International Journal of Computational Intelligence Systems
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcis20

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Mi Zhou [a b], Xin-Bao Liu [b], Jian-Bo Yang [c] & Chang Fang [b]

[a] Research Centre of Knowledge Economy and Business Management Innovation, Key Laboratory of Process Optimization and Intelligent Decision-making, Ministry of Education, 230009, Hefei, Anhui, P.R. China

[b] School of Management, Hefei University of Technology, 230009, Hefei, Anhui, P.R. China

[c] Manchester Business School, the University of Manchester, Manchester, M15 6PB, UK

Version of record first published: 05 Mar 2013.

To cite this article: Mi Zhou, Xin-Bao Liu, Jian-Bo Yang & Chang Fang (2013): Group Evidential Reasoning Approach for MADA under Fuzziness and Uncertainties, International Journal of Computational Intelligence Systems, 6:3, 423-441

To link to this article: http://dx.doi.org/10.1080/18756891.2013.780732

Group Evidential Reasoning Approach for MADA under Fuzziness and Uncertainties

Mi Zhou*(Corresponding Author)
Research Centre of Knowledge Economy and Business Management Innovation
Key Laboratory of Process Optimization and Intelligent Decision-making, Ministry of Education
School of Management, Hefei University of Technology
230009, Hefei, Anhui, P.R. China
Tel: +86 13856952434
zhoumi83@163.com

Xin-Bao Liu
School of Management, Hefei University of Technology
230009, Hefei, Anhui, P.R. China
lxinbao@126.com

Jian-Bo Yang
Manchester Business School, the University of Manchester
Manchester M15 6PB, UK
jian-bo.yang@manchester.ac.uk

Chang Fang
School of Management, Hefei University of Technology
230009, Hefei, Anhui, P.R. China
fangch1986@126.com

Received 2 February 2012
Accepted 15 December 2012

Abstract

Multiple attribute decision analysis (MADA) problems often include both qualitative and quantitative attributes which may be either precise or inaccurate. The evidential reasoning (ER) approach is one of reliable and rational methods for dealing with MADA problems and can generate aggregated assessments from a variety of attributes. In many real world decision situations, accurate assessments are difficult to provide such as in group decision situations. Extensive research in dealing with imprecise or uncertain belief structures has been conducted on the basis of the ER approach, such as interval belief degrees, interval weights and interval uncertainty. In this paper, the weights of attributes and utilities of evaluation grades are considered to be fuzzy numbers for the ER approach. Fuzzy analytic hierarchy process (FAHP) is used for generating triangular fuzzy weights for attributes from a triangular fuzzy judgment matrix provided by an expert. The weighted arithmetic mean method is proposed to aggregate the triangular fuzzy weights of attributes from a group of experts. α-cut is then used to transform the combined triangular fuzzy weights to interval weights for the purpose of dealing with the fuzzy type of weight and utility in a consistent way. Several pairs of group evidential reasoning based nonlinear programming models are then designed to calculate the global fuzzy belief degrees and the overall expected interval utilities of each alternative with interval weights and interval utilities as constraints. A case study is conducted to show the validity and effectiveness of the proposed approach and sensitivity analysis is also conducted on interval weights generated by different α-cuts.

Keywords: Evidential reasoning; Fuzzy weight; Interval utility; α-cut; Multiple attribute decision analysis.

1. Introduction

Multiple attribute decision analysis (MADA) with various types of attributes is common in practice.[1] For instance, in a project evaluation problem, both quantitative attributes measured by numerical values and qualitative attributes judged by linguistic variables need to be taken into account. In general, several quantified evaluation grades may be defined for assessing a qualitative attribute, for example, indifferent, average, good, and so on. Numerical values associated with these grades are used to transform the subjective judgment of an alternative on a qualitative attribute to a numerical value, or the numerical value assessed to a quantitative attribute is transformed to degrees on several evaluation grades. Then the values assessed on all attributes in a consistent form can be aggregated into a general assessment.

The evidential reasoning (ER) approach was introduced in 1990s[2,3] based on the Dempster-Shafer (D-S) theory[4,5] and is well-suited to dealing with complex MADA problems. It uses a distributed assessment based on several defined evaluation grades to present incomplete or fuzzy subjective judgments and it is convenient to combine different types of attributes.

There have been several development stages for the ER approach. Firstly, a basic framework of the ER approach is proposed[2,3] based on the evidence combination rule of the D-S theory for the combination of multiple uncertain subjective judgments. To facilitate data collection in real decision environments, Yang proposed an improved method to the ER approach for the transformation of different sets of linguistic evaluation grades associated with different qualitative attributes and certain values associated with quantitative attributes to one set of evaluation grades.[6] In dealing with the irrationality in the original ER framework, four synthesis axioms are discussed and an improved ER algorithm which could satisfy these axioms is proposed.[7] In the original ER recursive algorithm,[6,7] L-1 calculation steps are needed for the combination of L basic attributes. The analytical ER algorithm is then proposed[8] based on the recursive algorithm and Yen's combination rule,[9] in which only one step of calculation is to be conducted to generate the overall performance. Secondly, the ER approach has been applied to many real world decision making issues, for instance, motorcycle and car evaluation,[2,3,7] large engineering product evaluation,[10] general cargo ship design,[11] supplier selection in VMI alliance circumstance, [12] contractor selection,[13] safety analysis,[14,15] self assessment,[16, 17, 18] environmental impact assessment,[8] pipeline leak detection,[19] strategic R&D project assessment,[20] new product development,[21] bridge condition assessment,[22] consumer preference prediction,[23] and so on. Thirdly, on the implementation of the ER approach, a window-based and graphically designed decision support software package called intelligent decision system (IDS) is developed on the basis of the ER approach.[16, 17, 24] It not only provides a flexible and easy to use interface for modeling and decision analysis, but also a structured knowledge base to help assessors to make judgments more objectively.

In a MADA problem, although the relative weight of an attribute is considered in the ER algorithm, the relative importance of an attribute is not always provided precisely due to the lack of information or the limit of knowledge and experience. Consequently, subjective judgments may be provided by a group of assessors because an individual may be incapable of providing a reliable judgment. In this circumstance, crisp values are not appropriate to present the weights of attributes from a group of experts anymore. In the field of research in the ER approach under uncertainties, Xu *et al* studied the ER approach for MADA under interval uncertainty[25] where the frame of discernment comprising not only single evaluation grades such as 'good', 'average', but also any subset of consecutive evaluation grades. Yang studied the ER approach for MADA under both probabilistic and fuzzy uncertainties where every two adjacent evaluation grades are supposed to be possibly overlapped to some degree.[26] Wang *et al* also extended the ER approach where the belief degrees are supposed to be interval values and several programming models are constructed,[27] based on which Guo further extended the ER approach under both interval belief degrees and interval weights.[28]

In this paper, we investigate a decision situation where a group of experts are involved in providing uncertain weights, in particular triangular fuzzy weights. Firstly, a set of triangular fuzzy judgment matrices are constructed from a group of experts, based on which a set of triangular fuzzy weights are generated by the FAHP[29] method. Then the weighted arithmetic mean method is used to generate a combined triangular fuzzy weight of each attribute from the perspectives of all experts. The α-cut method is used to transform the

combined triangular fuzzy weights to interval weights. The global fuzzy belief degrees are then generated based on the interval weights calculated by the α-cut method and the ER algorithm through four groups of ER based nonlinear programming models.

From the calculation by the ER algorithm, an aggregated belief degree can be generated which needs to be transferred into a definite value through utility function.[6, 7, 20] Due to the different backgrounds and expertise, the utility estimations on the same evaluation grade by every two decision makers (DMs) may be different in group decision circumstances. It may also be changed by individual DM himself/herself at different decision points. For example, a DM may be risk taking at one period of time and risk averse at another time point due to the changes of external environment. In this paper, fuzzy utilities are used in the ER approach and assumed to be constraints in the nonlinear programming models for the computation of the general assessment value for the presentation of risk preferences from different DMs or the attitude changes of DMs towards risk.

The structure of this paper is summarized as follows. Section 2 and section 4 are a brief introduction about the Dempster-Shafer's evidence theory and the ER approach respectively. The concept of triangular fuzzy number which is the basis of our extension of the ER approach is to be introduced in section 3. Section 5 provides the extension of the ER approach to deal with triangular fuzzy weights in group decision making environment. In section 6, the approach to generate global belief degrees under triangular fuzzy weights are proposed, and section 7 provides the approach to generate overall expected utility under fuzzy weights and fuzzy utilities. Section 8 is a numerical example to illustrate our proposed approach. This paper is concluded in section 9.

2. Dempster–Shafer's Evidence Theory

Dempster-Shafer's evidence theory is introduced by Dempster[4] and refined by Shafer.[5] It is one of the powerful tools to deal with uncertainty and has been applied in many fields.[30-34] In the D-S evidence theory, a sample space is called the frame of discernment which is denoted by θ. A basic hypothesis in θ is represented by H_n $(n = 1, 2, \cdots, N)$.

The basic probability mass to the subset A of θ is denoted by $m(A)$, which measures the degree to which the evidence supports A. It satisfies the following two conditions:

$$\sum_{A \subseteq \Theta} m(A) = 1 \quad 0 \le m(A) \le 1 \quad \forall A \subset \theta \tag{1}$$

$$m(\phi) = 0 \tag{2}$$

where Θ is the power set of θ, consisting of all subsets of θ, and ϕ is an empty set. $m(A)$ expresses the portion of the total belief exactly committed to A given a piece of evidence, but does not include the portion of belief to the subsets of A. $m(\theta)$ measures the degree of ignorance that is the portion of the belief unassigned to any subsets of θ.

The probability assigned to A of θ that considering all the premises of A is denoted by $Bel(A)$ as defined below:

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad (\forall A \subset \theta) \tag{3}$$

The kernel of the D-S evidence theory is the D-S combination rule which can be used to aggregate different sources of evidence. Suppose there are n pieces of evidence in θ, and they each provide a basic probability mass to a subset A of θ, such as $m_1(A), m_2(A), \cdots, m_n(A)$. The D-S combination rule is defined as follows:

$$K = \left(1 - \sum_{\substack{A_1, \cdots, A_n \subset \theta \\ A_1 \cap \cdots \cap A_n = \phi}} m_1(A_1) \cdot \cdots \cdot m_n(A_n) \right)^{-1} \tag{4}$$

$$m(A) = \begin{cases} 0, & A = \phi \\ K \cdot \sum_{\substack{A_1, \cdots, A_n \subset \theta \\ A_1 \cap \cdots \cap A_n = A}} m_1(A_1) \cdot \cdots \cdot m_n(A_n), & A \ne \phi \end{cases} \tag{5}$$

Eq. (4) reflects the conflict among n pieces of evidence, which is called the normalization factor and satisfies $K \ge 1$.

3. Concept of Triangular Fuzzy Number

In this section, the concept of triangular fuzzy number will be briefly reviewed.[35-38] At first, the basic concept of fuzzy sets and fuzzy numbers will be briefly introduced.[38-41]

A fuzzy set M of the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$ can be denoted by

$$M = \mu_M(x_1)/x_1 + \mu_M(x_2)/x_2 + \cdots + \mu_M(x_n)/x_n \tag{6}$$

where μ_M denotes the membership function of the fuzzy set M with equation as follows:

$$\mu_M : X \to [0,1] \tag{7}$$

$\mu_M(x_i)$ $(i = 1, 2, \cdots, n)$ denotes the degree of membership of x_i to the fuzzy set M. If $\exists x_i \in X$

with $\mu_M(x_i)=1$, then the fuzzy set M is called a normal fuzzy set. According to Chen,[38] a fuzzy number is a fuzzy set in the universe of discourse X that is both normal and convex.

A generalized triangular fuzzy number M of the universe of discourse X could be characterized by a triangular membership function represented as $M=(l,m,u;w_M)$, where $w_M \in (0,1]$. l, m and u are all real numbers. l and u are the lower and upper bounds of the fuzzy number M respectively, and $l \le m \le u$. When $w_M=1$, then the generalized triangular fuzzy number M becomes a normal triangular fuzzy number, denoted as $M=(l,m,u;1)=(l,m,u)$. When $l=m=u$ and $w_M=1$, then M becomes a crisp value. The geometrical explanation of a generalized triangular fuzzy number is shown in Fig. 1.

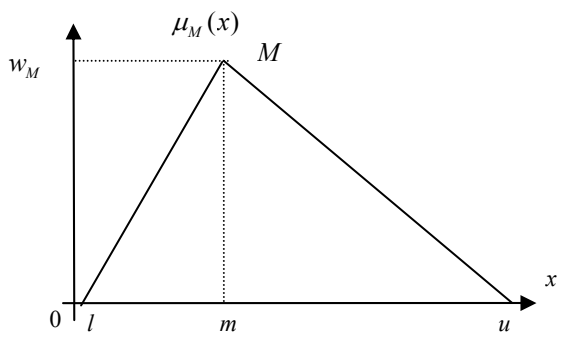

Fig. 1. Geometrical explanation of a generalized triangular fuzzy number $(l,m,u;w_M)$

The membership function of a normal triangular fuzzy number M can be represented as follows:

$$\mu_M(x)=\begin{cases} 0 & x\in(-\infty,l]\cup[u,+\infty) \\ \dfrac{1}{m-l}x-\dfrac{l}{m-l} & x\in[l,m] \\ \dfrac{1}{m-u}x-\dfrac{u}{m-u} & x\in[m,u] \end{cases} \tag{8}$$

In this paper, the relative importance of an attribute e_i is represented by $\omega_i=(\omega_i^l,\omega_i^m,\omega_i^u;1)=(\omega_i^l,\omega_i^m,\omega_i^u)$. ω_i^m denotes the most possible value of the relative importance ω_i confirmed to the ith attribute with the membership function $\mu_M(\omega_i^m)=w_M=1$, while ω_i^l and ω_i^u are the left and right extensions of ω_i respectively. They each denote the most pessimistic weight judgment and the most optimistic weight judgment assigned to e_i respectively, with $\mu_M(\omega_i^l)=\mu_M(\omega_i^u)=0$. The weight ω_i could be assigned to any values in the range of $[\omega_i^l,\omega_i^u]$ with a member function which can be calculated by Eq. (8). The distance of the two values is represented by the absolute value $\left|\omega_i^u-\omega_i^l\right|$ which denotes the largest fuzzy degree of the relative importance assigned to e_i. In other words, the larger the absolute value, the fuzzier the judgment is. $\left|\omega_i^u-\omega_i^l\right|=0$ indicates that the weight judgment to the ith attribute is not fuzzy at all.

4. The Analytical ER Algorithm

The ER approach can be used to deal with MADA problems to aggregate multiple attributes.[2, 3, 6-8] It uses a belief decision matrix (BDM) where each of its elements is a vector of belief degrees assessed to a set of evaluation grades in contrast with a conventional decision matrix where a single value is used to assess an alternative on each attribute in most other MADA methods. The ER approach is developed on the basis of the D-S evidence theory discussed in section 2. A set of linguistic evaluation grades for the assessment of an attribute on an alternative is defined in the ER approach as follows:

$$H=\{H_1,H_2,\cdots,H_N\} \tag{9}$$

where H_n $(n=1,2,\cdots,N)$ each denotes an evaluation grade and they together form a frame of discernment. They are assumed to be collectively exhaustive and mutually exclusive. H_1 and H_N are assumed to be the worst and best evaluation grade respectively. Without loss of generality, H_{n+1} is supposed to be preferred to H_n. The notations of some basic symbols in the ER approach are shown in table 1.

Table 1. Notation of the main symbols

Symbol	Notation
e_i	The ith attribute, $i=1,2,\cdots,L$, L is the total number of basic attributes in assessing the general performance
a_l	The lth alternative, $l=1,2,\cdots,S$, S is the number of alternatives to be assessed
ω_i	The weight of the ith attribute e_i, and $\sum_{i=1}^{L}\omega_i=1$
$\beta_{n,i}(a_l)$	The belief degree that a_l is assessed on e_i to evaluation grade H_n
$\beta_{H,i}(a_l)$	The degree of uncertainty that a_l is assessed on e_i

From the basic symbols presented in Table 1, the assessment of an alternative a_l on a basic attribute e_i to all evaluation grades can be denoted by the following statement:

$$S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)), n = 1,2,\cdots,N; H, \beta_{H,i}(a_l)\}$$
$$(i = 1,2,\cdots,L; l = 1,2,\cdots,S) \quad (10)$$

There are two basic conditions for $\beta_{n,i}(a_l)$ in the ER approach which are shown below:

$$0 \le \beta_{n,i}(a_l) \le 1, \quad \sum_{n=1}^{N} \beta_{n,i}(a_l) \le 1 \quad (11)$$

If $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1$, the information provided by DM is considered to be complete. Otherwise, it is said to be incomplete.

In the ER approach, each qualitative attribute is assigned with belief degrees on all evaluation grades as denoted in Eq. (9). The ER approach provides both the recursive[7] and analytical algorithms[8] to aggregate the assessments on multiple factors. Each of the algorithms has its own merits and can be applied in different decision situations. The analytical ER algorithm is described as follows:

$$m_{n,i}(a_l) = \omega_i \beta_{n,i}(a_l) \quad (12)$$

$$m_{H,i}(a_l) = 1 - \sum_{n=1}^{N} m_{n,i}(a_l) = 1 - \omega_i \cdot \sum_{n=i}^{N} \beta_{n,i}(a_l) \quad (13)$$

$$\bar{m}_{H,i}(a_l) = 1 - \omega_i, \tilde{m}_{H,i}(a_l) = \omega_i (1 - \sum_{n=1}^{N} \beta_{n,i}(a_l)) \quad (14)$$

$$m_n(a_l) = k'[\prod_{i=1}^{L}(m_{n,i}(a_l) + \bar{m}_{H,i}(a_l) + \tilde{m}_{H,i}(a_l)) - \prod_{i=1}^{L}(\bar{m}_{H,i}(a_l) + \tilde{m}_{H,i}(a_l))] \quad (15)$$

$$\tilde{m}_H(a_l) = k'\left[\prod_{i=1}^{L}(\bar{m}_{H,i}(a_l) + \tilde{m}_{H,i}(a_l)) - \prod_{i=1}^{L}\bar{m}_{H,i}(a_l)\right] \quad (16)$$
$$\bar{m}_H(a_l) = k'\left[\prod_{i=1}^{L}\bar{m}_{H,i}(a_l)\right]$$

$$k' = [\sum_{n=1}^{N}\prod_{i=1}^{L}(m_{n,i}(a_l) + \bar{m}_{H,i}(a_l) + \tilde{m}_{H,i}(a_l)) - (N-1)\prod_{i=1}^{L}(\bar{m}_{H,i}(a_l) + \tilde{m}_{H,i}(a_l))]^{-1} \quad (17)$$

$$i = 1,2,\cdots,L; \quad n = 1,2,\cdots,N; \quad l = 1,2,\cdots,S$$

where $m_{n,i}(a_l)$ is the basic probability assignment of a_l being assessed to H_n on the basic attribute e_i, $m_{H,i}(a_l)$ is the remaining probability mass unassigned to any individual grade after all the N grades have been considered for assessing the general attribute as far as e_i is concerned. $m_n(a_l)$ is the combined probability assignment to a_l on H_n generated by assessing all the L attributes.

From the above description, it is clear that only one step needs to be conducted for generating the overall assessment $m_n(a_l)$, $\tilde{m}_H(a_l)$ and $\bar{m}_H(a_l)$ by combining L basic attributes.

Let $\beta_n(a_l)$ be the combined belief degree to which a_l $(l = 1,2,\cdots,S)$ is assessed on H_n and $\beta_H(a_l)$ the belief degree unassigned to any individual evaluation grade after all the L basic attributes have been assessed. Then we have

$$\beta_n(a_l) = \frac{m_n(a_l)}{1 - \bar{m}_H(a_l)}, \quad n = 1,2,\cdots,N \quad (18)$$

$$\beta_H(a_l) = \frac{\tilde{m}_H(a_l)}{1 - \bar{m}_H(a_l)} \quad (19)$$

where

$$\beta_H(a_l) + \sum_{n=1}^{N} \beta_n(a_l) = 1 \quad (20)$$

After the aggregation of L basic attributes, a distributed assessment for a_l on the general level can then be presented as follows:

$$S(y(a_l)) = \{(H_n, \beta_n(a_l)), n = 1,2,\cdots,N; H, \beta_H(a_l)\} \quad (21)$$

5. Method for Generating the Aggregated Triangular Fuzzy Weights from Group of Experts

The analytic hierarchy process (AHP) can be used as a method for the comparison of relative importance for attributes.[42, 43] In the traditional AHP method, each element in the comparison matrix is an accurate value. With the decision making environments becoming more and more complex, the comparison of two attributes may not always be definitely fixed to a certain value anymore. The triangular fuzzy judgment matrix was proposed by Laarhoven,[29] which is a fuzzy extension of Saaty's pairwise comparison method[42] that was extended by Gran[44] and Lootsma.[45] Kwiesielewicz[46] further improved the fuzzy analytic hierarchy process (FAHP) proposed by Laarhoven. In this paper, the weights of attributes in the ER approach are supposed to be triangular fuzzy weights which are calculated using FAHP based on the triangular fuzzy judgment matrix.

5.1. *Triangular fuzzy judgment matrix*

Each element in a triangular fuzzy judgment matrix is a

triangular fuzzy number as defined in the following definition.

Definition 1. $A=(a_{ij})_{L\times L}$ $(i,j=1,2,\cdots,n)$ is a $L\times L$ triangular fuzzy judgment matrix if the elements a_{ij} are represented by triangular fuzzy number as follows:

$$a_{ij}=(l_{ij},m_{ij},u_{ij}) \tag{22}$$

where L is the total number of attributes involved in the comparison of two attributes.

The subjective preferences of experts have direct significant impact on the outcomes of decision making in the process of constructing a hierarchical assessment structure and judgment matrix, whilst expert judgments can be incorrect or inaccurate, which leads to wrong decisions. Individual experts may be incompetent for dealing with the complexity of decision making problems under multiple criteria. In complex real decision making processes, it is important to get a group of experts involved in order to make decisions more objectively and reliably. However, research shows that experts always achieve different degrees of consistency in decision making because of their different backgrounds or expertise and conflicting interests. A question then arises as to how to generate group attitude towards the importance of attributes from individual expert's judgments.

Suppose $\{A_1,A_2,\cdots,A_t,\cdots,A_T\}$ is a set of triangular fuzzy judgment matrices given by T experts, where A_t denotes the tth triangular fuzzy judgment matrix from the tth expert whose relative importance is assumed to be $\lambda_t(t=1,2,\cdots,T)$. A_t is defined as the following equation:

$$A_t=(a_{ij}^t)_{L\times L}=(l_{ij}^t,m_{ij}^t,u_{ij}^t)_{L\times L}$$
$$=\begin{bmatrix} (1,\ 1,\ 1) & (l_{12}^t,m_{12}^t,u_{12}^t) & \cdots & (l_{1L}^t,m_{1L}^t,u_{1L}^t) \\ (l_{21}^t,m_{21}^t,u_{21}^t) & (1,\ 1,\ 1) & \cdots & (l_{2L}^t,m_{2L}^t,u_{2L}^t) \\ \cdots & \cdots & \cdots & \cdots \\ (l_{L1}^t,m_{L1}^t,u_{L1}^t) & (l_{L2}^t,m_{L2}^t,u_{L2}^t) & \cdots & (1,\ 1,\ 1) \end{bmatrix}_{L\times L}$$
$$(t=1,2,\cdots,T) \tag{23}$$

5.2. *Generating the Aggregated Triangular Fuzzy Weights*

There are two ways for the generation of aggregated triangular fuzzy weights. One way is to generate a combined triangular fuzzy judgment matrix by aggregating the triangular fuzzy judgment matrices of all experts firstly, and then the combined triangular fuzzy weights of attributes can be calculated by FAHP from the combined judgment matrix. The other way is to use FAHP to calculate the triangular fuzzy weights of attributes from each triangular fuzzy judgment matrix firstly, and then the combined triangular fuzzy weights can be obtained by aggregating these calculated triangular fuzzy weights.

In the first way, the weighted geometric mean complex judgment matrix (WGMCJM) method[47, 48] can be used for the aggregation of multiple-matrices. Xu proved that WGMCJM satisfies the consistency theorem.[47] The proof shows that if a judgment matrix presented by each expert is of acceptable consistency, then the aggregated judgment matrix calculated using the weighted geometric mean method (WGMM) is of acceptable consistency as well. In this approach, however, elements in each judgment matrix are all assumed to be crisp values. When elements are extended to triangular fuzzy numbers, this approach becomes complicated.

In this paper, the second way is followed to generate aggregated triangular fuzzy weights from a group of experts. The FAHP approach is firstly used to calculate the triangular fuzzy weights of all L attributes from each expert. Suppose there are T experts involved in the process of assigning attributes' weights. The weight assigned to the ith attribute by the tth expert calculated by FAHP is assumed to be $\omega_{i,t}$ $(t=1,2,\cdots,T)$, represented by $(\omega_{i,t}^l,\omega_{i,t}^m,\omega_{i,t}^u)$. Fig. 2 shows the matrix of triangular fuzzy weight assignment to all L attributes by all the T experts. For example, the weight assignment by the tth expert is shown in the tth row of Fig.2 which is calculated from the tth triangular fuzzy judgment matrix A_t by FAHP.

The triangular fuzzy weight matrix can also be denoted by the following two vectors.

$$\left[\underline{\boldsymbol{\omega}}_{\mathbf{1,t}}\quad \underline{\boldsymbol{\omega}}_{\mathbf{2,t}}\quad \cdots\quad \underline{\boldsymbol{\omega}}_{\mathbf{i,t}}\quad \cdots\quad \underline{\boldsymbol{\omega}}_{\mathbf{L,t}}\right] \tag{24}$$

or

$$\left[\underline{\boldsymbol{\omega}}'_{\mathbf{i,1}}\quad \underline{\boldsymbol{\omega}}'_{\mathbf{i,2}}\quad \cdots\quad \underline{\boldsymbol{\omega}}'_{\mathbf{i,t}}\quad \cdots\quad \underline{\boldsymbol{\omega}}'_{\mathbf{i,T}}\right]^T \tag{25}$$

where $\underline{\boldsymbol{\omega}}_{\mathbf{i,t}}=(\omega_{i,1},\omega_{i,2},\cdots,\omega_{i,t},\cdots,\omega_{i,T})^T$ represents the weight vector of e_i assigned by all experts, and $\underline{\boldsymbol{\omega}}'_{\mathbf{i,t}}=(\omega_{1,t},\omega_{2,t},\cdots,\omega_{i,t},\cdots,\omega_{L,t})$ represents the weight vector assigned to all L attributes by the tth expert.

	e_1	e_2	…	e_i	…	e_L	
Expert 1	$(\omega_{1,1}^l, \omega_{1,1}^m, \omega_{1,1}^u)$	$(\omega_{2,1}^l, \omega_{2,1}^m, \omega_{2,1}^u)$	…	$(\omega_{i,1}^l, \omega_{i,1}^m, \omega_{i,1}^u)$	…	$(\omega_{L,1}^l, \omega_{L,1}^m, \omega_{L,1}^u)$	$\underline{\boldsymbol{\omega}}_{\mathbf{i,1}}^{'}$
Expert 2	$(\omega_{1,2}^l, \omega_{1,2}^m, \omega_{1,2}^u)$	$(\omega_{2,2}^l, \omega_{2,2}^m, \omega_{2,2}^u)$	…	$(\omega_{i,2}^l, \omega_{i,2}^m, \omega_{i,2}^u)$	…	$(\omega_{L,2}^l, \omega_{L,2}^m, \omega_{L,2}^u)$	$\underline{\boldsymbol{\omega}}_{\mathbf{i,2}}^{'}$
……	……	……		……		…	
Expert t	$(\omega_{1,t}^l, \omega_{1,t}^m, \omega_{1,t}^u)$	$(\omega_{2,t}^l, \omega_{2,t}^m, \omega_{2,t}^u)$	…	$(\omega_{i,t}^l, \omega_{i,t}^m, \omega_{i,t}^u)$	…	$(\omega_{L,t}^l, \omega_{L,t}^m, \omega_{L,t}^u)$	$\underline{\boldsymbol{\omega}}_{\mathbf{i,t}}^{'}$
……	……	……		……		…	
Expert T	$(\omega_{1,T}^l, \omega_{1,T}^m, \omega_{1,T}^u)$	$(\omega_{2,T}^l, \omega_{2,T}^m, \omega_{2,T}^u)$	…	$(\omega_{i,T}^l, \omega_{i,T}^m, \omega_{i,T}^u)$	…	$(\omega_{L,T}^l, \omega_{L,T}^m, \omega_{L,T}^u)$	$\underline{\boldsymbol{\omega}}_{\mathbf{i,T}}^{'}$
	$\underline{\boldsymbol{\omega}}_{\mathbf{1,t}}$	$\underline{\boldsymbol{\omega}}_{\mathbf{2,t}}$	…	$\underline{\boldsymbol{\omega}}_{\mathbf{i,t}}$	…	$\underline{\boldsymbol{\omega}}_{\mathbf{L,t}}$	

Fig. 2. Triangular fuzzy weights matrix of all L attributes by T experts

Let ω_i be the combined triangular fuzzy weight assigned to e_i generated from the weight assignments by T experts ($\underline{\boldsymbol{\omega}}_{\mathbf{i,t}}$), represented by $(\omega_i^l, \omega_i^m, \omega_i^u)$. In this paper, the method to obtain ω_i will be proposed and described in detail.

Suppose $\omega_{i,1}$ and $\omega_{i,2}$ are the triangular fuzzy weights assigned to the ith attribute e_i by two different experts, represented by $(\omega_{i,1}^l, \omega_{i,1}^m, \omega_{i,1}^u)$ and $(\omega_{i,2}^l, \omega_{i,2}^m, \omega_{i,2}^u)$ respectively as shown in Fig. 3. If $\omega_{i,1}^l = \omega_{i,2}^l$, $\omega_{i,1}^m = \omega_{i,2}^m$ and $\omega_{i,1}^u = \omega_{i,2}^u$, then, $\omega_{i,1}$ equals to $\omega_{i,2}$. In general, however, there is $\omega_{i,1} \neq \omega_{i,2}$ because the judgements of two experts are often inconsistent to some extent due to their differences in background, expertise and interest. Suppose the joint middle region of weight assigned to e_i is denoted by $[\omega_{i,1}^m, \omega_{i,2}^m]$, the joint minimum region of weight assigned to e_i is denoted by $[\omega_{i,1}^l, \omega_{i,2}^l]$, and $[\omega_{i,1}^u, \omega_{i,2}^u]$ represents the joint maximum region of weight assigned to e_i. Fig. 3 is just one of the cases in combining two triangular fuzzy weights. From Fig. 3, we could see that $\omega_{i,1}^l < \omega_{i,2}^l < \omega_{i,1}^m < \omega_{i,2}^m < \omega_{i,1}^u < \omega_{i,2}^u$.

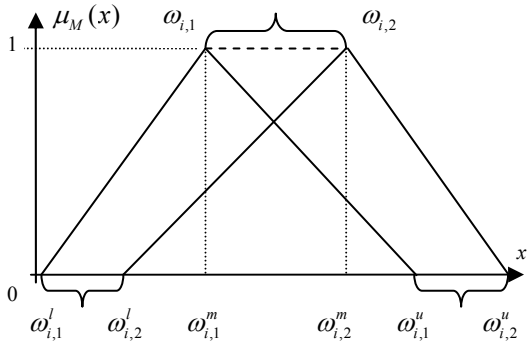

Fig. 3. Combining triangular fuzzy weights of the ith attribute derived from two experts

Another case is that the lower bound of the weight assigned to e_i by an expert (for instance $\omega_{i,2}^l$) may be larger than the middle weight assignment with membership of 1 by another expert (for instance $\omega_{i,1}^m$). In this case, the joint middle region of ω_i ($[\omega_{i,1}^m, \omega_{i,2}^m]$) is overlapped with the joint minimum region of ω_i ($[\omega_{i,1}^l, \omega_{i,2}^l]$) as shown in Fig. 4. The relative comparison of the three points in each of these two triangular fuzzy numbers in this case is $\omega_{i,1}^l < \omega_{i,1}^m < \omega_{i,2}^l < \omega_{i,2}^m < \omega_{i,1}^u < \omega_{i,2}^u$.

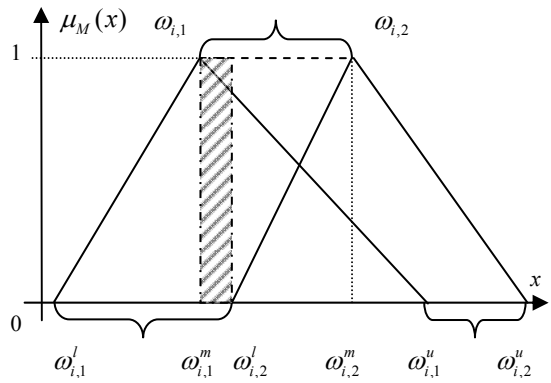

Fig. 4. One case of combining two triangular fuzzy weights from 2 experts

Many other cases may also occur, for example $\omega_{i,1}^l < \omega_{i,2}^l < \omega_{i,2}^m < \omega_{i,1}^m < \omega_{i,1}^u < \omega_{i,2}^u$ as shown in Fig. 5, $\omega_{i,1}^l < \omega_{i,2}^l < \omega_{i,2}^m < \omega_{i,1}^m < \omega_{i,2}^u < \omega_{i,1}^u$ as shown in Fig. 6 and so on.

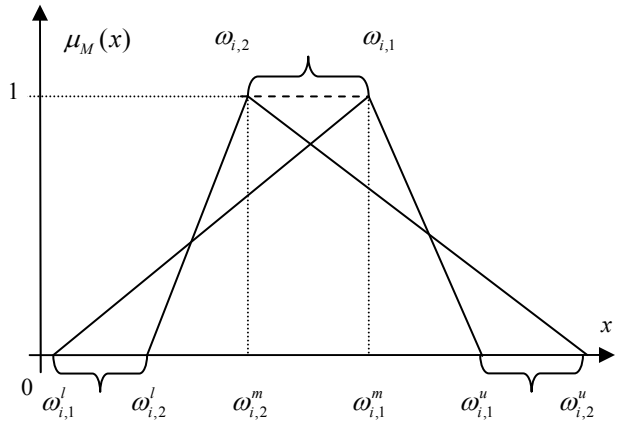

Fig. 5. One case of combining two triangular fuzzy weights from 2 experts

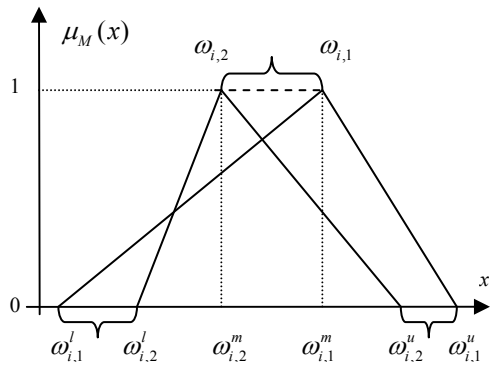

Fig. 6. One case of combining two triangular fuzzy weights from 2 experts

In group decision situations, there may be more complicated cases than shown in Fig. 3-Fig. 6 due to the differences of experts' judgments. A problem arises as to how to combine weights assigned to each attribute by all experts. Fig. 7 shows a triangular fuzzy weight assigned to e_i by T experts.

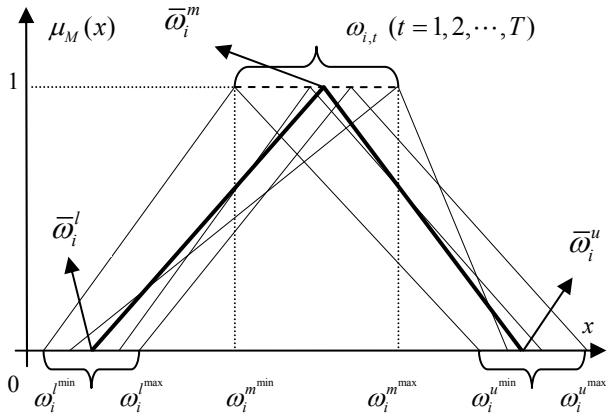

Fig. 7. Combining T triangular fuzzy weights from T different experts

Let $\omega_i^{l^{\min}}$ and $\omega_i^{l^{\max}}$ be the minimum and maximum values of the pessimistic weight judgment assigned to e_i from all T experts respectively, or in mathematical expression

$$\omega_i^{l^{\min}} = \min_{t=1,\cdots,T} \omega_{i,t}^l, \quad \omega_i^{l^{\max}} = \max_{t=1,\cdots,T} \omega_{i,t}^l \tag{26}$$

The minimum and maximum values of the most likely weight judgment $\omega_{i,t}^m$ $(t=1,2,\cdots,T)$ assigned to e_i by T experts are denoted by $\omega_i^{m^{\min}}$ and $\omega_i^{m^{\max}}$ respectively, or

$$\omega_i^{m^{\min}} = \min_{t=1,\cdots,T} \omega_{i,t}^m, \quad \omega_i^{m^{\max}} = \max_{t=1,\cdots,T} \omega_{i,t}^m \tag{27}$$

And the minimum and maximum values of the most optimistic weight judgment $\omega_{i,t}^u$ $(t=1,2,\cdots,T)$ assigned to e_i by T experts are calculated by

$$\omega_i^{u^{\min}} = \min_{t=1,\cdots,T} \omega_{i,t}^u, \quad \omega_i^{u^{\max}} = \max_{t=1,\cdots,T} \omega_{i,t}^u \tag{28}$$

Thus, in the case involving T experts shown in Fig. 7, the joint minimum, middle and maximum region of weight assigned to e_i are denoted by $[\omega_i^{l^{\min}}, \omega_i^{l^{\max}}]$, $[\omega_i^{m^{\min}}, \omega_i^{m^{\max}}]$ and $[\omega_i^{u^{\min}}, \omega_i^{u^{\max}}]$ respectively. The combined pessimistic weight judgment assigned to e_i represented by ω_i^l could be any value in the range of $[\omega_i^{l^{\min}}, \omega_i^{l^{\max}}]$ depending on real decision situations and decision maker's preferences. ω_i^m and ω_i^u could be any values in the ranges of $[\omega_i^{m^{\min}}, \omega_i^{m^{\max}}]$ and $[\omega_i^{u^{\min}}, \omega_i^{u^{\max}}]$ respectively. However, the above ranges only represent the largest possible ranges of values that ω_i^l, ω_i^m and ω_i^u can take. In this paper, we propose more realistic or likely ranges for ω_i^l, ω_i^m and ω_i^u.

For example, ω_i^l is supposed to take the weighted arithmetic mean of $\omega_{i,t}^l$ $(t=1,2,\cdots,T)$ in the region of $[\omega_i^{l^{\min}},\omega_i^{l^{\max}}]$ represented by $\bar{\omega}_i^l$ with standard deviation of δ_i^l. ω_i^m and ω_i^u are also supposed to take the weighted arithmetic mean of $\omega_{i,t}^m$ $(t=1,2,\cdots,T)$ and $\omega_{i,t}^u$ $(t=1,2,\cdots,T)$ respectively, represented by $\bar{\omega}_i^m$ and $\bar{\omega}_i^u$ with standard deviation of δ_i^m and δ_i^u. Thus, we have

$$\bar{\omega}_i^l=\sum_{t=1}^{T}\lambda_t\omega_{i,t}^l,\ \ \delta_i^l=\sqrt{\frac{1}{T}\sum_{t=1}^{T}(\omega_{i,t}^l-\bar{\omega}_i^l)^2} \tag{29}$$

$$\bar{\omega}_i^m=\sum_{t=1}^{T}\lambda_t\omega_{i,t}^m,\ \ \delta_i^m=\sqrt{\frac{1}{T}\sum_{t=1}^{T}(\omega_{i,t}^m-\bar{\omega}_i^m)^2} \tag{30}$$

$$\bar{\omega}_i^u=\sum_{t=1}^{T}\lambda_t\omega_{i,t}^u,\ \ \delta_i^u=\sqrt{\frac{1}{T}\sum_{t=1}^{T}(\omega_{i,t}^u-\bar{\omega}_i^u)^2} \tag{31}$$

$$\sum_{t=1}^{T}\lambda_t=1 \tag{32}$$

From Eq. (29)-(32), we can see that $\bar{\omega}_i^l$, $\bar{\omega}_i^m$ and $\bar{\omega}_i^u$ are bounded in the intervals $[\omega_i^{l^{\min}},\omega_i^{l^{\max}}]$, $[\omega_i^{m^{\min}},\omega_i^{m^{\max}}]$ and $[\omega_i^{u^{\min}},\omega_i^{u^{\max}}]$ respectively, and they may represent an average perspective on the weight assignment to e_i. The calculated combined triangular fuzzy weights of e_i can then be denoted by

$$\bar{\boldsymbol{\omega}}=(\omega_1,\omega_2,\cdots,\omega_i,\cdots,\omega_L) \tag{33}$$

where $\omega_i=(\bar{\omega}_i^l,\bar{\omega}_i^m,\bar{\omega}_i^u)$.

In combining triangular fuzzy weights from several experts, ω_i^l can take the middle value of the interval $[\omega_i^{l^{\min}},\omega_i^{l^{\max}}]$ as well. However, if $\omega_{i,t}^l$ assigned to e_i by most experts are near $\omega_i^{l^{\min}}$, the middle triangular fuzzy weights will be irrational.

ω_i^l, ω_i^m and ω_i^u can also take the weighted geometric mean of $\omega_{i,t}^l$, $\omega_{i,t}^m$ and $\omega_{i,t}^u$ $(t=1,2,\cdots,T)$ respectively as follows:

$$\bar{\omega}_i^l=\prod_{t=1}^{T}\omega_{i,t}^{l\ \lambda_t},\ \bar{\omega}_i^m=\prod_{t=1}^{T}\omega_{i,t}^{m\ \lambda_t},\ \bar{\omega}_i^u=\prod_{t=1}^{T}\omega_{i,t}^{u\ \lambda_t} \tag{34}$$

$$\sum_{t=1}^{T}\lambda_t=1 \tag{35}$$

The results generated by the weighted geometric mean method of Eqs. (34), (35) are different from the combined triangular fuzzy weights generated by the weighted arithmetic mean method (Eq. (29)-(32)). For example, suppose the triangular fuzzy weights of the *i*th attribute assigned by expert 1, expert 2 and expert 3 are (0.1, 0.2, 0.3), (0.3, 0.4, 0.5) and (0.15, 0.25, 0.3) respectively. These three experts are considered to be equally important ($\lambda_1=\lambda_2=\lambda_3=1/3$). Then the combined triangular fuzzy weight by the weighted arithmetic mean method illustrated above is (0.183, 0.283, 0.367) with standard deviations of (0.085, 0.085, 0.096). The result calculated by the weighted geometric mean method is (0.165, 0.271, 0.356).

Such difference also occurs in sampling the three values ω_i^l, ω_i^m and ω_i^u from the intervals $[\omega_i^{l^{\min}},\omega_i^{l^{\max}}]$, $[\omega_i^{m^{\min}},\omega_i^{m^{\max}}]$ and $[\omega_i^{u^{\min}},\omega_i^{u^{\max}}]$ using any other aggregation methods. Thus, sensitivity analysis needs to be conducted for different methods of combining triangular fuzzy weights for informative and reliable decision support.

The combined triangular fuzzy weights $\bar{\boldsymbol{\omega}}$ are then used in group evidential reasoning based programming models which is to be discussed in the next section.

6. Group Evidential Reasoning Approach to Compute Global Belief Degrees under Triangular Fuzzy Weights

6.1 *Using α-cut to Transform Triangular Fuzzy Weights to Interval Weights*

In this paper, *α*-cut is proposed to transform to interval weights the combined triangular fuzzy weights calculated using the method proposed in the previous section. The concept of *α*-cut is briefly reviewed in the following.[38]

The *k*-th *α*-cut M^{α_k} of triangular fuzzy number M is defined as follows:

$$M^{\alpha_k}=\{x\mid\mu_M(x)\geq\alpha_k,x\in X\}\quad k=1,2,\cdots,r \tag{36}$$

where $0<\alpha_k\leq 1$, r denotes the number of *α*-cuts. For instance, Fig. 8 shows a triangular fuzzy weights $\omega_i=(\omega_i^l,\omega_i^m,\omega_i^u)$ with 4 *α*-cuts ($r=4$), where $\alpha_1=0.5$, $\alpha_2=0.7$, $\alpha_3=0.9$ and $\alpha_4=1$.

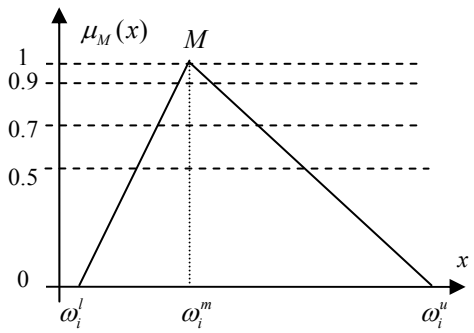

Fig. 8. Triangular fuzzy weights ω_i with four *α*-cuts

The minimum value $\omega_i^{l_k}$ and the maximum value $\omega_i^{u_k}$ of the k-th α-cut on the combined triangular fuzzy weight $\omega_i = (\omega_i^l, \omega_i^m, \omega_i^u)$ are defined as follows:

$$\omega_i^{l_k} = \underset{x \in X}{Inf}\{x \mid \mu_M(x) \geq \alpha_k\} \tag{37}$$

$$\omega_i^{u_k} = \underset{x \in X}{Sup}\{x \mid \mu_M(x) \geq \alpha_k\} \tag{38}$$

Fig. 9 shows the minimum value $\omega_i^{l_k}$ and the maximum value $\omega_i^{u_k}$ of the k-th α-cut on triangular fuzzy weight $\omega_i = (\omega_i^l, \omega_i^m, \omega_i^u)$. They can be calculated as follows:

$$\omega_i^{l_k} = \omega_i^l + \alpha_k(\omega_i^m - \omega_i^l) \tag{39}$$

$$\omega_i^{u_k} = \omega_i^u - \alpha_k(\omega_i^u - \omega_i^m) \tag{40}$$

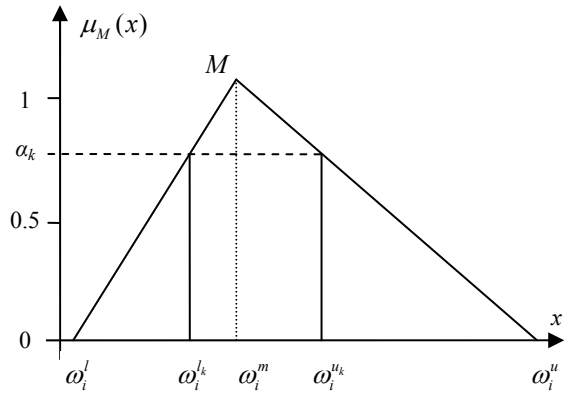

Fig. 9. Minimum and maximum value of the kth α-cut on triangular fuzzy weight ω_i

From the process of α-cut, we can see that if α_k $(0 < \alpha_k \leq 1)$ is selected, then an aggregated interval-valued weight assigned to an attribute e_i from a group of experts with a reliability of at least α_k will be confirmed, represented by $[\omega_i^{l_k}, \omega_i^{u_k}]$. From Fig. 8, it is clear that different α-cuts lead to different $\omega_i^{l_k}$ and $\omega_i^{u_k}$. In other word, with the increase of the value of α_k, the uncertainty of the weight assignment to e_i decreases because the interval region between $\omega_i^{l_k}$ and $\omega_i^{u_k}$ becomes smaller. In other words $\alpha_k \uparrow \Rightarrow \omega_i^{l_k} \uparrow, \omega_i^{u_k} \downarrow \Rightarrow |\omega_i^{u_k} - \omega_i^{l_k}| \downarrow$, and vice versa, where $\uparrow$ indicates increase and $\downarrow$ means decrease. So the sensitivity analysis of α_k being assigned to different values between 0 and 1 should be conducted for the purpose of informative and reliable decision support to the decision maker.

In the field of interval-valued D-S theory and ER approach, lots of research has been conducted. For instance, Lee and Zhu studied the combination of interval evidence.[49] Denoeux and Yager systematically explored the combination and normalization of interval evidence.[50, 51] But their approaches do not provide a satisfactory solution for dealing with interval-valued belief structures.[52] Guo studied the ER approach under both interval weights and interval belief degrees[28] based on Wang's research[27, 52] which was mentioned in section 1 where interval belief degrees are considered. In the following, the models for calculating global belief degrees are constructed according to Wang and Guo's models because the combination and normalization processes are optimized simultaneously which making the results more rational and appropriate.

6.2. *Group Evidential Reasoning Based Programming Models to Generate Global Belief Degrees under Fuzzy Weights*

Based on the analytical ER algorithm and the calculated interval weights from the combined triangular fuzzy weights by α-cut, several group evidential reasoning based programming models for computing the global fuzzy belief degrees are constructed below. Programming model <1> is constructed for the generation of the maximum value of $\beta_n(a_l)$.

$$<1> \qquad Max\ \beta_n(a_l) = \frac{m_n(a_l)}{1 - \bar{m}_H(a_l)} \tag{41}$$

$$(n = 1, 2, \cdots, N)$$

s.t. EQS. (12)-(17), (23), (29)-(32), (39)-(40)

$$\omega_i^{l_k} \leq \omega_i \leq \omega_i^{u_k} \quad (i = 1, 2, \cdots, L) \tag{42}$$

$$\sum_{i=1}^{L} \omega_i = 1 \tag{43}$$

There are L variables in the above programming model which can be solved using "Solver" in Excel. To calculate the minimum value of $\beta_n(a_l)$ $(l = 1, 2, \cdots, S)$, the following model is constructed:

$$<2> \qquad Min\ \beta_n(a_l) = \frac{m_n(a_l)}{1 - \bar{m}_H(a_l)} \tag{44}$$

$$(n = 1, 2, \cdots, N)$$

s.t. EQS. (12)-(17), (23), (29)-(32), (39)-(40), (42)-(43).

Let $\beta_n^{u_k}(a_l)$ and $\beta_n^{l_k}(a_l)$ $(n = 1, 2, \cdots, N)$ be the optimal value of the objective function in the above two models respectively. Then the aggregated fuzzy belief degree of a_l assigned to H_n under the k-th α-cut of the combined fuzzy weights can be generated and presented by $\beta_n(a_l) \in [\beta_n^{l_k}(a_l), \beta_n^{u_k}(a_l)]$.

For the generation of $\beta_H(a_l)$ $(l = 1, 2, \cdots, S)$, two programming models may also be constructed as follows:

$$<3>\quad Max\ \beta_H(a_l) = \frac{\tilde{m}_H(a_l)}{1-\bar{m}_H(a_l)} \tag{45}$$

s.t. EQS. (12)-(17), (23), (29)-(32), (39)-(40), (42)-(43).

$$<4>\quad Min\ \beta_H(a_l) = \frac{\tilde{m}_H(a_l)}{1-\bar{m}_H(a_l)} \tag{46}$$

s.t. EQS. (12)-(17), (23), (29)-(32), (39)-(40), (42)-(43).

Let $\beta_H^{u_k}(a_l)$ and $\beta_H^{l_k}(a_l)$ be the optimal values of programming model <3> and <4> respectively. Then, we have $\beta_H(a_l) \in [\beta_H^{l_k}(a_l), \beta_H^{u_k}(a_l)]$. Based on programming models <1> to <4>, the global fuzzy belief degrees assigned to a_l under the *k*-th *α*-cut of the combined triangular fuzzy weights in group decision situation can then be represented as follow

$$\begin{aligned} &S(y(a_l)) = \{(H_n, \beta_n(a_l) \in [\beta_n^{l_k}(a_l), \beta_n^{u_k}(a_l)]), \\ &n = 1,2,\cdots,N;\ H, \beta_H(a_l) \in [\beta_H^{l_k}(a_l), \beta_H^{u_k}(a_l)]\} \end{aligned} \tag{47}$$

where

$$\beta_H(a_l) + \sum_{n=1}^{N} \beta_n(a_l) = 1$$

Model <1> and model <2> each need to be solved *N* times to generate the maximum values of the combined fuzzy belief degrees of alternative a_l on all *N* evaluation grades. Models <3> and <4> each only need to be solved once to calculate $\beta_H^{u_k}(a_l)$ and $\beta_H^{l_k}(a_l)$ respectively.

7. Group Evidential Reasoning Approach to Compute Overall Utility under Fuzzy Weights and Utilities

7.1. *Using Fuzzy Value to Represent Evaluation Grade Utility*

In the existent ER approach where weights and utilities are considered to be crisp values, the maximum, minimum and average utility are calculated to transform Eq. (21) to a single value for the purpose of comparing several alternatives clearly.[6, 7] They are defined as follows:

$$u_{\max}(a_l) = \sum_{n=1}^{N} \beta_n(a_l) \cdot u(H_n) + \beta_H(a_l) \cdot u(H_N) \tag{48}$$

$$u_{\min}(a_l) = \sum_{n=1}^{N} \beta_n(a_l) \cdot u(H_n) + \beta_H(a_l) \cdot u(H_1) \tag{49}$$

$$u_{ave}(a_l) = \frac{u_{\max}(a_l) + u_{\min}(a_l)}{2} \tag{50}$$

In the above formulae, a function needs to be defined for each evaluation grade H_n $(n = 1,2,\cdots,N)$. Utility function represented by $u(H_n)$ is used in Refs. 6, 7, 8, 13, 20, where $u(H_n)$ is defined to be the utility of the evaluation grade H_n $(n = 1,2,\cdots,N)$. $u(H_{n+1})$ is assumed to be larger than $u(H_n)$ if H_{n+1} is preferred to H_n. From the definition of utility function, we could see that an alternative will be judged to be of high level if it gets a larger utility value.

In a utility function, the subjective judgment of the decision maker (DM) is taken into account. Risk preferences and utility functions are different for different DMs and decision situations. There are three basic types of utility functions which are risk taking, risk neutral and risk averse respectively. In real group assessment problems, the utility of an evaluation grade estimated by a group of experts is not a crisp value in general. In this paper, the utilities of all evaluation grades are assumed to be fuzzy (interval) numbers, whose values are represented below:

$$u(H_n) \in [u_n^l, u_n^u] \quad n=1, \ldots, N \tag{51}$$

7.2. *Group Evidential Reasoning Based Programming Models for Utility Function under Both Fuzzy Weights and Fuzzy Utilities*

Given that weights are presented as fuzzy numbers and utility estimation may be uncertain or ambiguous, if we use programming model <1>-<4> under fuzzy weights to generate a global belief degrees as Eq. (47) firstly and then utilize equation Eqs. (48)-(50) for the generation of overall utility values, the real utility intervals assessed on an alternative could not be obtained.[27, 28]

So, based on the analytical ER algorithm, fuzzy weights and utilities, the group ER based programming models for utility function could be constructed according to Wang and Guo's model as follows.

$$<5>\ Max\ u_{\max}(a_l) = \sum_{n=1}^{N} \beta_n(a_l) \cdot u(H_n) + \beta_H(a_l) \cdot u(H_N)$$

s.t. EQS. (12)-(19), (23), (29)-(32), (39)-(40), (42)-(43)

$$u_n^l \le u(H_n) \le u_n^u \quad n = 1,2,\cdots,N$$

$$<6>\ Min\ u_{\min}(a_l) = \sum_{n=1}^{N} \beta_n(a_l) \cdot u(H_n) + \beta_H(a_l) \cdot u(H_1)$$

s.t. EQS. (12)-(19), (23), (29)-(32), (39)-(40), (42)-(43)

$$u_n^l \le u(H_n) \le u_n^u \quad n = 1,2,\cdots,N$$

From the two group ER based programming models above, the overall maximum and minimum expected utilities under the *k*th *α*-cut of weights can then be generated as $U^{u_k}(a_l)$ and $U^{l_k}(a_l)$ respectively. Then the overall fuzzy utility assessed for a_l can be

represented as follows:

$$U(a_l) = [U^{l_k}(a_l), U^{u_k}(a_l)] \quad (52)$$

The whole process of this group evidential reasoning approach proposed in section 5-7 is shown in Fig. 10 as follows.

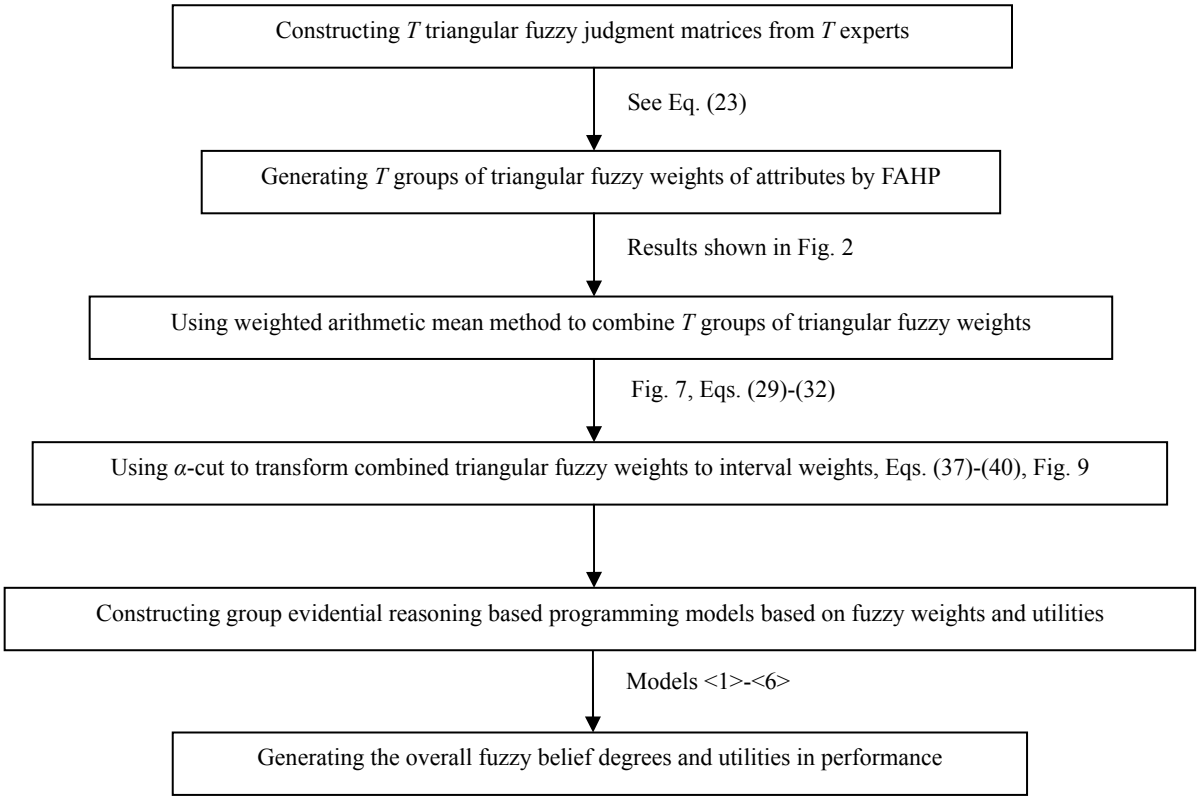

Fig. 10. Process of group evidential reasoning approach under fuzzy weights and utilities

8. Application

In this section, the group ER approach is applied to assess the performance of products in a manufacturing factory. In the life cycle of a project, our assessment is in the period that the development of a project has been completed or already in market for some time. As the result of assessment, the score assigned to a product can be obtained on the general level that could reflect the performance of completion or operation. A product can spread its performance in design and development whereas poor performance should be identified and discussed for future improvements. Five steps are involved in this process, summarized as follows:

8.1. *Construction of the assessment framework*

The research was conducted in close collaboration with the managers and faculties in the departments of a factory in China whose name is not mentioned here due to its business confidentiality. The constructed attribute framework is shown in Fig. 11.

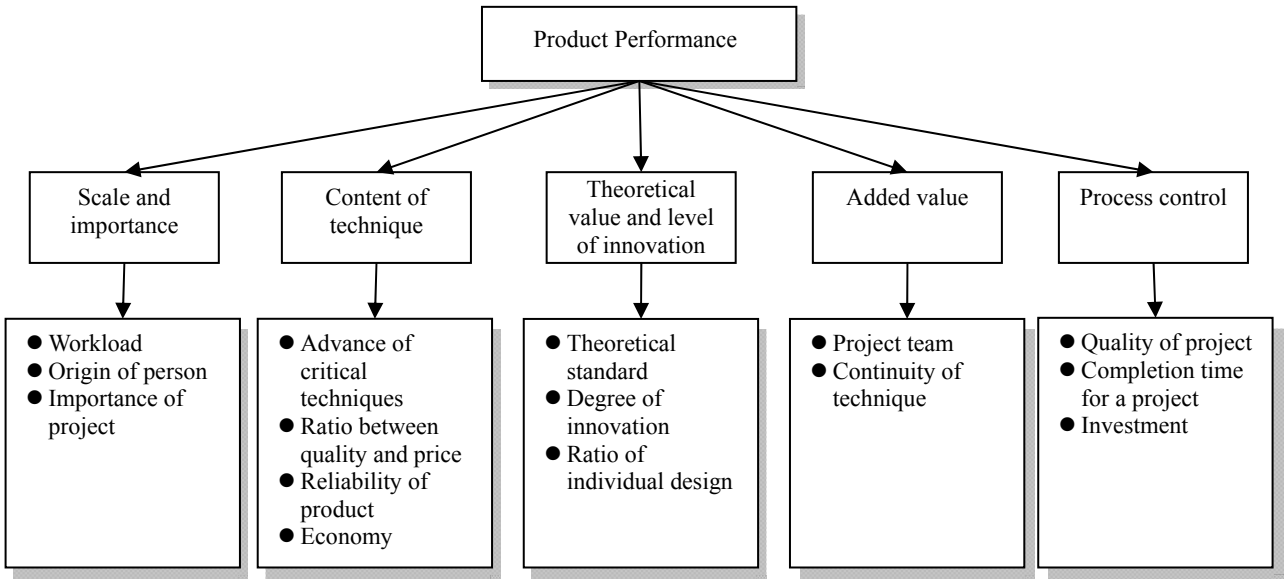

Fig. 11. Attribute framework for the assessment of product performance

8.2. *Assignment of belief degrees in performance*

Five evaluation grades are involved for the assessment of a product. They are Bad, Poor, Average, Good and Very good, which form the frame of discernment together as follows:

$$H = \{H_1, H_2, H_3, H_4, H_5\} = \{Bad, Poor, Average, Good, Very\ good\} \quad (53)$$

Table 2. Generalized decision matrix for product assessment and belief degrees of attributes

General attributes	Sub-attributes	Belief degrees	
		a_1	a_2
Scale and importance E_1 (ω_1)	Workload e_{11} (ω_{11})	(H_2, 0.6; H_3, 0.4)	(H_4, 0.5; H_5, 0.5)
	Origin of person e_{12} (ω_{12})	(H_4, 0.9; H_5, 0.1)	(H_5, 0.8)
	Importance of project e_{13} (ω_{13})	(H_2, 04.; H_3, 0.4)	(H_2, 0.3; H_3, 0.7)
Content of technique E_2 (ω_2)	Advance of critical techniques e_{21} (ω_{21})	(H_5,0.9)	(H_2, 0.1; H_3, 0.4; H_4, 0.5)
	Ratio between quality and price e_{22} (ω_{22})	(H_4,1.0)	(H_3, 0.2; H_4, 0.7; H_5, 0.1)
	Reliability of product e_{23} (ω_{23})	(H_3, 0.2; H_4, 0.3; H_5, 0.5)	(H_4, 1.0)
	Economy e_{24} (ω_{24})	(H_2, 0.2; H_3, 0.8)	(H_1, 0.1; H_2, 0.2; H_3,0.4; H_4,0.3)
Theoretical value and level of innovation E_3 (ω_3)	Theoretical standard e_{31} (ω_{31})	(H_3, 0.3; H_4, 0.5; H_5, 0.2)	(H_3,0.2; H_4,0.7)
	Degree of innovation e_{32} (ω_{32})	(H_2, 0.1; H_3, 0.1; H_4, 0.2; H_5, 0.6)	(H_4, 0.7;H_5, 0.3)
	Ratio of individual design e_{33} (ω_{33})	(H_3, 0.3, H_4, 0.6)	(H_3,0.9)
Added value E_4 (ω_4)	Project team e_{41} (ω_{41})	(H_1, 0.3; H_2, 0.2; H_3, 0.5)	(H_4,0.7; H_5,0.3)
	Continuity of technique e_{42} (ω_{42})	(H_4, 1.0)	(H_2,0.2; H_3, 0.7)
Process control E_5 (ω_5)	Quality of project e_{51} (ω_{51})	(H_3, 0.4; H_4, 0.6)	(H_3,0.3; H_4, 0.5; H_5, 0.2)
	Completion time for a project e_{52} (ω_{52})	(H_1, 0.1; H_2, 0.1; H_3, 0.5; H_4, 0.3)	(H_3,0.3; H_4, 0.7)
	Investment e_{53} (ω_{53})	(H_4, 0.9)	(H_2,0.3; H_3, 0.3; H_4, 0.3)

Two products which form the set of alternatives $A = \{a_1, a_2\}$ are involved in the case study. The belief degrees assigned to the two products on each attribute are shown in Table 2.

Table 2 shows a distributed view of the assessment of each product on every attribute. For example, "workload" for a_1 is assessed to be poor with a belief degree of 0.6 and to be average with a belief degree of 0.4". The above statement can also be represented by the following expectation:

$$S(e_{11}(a_1)) = \{(Poor, 0.6), (Average, 0.4)\} \quad (54)$$

8.3. *Generating the combined triangular fuzzy weight of each attribute*

Just as discussed in section 5.1, a group of experts who may have different backgrounds or expertise and may represent conflicting interests are involved in assigning the weights of attributes.

Firstly, pairwise comparisons between every two attributes associated with their upper level attribute are provided by each expert in the form of triangular fuzzy number to construct his judgment matrix. For example, a five-dimensional triangular fuzzy judgment matrix can be generated to compare the relative importance of the five general level attributes (E_1, E_2, E_3, E_4 and E_5) by an expert. For the comparison of the importance of the three sub-level attributes (e_{11}, e_{12}, e_{13}) associated with their upper level attribute (E_1), a three-dimensional triangular fuzzy judgment matrix can be constructed. Similarly, a four-dimensional matrix, three-dimensional matrix, two-dimensional matrix and three-dimensional matrix can be formed for comparing the importance of the sub-level attributes associated with E_2, E_3, E_4 and E_5. So, six triangular fuzzy judgment matrices are generated by each expert.

Secondly, FAHP is used to calculate the triangular fuzzy weights of attributes by each expert. In this case study, there are 4 experts involved in generating the combined triangular fuzzy weights of attributes. Due to the limited space of this paper, the detailed process to calculate triangular fuzzy weights from each expert using FAHP is not described here. The triangular fuzzy weights calculated from these 4 experts are presented in Table 3.

Table 3. The abstract triangular fuzzy weights of attributes generated from 4 experts by FAHP

Expert 1	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}
$\omega_{i,1}^{l}$	0.0231	0.0277	0.0525	0.0242	0.0267	0.0409	0.0277	0.0231	0.0351	0.0234	0.0899	0.0955	0.1220	0.0522	0.0698
$\omega_{i,1}^{m}$	0.0298	0.0362	0.0660	0.0346	0.0407	0.0578	0.0409	0.0328	0.0503	0.0349	0.1215	0.1255	0.1658	0.0776	0.0855
$\omega_{i,1}^{u}$	0.0422	0.0493	0.0915	0.0457	0.0543	0.0753	0.0508	0.0443	0.0654	0.0475	0.1511	0.1578	0.1971	0.1006	0.1068

Expert 2	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}
$\omega_{i,2}^{l}$	0.0214	0.0197	0.0403	0.0219	0.0283	0.0545	0.0183	0.0190	0.0281	0.0253	0.0696	0.0994	0.1173	0.0664	0.0503
$\omega_{i,2}^{m}$	0.0310	0.0284	0.0537	0.0353	0.0442	0.0779	0.0316	0.0291	0.0380	0.0339	0.1058	0.1443	0.1617	0.0992	0.0861
$\omega_{i,2}^{u}$	0.0436	0.0398	0.0686	0.0484	0.0630	0.1007	0.0432	0.0464	0.0591	0.0533	0.1363	0.1779	0.1998	0.1287	0.1146

Expert 3	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}
$\omega_{i,3}^{l}$	0.0211	0.0070	0.0409	0.0324	0.0231	0.0491	0.0223	0.0338	0.0400	0.0301	0.0746	0.1189	0.1092	0.0474	0.0870
$\omega_{i,3}^{m}$	0.0316	0.0117	0.0548	0.0473	0.0352	0.0701	0.0344	0.0427	0.0500	0.0403	0.1029	0.1631	0.1375	0.0673	0.1112
$\omega_{i,3}^{u}$	0.0539	0.0217	0.0796	0.0697	0.0591	0.0968	0.0569	0.0654	0.0734	0.0613	0.1333	0.2012	0.1870	0.1048	0.1591

Expert 4	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}
$\omega_{i,4}^{l}$	0.0165	0.0187	0.0568	0.0453	0.0280	0.0642	0.0232	0.0132	0.0300	0.0195	0.0895	0.0507	0.1069	0.0187	0.0620
$\omega_{i,4}^{m}$	0.0310	0.0329	0.0920	0.0672	0.0479	0.0858	0.0341	0.0234	0.0483	0.0293	0.1391	0.0849	0.1644	0.0321	0.0875
$\omega_{i,4}^{u}$	0.0542	0.0665	0.1562	0.0975	0.0695	0.1199	0.0550	0.0416	0.0881	0.0596	0.1932	0.1200	0.2117	0.0504	0.1210

Thirdly, the weighted arithmetic mean method introduced in section 5.2 is to be used to aggregate the triangular fuzzy weights of each attribute generated by all the four experts. Suppose the relative importance of each expert is λ_t $(t = 1,2,3,4)$, and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.25$. The combined triangular fuzzy weight of each attribute is shown in Table 4.

Table 4. The combined triangular fuzzy weights of attributes

Combined weights	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}
$\bar{\omega}_i^l$	0.0205	0.0183	0.0476	0.0310	0.0265	0.0522	0.0229	0.0223	0.0333	0.0246	0.0809	0.0911	0.1139	0.0462	0.0673
$\bar{\omega}_i^m$	0.0309	0.0273	0.0666	0.0461	0.0420	0.0729	0.0353	0.0320	0.0467	0.0346	0.1173	0.1295	0.1574	0.0691	0.0926
$\bar{\omega}_i^u$	0.0485	0.0443	0.0990	0.0653	0.0615	0.0982	0.0515	0.0494	0.0715	0.0554	0.1535	0.1642	0.1989	0.0961	0.1254

8.4. *Generating the global fuzzy belief degrees*

After the generation of combined triangular fuzzy weights, the group ER based programming model described in section 6 and 7 can then be used to calculate the global fuzzy belief degrees. For the convenience of constructing models, α-cut is used to transform the combined triangular fuzzy weights to interval-valued weights. Here, $\alpha_k (k = 1)$ is firstly assumed to be 0.5. The calculated interval weights generated under this α-cut are shown in the first chart of Table 5.

Table 5. Interval weights generated under the three α-cuts

α_1=0.5	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}	SUM
$\omega_i^{l_1}$	0.0257	0.0228	0.0571	0.0385	0.0343	0.0625	0.0291	0.0271	0.0400	0.0296	0.0991	0.1103	0.1356	0.0576	0.0799	0.8492
$\omega_i^{u_1}$	0.0397	0.0358	0.0828	0.0557	0.0517	0.0855	0.0434	0.0407	0.0591	0.0450	0.1354	0.1468	0.1781	0.0826	0.1090	1.1914

α_2=0.7	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}	SUM
$\omega_i^{l_2}$	0.0278	0.0246	0.0609	0.0416	0.0374	0.0667	0.0315	0.0291	0.0426	0.0316	0.1064	0.118	0.1443	0.0622	0.0850	0.9095
$\omega_i^{u_2}$	0.0361	0.0324	0.0763	0.0519	0.0478	0.0805	0.0401	0.0372	0.0541	0.0408	0.1282	0.1399	0.1698	0.0772	0.1024	1.1148

α_3=0.9	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{24}	e_{31}	e_{32}	e_{33}	e_{41}	e_{42}	e_{51}	e_{52}	e_{53}	SUM
$\omega_i^{l_3}$	0.0298	0.0264	0.0647	0.0446	0.0405	0.0708	0.0340	0.0310	0.0453	0.0336	0.1137	0.1256	0.1530	0.0668	0.0900	0.9699
$\omega_i^{u_3}$	0.0326	0.0290	0.0699	0.0480	0.0439	0.0754	0.0369	0.0337	0.0491	0.0367	0.1209	0.1329	0.1615	0.0718	0.0959	1.0383

The combined fuzzy belief degrees of the two assessed products under this α-cut can then be generated and shown in the first chart of Table 6 and Fig. 12-13 as follows:

Table 6. The global fuzzy belief degrees of the 2 assessed products under α_1=0.5

	a_1		a_2	
	$\beta_n^{l_1}$	$\beta_n^{u_1}$	$\beta_n^{l_1}$	$\beta_n^{u_1}$
H_1	0.029496	0.044523	0.002289	0.003590
H_2	0.061066	0.097372	0.061672	0.092249
H_3	0.194210	0.267619	0.263761	0.354351
H_4	0.432373	0.568410	0.411473	0.521357
H_5	0.081055	0.127194	0.091288	0.135941
H	0.04604	0.06461	0.02925	0.04131

The global fuzzy belief degrees of the 2 assessed products under α_2=0.7

	a_1		a_2	
	$\beta_n^{l_2}$	$\beta_n^{u_2}$	$\beta_n^{l_2}$	$\beta_n^{u_2}$
H_1	0.032358	0.041397	0.002506	0.003287
H_2	0.066920	0.088611	0.066975	0.085295
H_3	0.208799	0.252852	0.281409	0.335707
H_4	0.460531	0.542142	0.434727	0.500593
H_5	0.088566	0.116182	0.098888	0.125663
H	0.04928	0.06041	0.03129	0.03851

The global fuzzy belief degrees of the 2 assessed products under α_3=0.9

	a_1		a_2	
	$\beta_n^{l_3}$	$\beta_n^{u_3}$	$\beta_n^{l_3}$	$\beta_n^{u_3}$
H_1	0.035278	0.038275	0.002732	0.002995
H_2	0.072862	0.080107	0.072320	0.078472
H_3	0.223644	0.238381	0.299187	0.317299
H_4	0.488291	0.515555	0.457746	0.479725
H_5	0.096130	0.105261	0.106546	0.115452
H	0.05250	0.05622	0.03330	0.03572

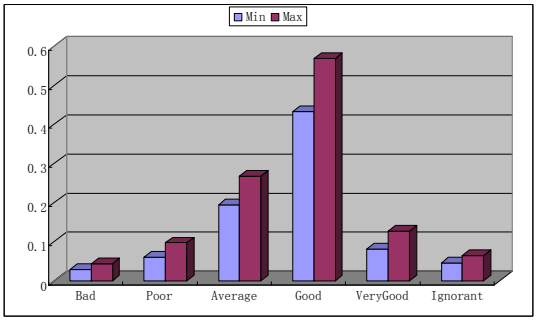

Fig. 12. Global fuzzy belief degrees of a_1 under $\alpha_1 = 0.5$

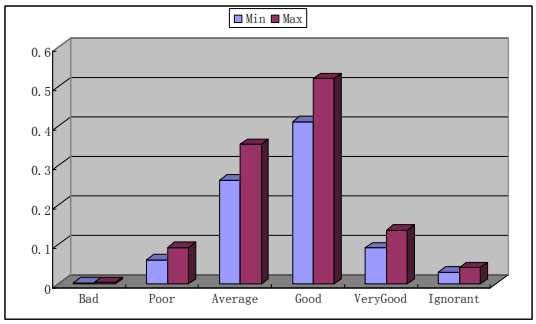

Fig. 13. Global fuzzy belief degrees of a_2 under $\alpha_1 = 0.5$

Table 6 provides us a general distributed assessment on each product under a given α-cut. For example, with more than 50% confidence of the weight assignments, the global assessment to a_1 is generated to be bad (H_1) with belief degree between 0.029496 and 0.044523, poor (H_2) with belief degree between 0.061066 and 0.097372, average (H_3) with belief degree between 0.19421 and 0.267619, good (H_4) with belief degree between 0.432373 and 0.56841, very good (H_5) with belief degree between 0.081055 and 0.127194, and the belief degree not assigned to any evaluation grade is between 0.04604 and 0.06461.

8.5. *Generating the overall fuzzy utility of performance*

As mentioned in section 7.1, the global fuzzy belief degrees generated above are not easy to be used for the comparison of the performance of all products. From the group ER based programming models for utility function discussed in section 7.2, the fuzzy utility of each assessed product can be obtained on a general level. Here, the utility value is assumed to be in the interval between 0 and 100, and the bounds of all the 5 evaluation grades are shown below:

$$17 \le u(H_1) \le 23,\ \ 35 \le u(H_2) \le 43,\ \ 58 \le u(H_3) \le 65,$$
$$76 \le u(H_4) \le 82,\ \ 95 \le u(H_5) \le 100$$

After calculation, we can obtain the overall fuzzy utilities of all assessed products which are shown in Table 7 and Fig. 14 below.

Table 7. Overall fuzzy utilities of the two assessed products under α= 0.5, 0.7, and 0.9

k	α_k	a_1		a_2	
		U^{l_k}	U^{u_k}	U^{l_k}	U^{u_k}
1	0.5	62.84451	77.46958	65.14661	77.87031
2	0.7	63.82331	76.77284	66.00637	77.21913
3	0.9	64.78698	76.06942	66.85651	76.59481

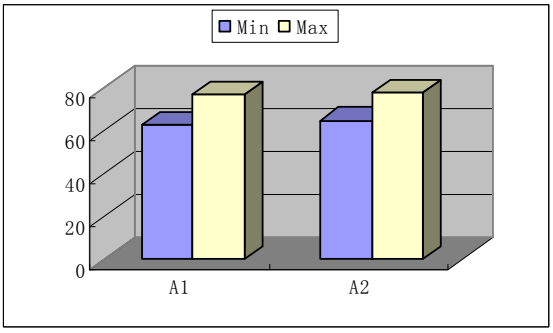

Fig. 14. Overall fuzzy utilities of the 2 products under $\alpha_1 = 0.5$

Based on the overall fuzzy expected utility generated, the ranking of the two products can then be conducted next. Let $P_{\alpha_k}(a_s \succ a_t)$ be the degree that a_s prefers to a_t under the interval weights calculated by the k-th α-cut (α_k), then we will have:

$$P_{\alpha_k}(a_s \succ a_t) = \max\{1 - \max\left[\frac{U^{u_k}(a_t) - U^{l_k}(a_s)}{[U^{u_k}(a_t) - U^{l_k}(a_t)] + [U^{u_k}(a_s) - U^{l_k}(a_s)]}, 0\right], 0\} \tag{55}$$

where the symbol "$\succ$" means "better than". From formula (55), the comparison between a_1 and a_2 under $\alpha_1 = 0.5$ can be generated as follows:

$$P_{0.5}(a_1 \succ a_2) = 45.1\%$$

The final assessment results still need further consideration, but the information provided by the result is valuable and supportive to the leader in a company for his future decision and plan.

In the above case study, the sensitivity analysis of fuzzy weights and fuzzy utilities are also conducted. When α_k increases, the fuzziness of interval weight assigned to each attribute decreases, and the fuzziness of global belief degrees on each evaluation grade for a product generated by the programming model becomes more accurate. In other words, if $\alpha_k \uparrow$, then $\forall i \in (1,2,\cdots,L)$, $\left|\omega_i^{u_k} - \omega_i^{l_k}\right|$ reduces, and $\forall n \in (1,2,\cdots,N)$, $|\beta_n^{u_k}(a_l) - \beta_n^{l_k}(a_l)|$ will decrease. Another two different α-cuts ($\alpha_2 = 0.7$, $\alpha_3 = 0.9$) are introduced for the transformation of triangular fuzzy weights to interval weights which are shown in the second and third chart of table 5. Based on them the global fuzzy belief degrees assessed on each alternative are calculated and shown in the second and third chart of table 6. The fuzziness of the overall utility assessed on an alternative will also decrease when α_k increases or utility interval $|u_n^u - u_n^l| (\forall n = 1,2,\cdots,N)$ reduces.

From the analysis of the result, we could see that for the purpose of providing the decision maker a more accurate result, it is important to minimize the uncertainties or ambiguities of information provided by experts in the condition that the information are used sufficiently and not distorted. It is really an opposition to the fuzzy decision making environment, and the balance between certainty and uncertainty/fuzziness should be operated according to real decision making problems. The choice of α_k is also significant because it influences the fuzziness of assessment result directly. So several or more different α-cuts can be conducted, from which the real satisfactory assessment results can be selected according to real decision making circumstances.

9. Concluding remarks

Due to the complexity of real life decision environments, the decision made by an individual is not always reliable or rational. It is common that a group of DMs are involved in a decision making process to reduce the risk of making poor decisions. On the other hand, a DM's judgement may be inaccurate due to various types of uncertainties. So, there is a need to represent fuzzy or inaccurate information of an expert. In this paper, both attribute weights and evaluation grade utilities are considered to be fuzzy numbers. In our approach, FAHP is firstly used to compute the triangular fuzzy weights of attributes from each triangular fuzzy judgment matrix constructed by an expert. Then, the weighted arithmetic mean method is used to generate combined triangular fuzzy weights for a group of experts. Thirdly, α-cut is used to transform the combined triangular fuzzy weights to interval weights in a reliable and rational way. Finally, both the interval weights under α-cut and interval utilities are considered together to construct several group ER based programming models for the generation of the global fuzzy belief degrees and overall fuzzy utility values of each assessed alternative. Sensitivity analysis is also conducted on α-cuts of fuzzy weights with different α values. A numerical example to assess the performance of product is introduced for the illustration of the group ER approach under uncertainties.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Nos. 71171071, 71231004, 71201042 and 60736026), High Educational Humanities & Social Science Research Base Project of Anhui Province (No. 2012AJRW0293), and the Fundamental Research Funds for the Central Universities (Nos. 2011HGQC1000 and 2010HGBZ0617).

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