar{\varepsilon}_{ij}^{l}\right| \\
&= \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \left|\log \left[\left(\varepsilon_{ij}^{(1)l}\right)^{\lambda_1} \left(\varepsilon_{ij}^{(2)l}\right)^{\lambda_2} \cdots \left(\varepsilon_{ij}^{(m)l}\right)^{\lambda_m}\right]\right| \\
&= \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \left|\sum_{t=1}^{m} \lambda_t \log \varepsilon_{ij}^{(t)l}\right| \le \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \sum_{t=1}^{m} \left|\lambda_t \log \varepsilon_{ij}^{(t)l}\right| \\
&= \frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \sum_{t=1}^{m} \lambda_t \left|\log \varepsilon_{ij}^{(t)l}\right| = \frac{1}{n(n-1)} \sum_{t=1}^{m} \lambda_t \left(\sum_{1 \le i < j \le n} \left|\log \varepsilon_{ij}^{(t)l}\right|\right) \\
&= \sum_{t=1}^{m} \lambda_t \left[\frac{1}{n(n-1)} \left(\sum_{1 \le i < j \le n} \left|\log \varepsilon_{ij}^{(t)l}\right|\right)\right].
\end{aligned} \tag{A.7}$$

Similarly,

$$\frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \left|\log \overline{\varepsilon_{ij}^{m}}\right| \le \sum_{t=1}^{m} \lambda_t \left[\frac{1}{n(n-1)} \left(\sum_{1 \le i < j \le n} \left|\log \varepsilon_{ij}^{(t)m}\right|\right)\right]. \tag{A.8}$$

$$\frac{1}{n(n-1)} \sum_{1 \le i < j \le n} \left|\log \bar{\varepsilon}_{ij}^{u}\right| \le \sum_{t=1}^{m} \lambda_t \left[\frac{1}{n(n-1)} \left(\sum_{1 \le i < j \le n} \left|\log \varepsilon_{ij}^{(t)u}\right|\right)\right]. \tag{A.9}$$

According to the logarithm algorithm of triangular fuzzy number,

$$|\log \varepsilon_{ij}| = \left[v_{\varepsilon_{ij}}^{l}, v_{\varepsilon_{ij}}^{m}, v_{\varepsilon_{ij}}^{u}\right], \quad \text{so} \left|\log \varepsilon_{ij}^{(t)}\right| = \left[v_{\varepsilon_{ij}^{(t)}}^{l}, v_{\varepsilon_{ij}^{(t)}}^{m}, v_{\varepsilon_{ij}^{(t)}}^{u}\right]$$

where $v_{\varepsilon_{ij}^{(t)}}^{l}$, $v_{\varepsilon_{ij}^{(t)}}^{m}$, $v_{\varepsilon_{ij}^{(t)}}^{u}$ are the minimum, middle, and maximum values of $\left|\log \varepsilon_{ij}^{(t)l}\right|$, $\left|\log \varepsilon_{ij}^{(t)m}\right|$, and $\left|\log \varepsilon_{ij}^{(t)u}\right|$, respectively.

Since $\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \varepsilon_{ij}^{(t)}\right| \le R$, we have

$$\frac{1}{n(n-1)}\sum_{1\le i<j\le n}[v^l_{\varepsilon_{ij}^{(t)}}, v^m_{\varepsilon_{ij}^{(t)}}, v^u_{\varepsilon_{ij}^{(t)}}] \le [R^l, R^m, R^u].$$

According to the comparison method of triangular fuzzy number, we have

$$\frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^l_{\varepsilon_{ij}^{(t)}} + \frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^m_{\varepsilon_{ij}^{(t)}} + \frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^u_{\varepsilon_{ij}^{(t)}} \le R^l + R^m + R^u \tag{A.10}$$

Also, $\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \overline{\varepsilon_{ij}}\right| = \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left[v^l_{\overline{\varepsilon_{ij}}}, v^m_{\overline{\varepsilon_{ij}}}, v^u_{\overline{\varepsilon_{ij}}}\right]$ where $v^l_{\overline{\varepsilon_{ij}}}$, $v^m_{\overline{\varepsilon_{ij}}}$, $v^u_{\overline{\varepsilon_{ij}}}$ are the minimum, middle, and maximum values of $\left|\log \overline{\varepsilon^l_{ij}}\right|$, $\left|\log \overline{\varepsilon^m_{ij}}\right|$, and $\left|\log \overline{\varepsilon^u_{ij}}\right|$ respectively. From formula (A.7) to (A.10),

$$\begin{aligned}
&\frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^l_{\overline{\varepsilon_{ij}}} + \frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^m_{\overline{\varepsilon_{ij}}} + \frac{1}{n(n-1)}\sum_{1\le i<j\le n} v^u_{\overline{\varepsilon_{ij}}} \\
&= \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \overline{\varepsilon^l_{ij}}\right| + \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \overline{\varepsilon^m_{ij}}\right| \\
&\quad + \frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \overline{\varepsilon^u_{ij}}\right| \\
&\le \sum_{t=1}^{m}\lambda_t\left[\frac{1}{n(n-1)}\left(\sum_{1\le i<j\le n}\left|\log \varepsilon_{ij}^{(t)l}\right|\right)\right] \\
&\quad + \sum_{t=1}^{m}\lambda_t\left[\frac{1}{n(n-1)}\left(\sum_{1\le i<j\le n}\left|\log \varepsilon_{ij}^{(t)m}\right|\right)\right] \\
&\quad + \sum_{t=1}^{m}\lambda_t\left[\frac{1}{n(n-1)}\left(\sum_{1\le i<j\le n}\left|\log \varepsilon_{ij}^{(t)u}\right|\right)\right] \\
&\le \sum_{t=1}^{m}\lambda_t[R^l + R^m + R^u] = R^l + R^m + R^u.
\end{aligned}$$

So, we have $\frac{1}{n(n-1)}\sum_{1\le i<j\le n}\left|\log \overline{\varepsilon_{ij}}\right| \le R$.