

Evidential reasoning approach with multiple kinds of attributes and entropy-based weight assignment

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ABSTRACT

Multiple attribute decision making (MADM) problems often consists of quantitative and qualitative attributes which can be assessed by numerical values and subjective judgments. Subjective judgments can be evaluated by linguistic variables, and both numerical values and subjective judgments can be accurate or uncertain. The evidential reasoning (ER) approach provides a process for dealing with MADM problems of both a quantitative and qualitative nature under uncertainty. The existing ER approach considers both benefit and cost attributes in the evidence combination process. In this paper, deviated interval and fixed interval attributes are introduced into ER based MADM approach and the frames of discernment for representing these two kinds of attributes are given. The transformation rules from the assessment values of deviated interval attributes to belief degrees in the ER structure are then studied. An ave-entropy based weight assignment method considering the risk preference of decision maker is also shown to deal with uncertain assessment situation, such as belief distribution with qualitative attribute and uncertain utility function. Some programming models to generate interval weights and utilities are constructed. The rationality and efficiency of the methods in supporting MADM problems are discussed. Two case studies are provided to demonstrate the applicability and validity of the proposed approaches and the potential in supporting MADM under uncertainty.

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1. Introduction

Multiple attribute decision making (MADM) problem often include both quantitative and qualitative attributes. The Evidential reasoning (ER) approach [1–6] is well suited in dealing with MADM problems where ambiguity, incompleteness and fuzziness are involved in. The unique characteristic of the ER approach is that it can represent uncertainty and ignorance in a MADM problem in a systematic and consistent way.

In recent years, the ER approach has been developed in many applications, for instance, R&D projects assessment [3], consumer preferences extraction [7], data classification [8], medical quality assessment [9], fault diagnosis [10–13], belief rule-based inference [14–18], system reliability prediction [19], failure mode and effects analysis [20], the performance of VMI alliance [4], life cycle assessment [21], optimal power system dispatch [22], urban bus transit network assessment [23] and so on.

In a MADM problem, benefit and cost attributes are always considered. It is well-known that benefit attribute (BA) is that the higher value an attribute is assessed to, the better it is considered. On the contrary, cost attribute (CA) is just on the opposite

compared with BA. These two kinds of attributes are very common in real world. In addition to the BA and CA, there are other four kinds of attributes in real life MADM problems, which are deviated attribute (DA), deviated interval attribute (DIA), fixed attribute (FA) and fixed interval attribute (FIA) [24,25]. FIA means that there exists a desired interval value, if the distance between an assessment value and the desired interval value is small, it is assumed to be better, no matter whether the value is larger or less than the desired interval value. In the life cycle assessment (LCA) of a product, e.g., a mobile phone, the designed life length for it is a FIA because it should neither be too short nor too long. If the life length is too short, it may have no competitiveness in the market; while it is designed for an extremely long life length, the R&D cost would increase greatly, and it is also not necessary because customer want to exchange for a new mobile phone when the old one has been used for a period of time. In the case study of [23], ‘employees per bus’ and ‘passengers carried per bus’ are just two FIAs because they are regarded to be good neither with a too small nor too large value. The well-known ‘asset-liability ratio’ is a typical FIA because it is supposed to be appropriate between 40%–60%. When the value is lower than the interval, the utilization rate of funds is low; while the risk is high if the value is larger than the interval. In fault diagnosis of industrial equipment, a fault model of

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equipment can be reflected by a given historical sampling data set (HSDS for short). The smaller distance between a testing sample and the HSDS of a certain fault model, the more probable that this testing sample points to this fault model, namely, this fault model perhaps happens [10–13]. DIA is just opposite to FIA in the sense that the larger the distance between the assessment value and the undesired interval value, the better the value is considered. In an earthquake, ‘the longitude and latitude’ is a DIA because the farther the distance from the epicenter area, the safer it is. ‘The travel time’ in a city is also a DIA because we should avoid the rush hour to save the travel time no matter by car or bus. It can be seen that there are a lot of such two kinds of attributes in real life problems. In the previous studies, the ER approach has not yet been explored to take into account these attributes. The existing researches on these four kinds of attributes are mainly focused on the normalization of the values of these attributes. For example, Hwang summarized the methods to normalizing the values of BA and CA in [26]. In [24], the concept of DIA was proposed and the normalization process of it was given, while the normalization method of FIA was presented in [27]. In [28] and [29], the normalization process of FA which is a special case of FIA was discussed. Zhou et al. provided the transformation methods from the values of FA to belief distributions in [30], but only complete assessment is considered. In summary, the above studies have not been devoted to coping with decision making problems under ignorance and fuzziness. In this paper, the frames of discernment of DIA and FIA are proposed in the ER framework. Then the evidence transformation rules from values of DIA to belief degrees on evaluation grades of the general frame of discernment are studied.

There are different kinds of techniques to represent uncertainties in MADM problems, for example, interval value [23,31–35], fuzzy number [4,36], belief distribution in the ER approach and so on. As we all know, entropy method [26,37–43] for assigning attribute weight is a data-driven objective approach. In the traditional entropy method, the values of all attributes are assumed to be precise values that can be transformed to entropies no matter the attributes are quantitative or qualitative ones. When the decision situation is uncertain and the quantitative attributes are assigned with interval or fuzzy values, how can the entropy method be applied to generate the objective weights in an appropriate way is significant. Some studies have been devoted to it [37,40,44]. Moreover, when qualitative attributes are assessed by subjective judgments such as belief distributions, the method for measuring the discrepancy among the distributions of all alternatives on a certain attribute will lead to the credibility of the generated weights. The kernel of the problem lies in that how to tackle with uncertainties and ignorance included in the assessment to quantitative or qualitative attributes when we generate the objective weights of attributes. The ER approach has been studied in situations of fuzziness, uncertainties and ambiguities to dealing with different real life MADM problems. For example, the situation of interval uncertainty [32,45], fuzzy evaluation grade [46], interval value [23,31], interval belief degree [31], interval weight [47], fuzzy utility [4], interval difference [48] and interval reliability [21] have all been discussed. How to obtain the attribute weights from these different types of uncertain information still needs to be discussed. In this paper, an ave-entropy based weight assignment process from uncertain and incomplete subjective judgments is studied.

Furthermore, the objective weight assignment methods focus on the differences among alternatives and generate attribute weights from data alone without requiring any preference information from the DMs [49]. Utility represents the degree of preference that a decision maker (DM) considers the value of an option. In a utility function, the subjective judgment of DM is taken into account. Different types of utility function may be constructed to show the attitude of different DMs towards risk. There are three

basic types of utility function which are risk taking, risk neutral and risk averse respectively. In real group MADM problems, the utility of an evaluation grade estimated by a group of experts is not a crisp value in general. Some studies have focused on the risk preferences of different DMs in generating utility function. For instance, Zhou discussed how to generate a general assessment under fuzzy utilities [4,36], but the weight assignment method is FAHP which is a subjective process. In this paper, the dissimilarity of risk preferences among different DMs is considered in generating the attribute weights.

The main contributions of the paper can be summarized as follows:

(1) The frames of discernment of DIA and FIA are proposed respectively in the ER based MADM framework. The equivalent rules between assessment values of DIA or FIA and evaluation grades in the frame of discernment are presented.

(2) Transformation rules from values of DIA to belief degrees on evaluation grades of the general frame of discernment are constructed.

(3) An ave-entropy based weight assignment process considering the risk preference of DM is shown to tackle with MADM problems where uncertain subjective judgments such as belief distributions are included.

The remainder of this paper is organized as follows. Section 2 is a brief introduction about the ER approach. In Section 3, the frames of discernment of DIA and FIA are firstly constructed, and then the extended transformation rules for DIA are given. In Section 4, ave-entropy based weight assignment process based on uncertain and incomplete subjective judgment considering the risk preferences of DMs is proposed. Section 5 presents two case studies to illustrate the approaches in Sections 3 and 4. This paper is concluded in Section 6.

2. Preliminaries

There are many MADM approaches such as AHP [50,51], TOPSIS [26,52,53], ELECTRE [54–56] and PROMETHEE [57,58]. Different from these MADM methods, the ER approach is specifically effective in the situation where uncertainties, ignorance or incompleteness are included. The ER approach was proposed by Yang et al. [1] in 1994 based on the general framework of Dempster–Shafer (D–S) theory [59] and decision theory for the combination of uncertain and incomplete subjective assessments. To facilitate data collection in real decision situations, Yang proposed the rules for the transformation of different sets of linguistic evaluation grades associated with different qualitative attributes and certain values associated with quantitative attributes to a set of common evaluation grades [2]. In [5], an updated ER algorithm was proposed to deal with the irrationality of the original ER framework, referred to as the ER recursive algorithm where $L - 1$ calculation steps are needed for the combination of L basic attributes. Based on the recursive algorithm, the analytical ER algorithm was then proposed [6] in which only one step of calculation is needed to generate the combined performance of assessment. The ER approach has now been developed to the ER rule [21,60] where both weight and reliability of attribute and DM are considered.

Suppose N evaluation grades are involved in the assessment to a qualitative attribute, represented by $H_1, H_2, \dots, H_n, H_{n+1}, \dots, H_N$, where the subscript $n(n = 1, 2, \dots, N)$ represents the n th grade. In general, H_n is supposed to be worse than H_{n+1} , or denoted by $H_{n+1} > H_n$ where $>$ represents “prefer to”. If the utility of H_n is represented by $u(H_n)$, then $u(H_{n+1}) > u(H_n)$ which means that $u(H_{n+1})$ is assumed to be larger than $u(H_n)$ if H_{n+1} is preferred to H_n . Then the frame of discernment is defined as follows:

$$H = \{H_1, H_2, \dots, H_N\} \quad (1)$$

It should be mentioned that the frame of discernment for each qualitative attribute may be different or unique for the purpose of original data collection. In other words, the number of evaluation grades to assess a qualitative attribute may be more or less than N . Suppose the general frame of discernment contains N evaluation grades, then the evaluation grades related to each qualitative attribute should be interpreted and transformed to the general framework of discernment according to specific rules [2]. For each quantitative attribute, the values corresponding to all the N evaluation grades in the general frame of discernment should be firstly determined, and then the numerical assessment value could be transformed to belief degrees on the evaluation grades in Eq. (1). When a CA is assessed, a smaller value is projected to a better evaluation grade, which means that a small value for a cost attribute is more preferred. For a BA, a large value is projected to a better evaluation grade.

Let $E = \{e_1, e_2, \dots, e_i, \dots, e_L\}$ be the set of attributes for the assessment, and the relative weight of $e_i (i = 1, 2, \dots, L)$ is denoted by w_i such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^L w_i = 1$. $a_l (l = 1, 2, \dots, S)$ represents the l th assessed alternative, where S indicates the number of assessed alternatives. The belief degree that a_l be assessed to e_i on evaluation grade $H_n (n = 1, 2, \dots, N)$ is denoted by $\beta_{n,i}(a_l)$ (abbreviated by $\beta_{n,i}$) such that $0 \leq \beta_{n,i}(a_l) \leq 1$ and $\sum_{n=1}^N \beta_{n,i}(a_l) \leq 1$. The subscript n, i and l represent the n th grade, the i th attribute and the l th alternative respectively. $\beta_{H,i}(a_l)$ (abbreviated by $\beta_{H,i}$) is supposed to be the incompleteness of a_l being assessed to e_i , also called the degree of global ignorance. After the transformation of each attribute from the original value to belief degree, we will have the distribution for $e_i (i = 1, 2, \dots, L)$ as follows:

$$S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)), n = 1, 2, \dots, N; (H, \beta_{H,i}(a_l))\} \quad (2)$$

If $S(e_i(a_l))$ contains at least two evaluation grades for which the belief degrees are not zero, the assessment is uncertain. In the ER approach, each attribute is assigned with belief degrees on one or several linguistic evaluation grades as denoted by Eq. (2) so that uncertainty and ambiguity for assessing both quantitative and qualitative attributes can be taken into account simultaneously. In [5] and [6], ER algorithms are proposed in recursive and analytical forms respectively for the aggregation of belief degrees of all L attributes for a_l . Each attribute is considered to be a piece of evidence with its weight w_i . After the aggregation of L attributes, a distributed assessment for a_l on the general level can be presented as follows:

$$S(a_l) = \{(H_n, \beta_n(a_l)), n = 1, 2, \dots, N; (H, \beta_H(a_l))\} \quad (3)$$

In Eq. (3), $\beta_n(a_l)$ is the total belief degree that a_l be assessed on H_n . The combined belief degree presents a panoramic view about the total assessment to an alternative explicitly. It has been proved that even if there exists only one incomplete assessment attribute, the total belief degree will be incomplete as well [5]. So the information contained in the original data could be preserved after the combination process of all attributes.

3. Transformation rules for DIA and FIA

In this section, the frames of discernment of DIA and FIA are to be constructed firstly, and the transformation rules from assessment value to belief degrees assigned to evaluation grades on the frame of discernment for DIA is analyzed.

Definition 1. Suppose $h_{w,i} = [h_{w,i}^-, h_{w,i}^+]$ is the worst interval value for attribute e_i corresponding to the worst evaluation grade H_1 . The subscript w and i represent ‘Worst’ and the i th attribute respectively. e_i is a DIA if the larger the distance between an assessment value and $h_{w,i}$, the better the assessment value, no matter whether it is larger than $h_{w,i}^+$ or smaller than $h_{w,i}^-$.

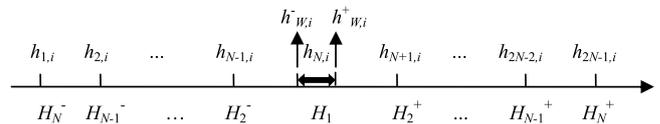


Fig. 1. Relationship between DIA values and the frame of discernment.

The distance between an assessment value and the worst interval value of a DIA measures the performance of an alternative on the attribute. If the value of a DIA is larger than $h_{w,i}^+$, it is assumed to be better when it increases. On the contrary, if the value is less than $h_{w,i}^-$, it is assumed to be better when the value decreases. If the value is located in the worst interval value, it is assumed to be the worst. A special case is that the worst interval value becomes a crisp value that reduces to the DA.

Definition 2. Suppose $h_{B,i} = [h_{B,i}^-, h_{B,i}^+]$ is the best interval value for e_i corresponding to the best evaluation grade H_N . The subscript B and i represent ‘Best’ and the i th attribute respectively. e_i is a FIA if the smaller the distance between an assessment value and $h_{B,i}$, the better the assessment value, no matter whether it is larger than $h_{B,i}^+$ or smaller than $h_{B,i}^-$.

For a FIA, it is assumed to be best if the assessment value is located between $h_{B,i}^-$ and $h_{B,i}^+$. When $h_{B,i}^- = h_{B,i}^+$, it reduces to the FA.

The details of DIA and FIA can be referred to [25]. When an attribute is quantitative, the value assigned to it may be precise or uncertain. If an assessment is uncertain, the representation on e_i may be several crisp values, an interval value, several interval values, fuzzy numbers and so on. In the following, we will discuss the transformation rules of crisp value for quantitative DIA to belief degrees on the frame of discernment. Firstly, the frames of discernment of DIA and FIA will be constructed.

3.1. Frames of discernment of DIA and FIA

The frame of discernment of DIA is constructed and denoted by Eq. (4) as follows:

$$H^{DIA} = \{H_N^-, \dots, H_2^-, H_1, H_2^+, \dots, H_N^+\} \quad (4)$$

where H_1 and $H_N^-(H_N^+)$ are the worst and best evaluation grade respectively, and $u(H_n^-) = u(H_n^+)$ ($n = 2, \dots, N$). Here, a set of assessment values of DIA e_i related to the evaluation grades in Eq. (4) should firstly be identified according to a real decision situation which is denoted by Eq. (5) as follows:

$$h^i = \{h_{n,i}, n = 1, 2, \dots, 2N - 1\} \quad (5)$$

Compared with BA or CA as Eq. (1) denoted, when confirming the corresponding values of a quantitative attribute to evaluation grades in the frame of discernment, $2N - 1$ values need to be identified for DIA while only N values are to be confirmed for BA or CA. The relationship between the values of DIA and the frame of discernment is shown in Fig. 1.

Hypothesis 1. The difference between $h_{n,i}$ and $h_{n+1,i}$ ($1 \leq n \leq N - 1$) is supposed to be the same as the difference between $h_{2N-n-1,i}$ and $h_{2N-n,i}$, so the preferences between the worst interval value for DIA is symmetrical.

It is a basic and important task for a DM to provide the rules linking the evaluation grades with the particular values of each attribute. If the rules are not extracted scientifically or objectively, the assessment result will not be convincing or rational. $h_{N,i} = h_{w,i} = [h_{w,i}^-, h_{w,i}^+]$ is the worst possible interval value of e_i that

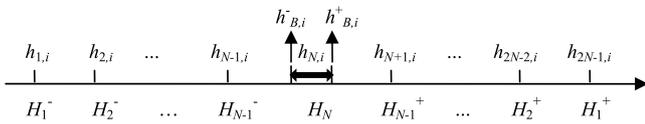


Fig. 2. Relationship between values of FIA and the frame of discernment.

is equal to the worst evaluation grade H_1 , $h_{1,i}$ and $h_{2N-1,i}$ are the smallest and largest possible value which represent the best feasible value of attribute e_i corresponding to H_N . From Fig. 1, we will have the following equivalent rule:

$$h_{n,i} \text{ is equivalent to } H_{N-n+1}^- \text{ when } 1 \leq n \leq N$$

$$h_{n,i} \text{ is equivalent to } H_{n-N+1}^+ \text{ when } N + 1 \leq n \leq 2N - 1 \quad (6)$$

Eq. (6) can also be interpreted from the utility aspect that $u(h_{n,i}) = u(H_{N-n+1}^-)$ ($1 \leq n \leq N$)

$$u(h_{n,i}) = u(H_{n-N+1}^+) \quad (N + 1 \leq n \leq 2N - 1) \quad (7)$$

From Eqs. (6) and (7), we will also have the following rule:

$$u(h_{n,i}) = u(h_{2N-n,i}) \quad (n = 1, 2, \dots, N) \quad (8)$$

It is obvious that $u(H_n^-) < u(H_{n+1}^-)$, $u(H_n^+) < u(H_{n+1}^+)$ ($1 \leq n \leq N - 1$), and the utility between every two adjacent evaluation grades may be linear or nonlinear. Here, we assume that the utility is piecewise linear. If h_j is an accurate assessment value for e_i , then h_j is called the right value of e_i when $h_{w,i}^+ < h_j \leq h_{2N-1,i}$, and when $h_{1,i} \leq h_j < h_{w,i}^-$, h_j is called the left value of e_i . The right values and left values relative to the worst interval value $h_{N,i}$ may be non-symmetrical because the difference between $h_{n,i}$ and $h_{n+1,i}$ is not necessarily the same as the difference between $h_{2N-n-1,i}$ and $h_{2N-n,i}$ ($1 \leq n \leq N - 1$). The assessment to a DIA e_i could be represented by the following expectation:

$$S(e_i(a_l)) = \{(h_{n,i}, \tilde{r}_{n,i}(a_l)); (H, \tilde{r}_{H,i}(a_l))\} \quad (n = 1, 2, \dots, 2N - 1) \quad (9)$$

The calculation of $\tilde{r}_{n,i}(a_l)$ (abbreviated by $\tilde{r}_{n,i}$) which is the belief degree that e_i be assessed to $h_{n,i}$ on a_l will be discussed in the next subsection. It could also be denoted by Eq. (2) where

$$\beta_{n,i}(a_l) = \begin{cases} \tilde{r}_{N,i}(a_l) & n = 1 \\ \tilde{r}_{N+1-n,i}(a_l) + \tilde{r}_{N-1+n,i}(a_l) & n = 2, \dots, N, \end{cases}$$

$$\beta_{H,i}(a_l) = \tilde{r}_{H,i}(a_l) = 1 - \sum_{n=1}^{2N-1} \tilde{r}_{n,i}(a_l) \quad (10)$$

In Eq. (10), $\tilde{r}_{N+1-n,i}$ and $\tilde{r}_{N-1+n,i}$ are the belief degrees of evaluation grade H_n^- and H_n^+ ($n = 2, \dots, N$) which corresponding to left value $h_{N+1-n,i}$ and right value $h_{N-1+n,i}$ respectively considering the relationship in Fig. 1. From the equivalent rules of Eqs. (6) and (8) for DIA, Eq. (10) is generated from Eq. (9).

Similarly, the frame of discernment of FIA could be denoted as follows:

$$H^{FIA} = \{H_1^-, \dots, H_{N-1}^-, H_N, H_{N-1}^+, \dots, H_1^+\} \quad (11)$$

There are $2N - 1$ corresponding values to be identified for FIA. The corresponding relationship between the values of FIA and the frame of discernment should be given by the DM which is shown in Fig. 2.

Hypothesis 2. The difference between $h_{n,i}$ and $h_{n+1,i}$ ($1 \leq n \leq N - 1$) is supposed to be the same as the difference between $h_{2N-n-1,i}$ and $h_{2N-n,i}$, so the preferences between the best interval value for FIA is symmetrical.

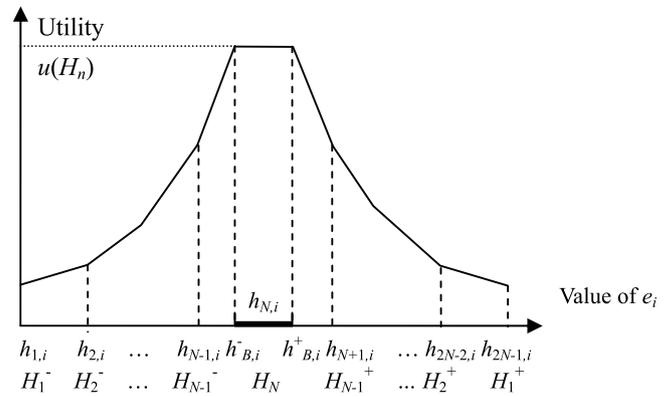


Fig. 3. Utility function of FIA when DM is risk taking.

The equivalent rule for FIA is as follows:

$$h_{n,i} \text{ is equivalent to } H_n^- \text{ when } 1 \leq n \leq N$$

$$h_{n,i} \text{ is equivalent to } H_{2N-n}^+ \text{ when } N + 1 \leq n \leq 2N - 1 \quad (12)$$

Here, $h_{N,i}$ is the best possible interval value of e_i corresponding to H_N , $h_{1,i}$ and $h_{2N-1,i}$ are the smallest and largest possible value which represent the worst feasible values of e_i corresponding to H_1^- and H_1^+ . Similar to DIA, the difference between $h_{n,i}$ and $h_{n+1,i}$ is not necessarily the same as the difference between $h_{2N-n-1,i}$ and $h_{2N-n,i}$ ($1 \leq n \leq N - 1$) which means the right values and left values relative to the best interval value $h_{N,i}$ may be non-symmetrical. If the utility function is supposed to be piecewise linear [2] in each evaluation grade interval $[h_{n,i}, h_{n+1,i}]$ ($1 \leq n \leq 2N - 2$), it could be depicted in Fig. 3. The utility is linearly decreasing between $[h_{n,i}, h_{n+1,i}]$ when $N \leq n \leq 2N - 2$, whereas linearly increasing when $1 \leq n \leq N - 1$. So given the equivalent rule as Eq. (12), we could get the utility of any value between $h_{1,i}$ and $h_{2N-1,i}$.

As described in [25], FIA and DIA are the extensions of CA and BA respectively. So only half of the identified values corresponding to evaluation grades exist in CA or BA. Then a FIA could be denoted by the expectation as Eq. (9) or by Eq. (2) where

$$\beta_{n,i}(a_l) = \begin{cases} \tilde{r}_{N,i}(a_l) & n = N \\ \tilde{r}_{n,i}(a_l) + \tilde{r}_{2N-n,i}(a_l) & n = 1, 2, \dots, N - 1, \end{cases}$$

$$\beta_{H,i}(a_l) = \tilde{r}_{H,i}(a_l) = 1 - \sum_{n=1}^{2N-1} \tilde{r}_{n,i}(a_l) \quad (13)$$

In Eq. (13), $\tilde{r}_{n,i}$ and $\tilde{r}_{2N-n,i}$ are the belief degrees of evaluation grade H_n^- and H_n^+ ($n = 1, 2, \dots, N - 1$) which corresponding to left value $h_{n,i}$ and right value $h_{2N-n,i}$ respectively considering the relationship in Fig. 2.

After setting the frame of discernment and equivalent rules between assessment values of DIA or FIA and evaluation grades in the frame of discernment, we should calculate the belief degrees assessed to the attribute on all evaluation grades in the frame of discernment for the purpose of combining all the attributes in the ER framework. In [2], the transformation rules that values of quantitative and qualitative BA to belief degrees are proposed, whereas the method about interval value of quantitative BA transformed to belief degrees is given in [23] and [31]. Here, we will discuss the situation where the assessment to a quantitative DIA is an accurate value.

3.2. Transformation rules for DIA

When the assessment to e_i is a crisp value represented by $S^i(e_i) = \{(h_j, p_i)\}$ as mentioned in [2], three cases may appear.

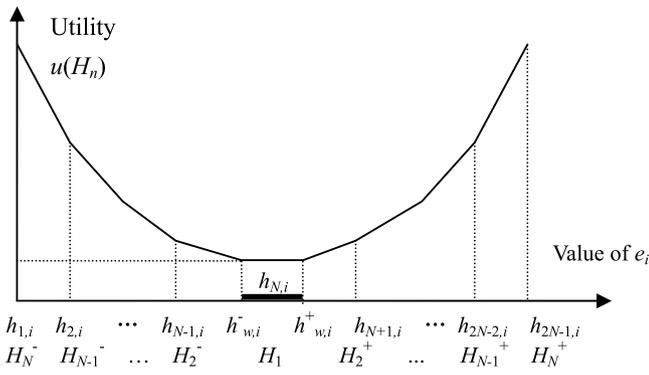


Fig. 4. Utility function of DIA when DM is risk taking.

Here, p_i is interpreted as the reliability or confidence of getting the crisp value h_j . It may be affected by the condition to acquire the assessment information or the reliability of the equipment for getting assessment data [21]. When $p_i < 1$, it means the reliability or confidence that the data we get is not 100%, then the assessment to e_i is incomplete. A special case is that $p_i = 1$ which means the assessment is complete. It is assumed that the utility function of e_i is piecewise linear between every two adjacent evaluation grades [2], which could be represented by Fig. 4. For instance, the utility function is linearly decreasing with the assessment to e_i in interval $[h_{n,i}, h_{n+1,i}]$ when $1 \leq n \leq N - 1$, and linearly increasing in $[h_{n,i}, h_{n+1,i}]$ when $N \leq n \leq 2N - 2$. It attains to the maximum utility at both $h_{1,i}$ and $h_{2N-1,i}$ represented by $u(H_N^-)$ and $u(H_N^+)$, and the minimum value $u(H_1)$ between $h_{w,i}^-$ and $h_{w,i}^+$. If the DM is risk taking, the curve of utility function is convex.

① In the first case, e_i is just exactly assessed on an identified value $h_{n,i} (1 \leq n \leq 2N - 1)$ in Eq. (5) corresponding to an evaluation grade in Eq. (4), then we will have $h_j = h_{n,i}$

$$(e_i) = \{(h_{n,i}, p_i)\} \Rightarrow (e_i) = \begin{cases} \{H_{N-n+1}^-, p_i; H, 1 - p_i\} & 1 \leq n \leq N - 1 \\ \{H_1, p_i; H, 1 - p_i\} & n = N \\ \{H_{n-N+1}^+, p_i; H, 1 - p_i\} & N + 1 \leq n \leq 2N - 1 \end{cases} \quad (14)$$

② In the second case, the assessment to e_i is larger than the worst value of e_i that is just located between $h_{n,i}$ and $h_{n+1,i} (N \leq n \leq 2N - 2)$ which is depicted in Fig. 5, or denoted as follows:

$$h_j > h_{N,i}(h_{w,i}^+), \text{ and } h_j \in (h_{n,i}, h_{n+1,i}](N \leq n \leq 2N - 2) \quad (15)$$

Here, $h_{n+1,i}$ is preferred to $h_{n,i}$. Then according to [2], we will have

$$S(e_i) = \{(h_{n,i}, p_i \cdot r_{n,j}; h_{n+1,i}, p_i \cdot r_{n+1,j})\} \quad (16)$$

where

$$r_{n,j} = \frac{h_{n+1,i} - h_j}{h_{n+1,i} - h_{n,i}}, r_{n+1,j} = \frac{h_j - h_{n,i}}{h_{n+1,i} - h_{n,i}} = 1 - r_{n,j} \quad (17)$$

$r_{t,j} (t = n, n + 1)$ is the belief degree that h_j be confirmed to $h_{n,i}$ and $h_{n+1,i}$. Since in this case, $h_{n,i}$ is equivalent to $H_{N-n+1}^+ (N \leq n \leq 2N - 1)$, the belief degrees assessed to e_i could be denoted by

$$S(e_i) = \{H_{N-n+1}^+, \beta_{N-n+1,i}; H_{N-n+2}^+, \beta_{N-n+2,i}; H, 1 - p_i\} \quad (18)$$

where

$$\beta_{N-n+1,i} = p_i \cdot r_{n,j}, \beta_{N-n+2,i} = p_i \cdot r_{n+1,j} \quad (19)$$

③ In the third case, the assessed value of e_i is less than $h_{N,i}$, and it is located between $h_{n,i}$ and $h_{n+1,i} (1 \leq n \leq N - 1)$, or denoted as follows:

$$h_j < h_{N,i}(h_{w,i}^-), \text{ and } h_j \in (h_{n,i}, h_{n+1,i}](1 \leq n \leq N - 1) \quad (20)$$

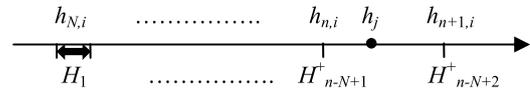


Fig. 5. Accurate right value for DIA.

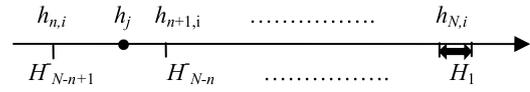


Fig. 6. Accurate left value for DIA.

It is depicted in Fig. 6. Then we will have the expectation represented by Eqs. (16) and (17) where $h_{n,i}$ is preferred to $h_{n+1,i}$. The belief degrees assessed to e_i in this case could then be denoted by

$$S(e_i) = \{H_{N-n}^-, \beta_{N-n,i}; H_{N-n+1}^-, \beta_{N-n+1,i}; H, 1 - p_i\} \quad (21)$$

where

$$\beta_{N-n,i} = p_i \cdot r_{n+1,j}, \beta_{N-n+1,i} = p_i \cdot r_{n,j} \quad (22)$$

If the utility function is not piecewise linear, the transformation rules will be more complex. Just as mentioned in [2], the assessment of a quantitative attribute may not always be certain. In this case, the assessment may be several crisp values with given probabilities as follows:

$$S^i(e_i) = \{(h_j, p_j), j = 1, 2, \dots, M_i\} \quad (23)$$

where p_j is the probability of e_i been assessed to the j th possible value, and $M_i (M_i \geq 2)$ is the number of possible values assessed to e_i . In this case, the adjacent values of e_i may be actually located in two adjacent evaluation grade intervals $(h_{n,i}, h_{n+1,i})$ and $(h_{n+1,i}, h_{n+2,i})$; meanwhile, an evaluation grade interval may also include two or more possible quantitative values assessed to e_i . The transformation rules for DIA and FIA in this situation can be extended according to the rules for BA in [2].

After the transformation of assessment values to belief distributions according to the rules given in this section for DIA and FIA together with the rules in [2] for BA and CA, the ER approach can then be used to aggregate the belief distributions of multiple kinds of attributes represented by Eq. (2) to generate a general assessment represented by Eq. (3). Since attribute weights are an important factor in the aggregating process of the ER approach, the weight assignment method under subjective assessments is discussed in Section 4.

4. Entropy-based weight assignment under subjective assessments

As we all know, the objective methods to determine attribute weights use the assessed values of attributes on different alternatives. When there exist a lot of qualitative attributes which are presented by uncertain and incomplete subjective judgments such as belief distributions, attribute weights should better be generated by objective method. Otherwise, two kinds of subjective judgments including the subjective belief distributions denoted by Eq. (2) and the subjective judgments for assigning weights are involved in a decision process. To reduce the influence of excessive subjective judgments to the final result, the subjective belief distributions assessed on qualitative attributes along with the numerical values on quantitative attributes can be used to generate the attribute weights. So only one kind of subjective judgment is contained under this circumstance. Here, we will discuss how to use the entropy method to generate relatively rational weights from uncertain and incomplete subjective judgments. Strictly speaking, when

the entropy method is used to calculate weights from subjective judgments, the generated weights should not be called ‘objective weights’. Absolutely ‘objective weights’ can only be titled when all attributes are assessed from objective numerical values.

4.1. Literature review

Just as Fu [61] said, different sets of attribute weights may generate different solutions to a decision problem. In general, weight assignment methods can be categorized into three types: subjective, objective and hybrid [48,49,62,63]. Subjective methods depend on the preference of DMs whose knowledge backgrounds may be different, such as AHP [50,51], GAHP [3,4,36] and Delphi method [64,65]. When a DM is not capable of providing subjective judgments on the importance of attributes on account of some reasons, it may not be used. Objective methods such as the Principal Component Analysis [66], Entropy method [26,37–43], CRITIC method [39,67,68], standard deviation (SD) method [39,62], correlation coefficient and standard deviation integrated (CCSD) method [61,63], and multiple objective programming model [69] generate attribute weights from the differences among the assessment information of alternatives through each attribute. They reflect the discriminating power [61] or contrast intensity [67] of each attribute on all alternatives. It is not applicable when only one alternative is assessed. Furthermore, the weights are changeable when different sets of alternatives are assessed. Hybrid methods [38,70] can reflect both the preferences of DMs and the intrinsic features contained in the assessment information.

In recent years, many methods have been proposed to determine weights, especially based on subjective judgments. In [43], although the assessments of qualitative attributes were transformed to scores, uncertainties were not fully discussed in the assessment. Wu et al. [37], Chen et al. [40], Zhang et al. [44] proposed entropy based objective weighting methods, but they are applicable when the judgment provided by the DM is intuitionistic fuzzy matrix. Just as demonstrated above, the entropy, CRITIC, SD and CCSD methods are entirely or partly based on the differences among alternatives on a specific attribute. When the attributes are all quantitative, the objective methods can be easily applied, e.g., [39,41,63,67]. Otherwise, subjective judgments on qualitative attributes should be transformed to accurate or interval values according to the preferences of DMs. Different preferences of DMs will probably lead to various results. The existing objective methods rarely handle different risk preferences of DMs [48]. From the above analysis, we can see that when the original assessments contain uncertainties and ignorance, how to generate a relatively rational objective weights need to be further studied.

4.2. The entropy method

A MADM problem which includes S alternatives and L attributes can be represented by the matrix in Eq. (24). In Eq. (24), each row represents the assessments to all the L attributes on a specific alternative, while each column indicates the assessments to an attribute on all S alternatives.

$$[S(e_i(a_l))]_{S \times L} = \begin{matrix} & e_1 & \dots & e_i & \dots & e_L \\ \begin{matrix} a_1 \\ \dots \\ a_i \\ \dots \\ a_S \end{matrix} & \begin{bmatrix} S(e_1(a_1)) & \dots & S(e_i(a_1)) & \dots & S(e_L(a_1)) \\ \dots & \dots & \dots & \dots & \dots \\ S(e_1(a_i)) & \dots & S(e_i(a_i)) & \dots & S(e_L(a_i)) \\ \dots & \dots & \dots & \dots & \dots \\ S(e_1(a_S)) & \dots & S(e_i(a_S)) & \dots & S(e_L(a_S)) \end{bmatrix} & \end{matrix} \quad (24)$$

The value of $S(e_i(a_l))$ is denoted by x_{li} when e_i is a quantitative attribute; otherwise, Eq. (2) can be used to represent $S(e_i(a_l))$ when e_i is a qualitative attribute. The entropy method [26,38] consists of the following three steps:

(1) Linear proportional transformation or standard 0–1 transformation:

For a BA, $y_{li} = \frac{x_{li}}{\max_{1 \leq l \leq S} x_{li}}$ or $y_{li} = \frac{x_{li} - \min_{1 \leq l \leq S} x_{li}}{\max_{1 \leq l \leq S} x_{li} - \min_{1 \leq l \leq S} x_{li}}$ ($1 \leq l \leq S, 1 \leq i \leq L$); while for a CA, $y_{li} = \frac{\min_{1 \leq l \leq S} x_{li}}{x_{li}}$ or $y_{li} = \frac{\max_{1 \leq l \leq S} x_{li} - x_{li}}{\max_{1 \leq l \leq S} x_{li} - \min_{1 \leq l \leq S} x_{li}}$ ($1 \leq l \leq S, 1 \leq i \leq L$).

(2) Normalization of the assessment value for $S(e_i(a_l))$:

$$p_{li} = \frac{y_{li}}{\sum_{l=1}^S y_{li}} \quad (l = 1, 2, \dots, S; i = 1, 2, \dots, L) \quad (25)$$

where $\sum_{l=1}^S p_{li} = 1$.

(3) Calculating the entropy and weight of e_i

The entropy of e_i derived from all S alternatives is measured by

$$E_i = -\frac{1}{\ln(S)} \sum_{l=1}^S p_{li} \ln p_{li} \quad (i = 1, 2, \dots, L) \quad (26)$$

The denominator $\frac{1}{\ln(S)}$ is used to limit E_i to $[0, 1]$. It is well known that an attribute with higher entropy will contribute less to the final solution, so the weight of e_i is generated as

$$w_i = \frac{1 - E_i}{L - \sum_{i=1}^L E_i} \quad (i = 1, 2, \dots, L) \quad (27)$$

where $\sum_{i=1}^L w_i = 1$.

The above process assumes that all the attributes are assigned with accurate numerical values. When a quantitative attribute is assigned with interval value or a qualitative attribute is given a belief distribution denoted by Eq. (2), how could the entropy method be used to generate relatively rational weights is significant.

4.3. Ave-entropy based weight assignment under subjective assessments

When e_i is a qualitative attribute with the assessment represented by Eq. (2), how to quantify the incompatibility of S alternatives on e_i and then apply the entropy method to generate the weight? There are two different processes to obtain the attribute weights based on the assessments of Eq. (24) provided that subjective belief distributions are given to qualitative attributes.

The 1st way is to directly measure the dissimilarity of any two belief distributions on a_l and a_k ($l, k = 1, 2, \dots, S; l \neq k$) for a certain qualitative attribute e_i , represented by $\text{Diss}(S(e_i(a_l)), S(e_i(a_k)))$, followed by the calculation of the entropy and weight of e_i . There are some existing methods to calculate the dissimilarity of two belief distributions. Smets (1990, 1994) introduced pignistic probability function [71,72] to measure the evidence distance. When belief degrees are only given to single evaluation grades presented by Eq. (2), it reflects the maximum difference between the belief degrees of two evidences assigned to the same evaluation grade. Fu et al. (2010) developed compatibility measure between two belief distributions based on pignistic probability function [73], then the concept was improved to dissimilarity measure that the utilities of evaluation grades are considered [74]. Chen et al. (2017) used pignistic probability function to determine alliances where the judgments of DMs are similar to some extent [75].

The 2nd way is to transform $S(e_i(a_l))$ into a definite value through utility function [2,6] with the help of $u(H_n)$ as follows:

$$u_{li}^{Max} = \sum_{n=1}^N \beta_{n,i}(a_l) u(H_n) + \beta_{H,i}(a_l) u(H_N) \quad (28)$$

$$u_{li}^{Min} = \sum_{n=1}^N \beta_{n,i}(a_l) u(H_n) + \beta_{H,i}(a_l) u(H_1) \quad (29)$$

$$u_{li}^{Ave} = \frac{(u_{li}^{Max} + u_{li}^{Min})}{2} \quad (30)$$

u_{li}^{Max} , u_{li}^{Min} and u_{li}^{Ave} are the measurements of the utility of e_i on a_l denoted by u_{li} . Then u_{li} can be used to calculate the discrepancy of e_i on all S alternatives. If $S(e_i(a_l))$ is incomplete such that $\beta_{H,i}(a_l) > 0$, we will have $u_{li}^{Max} > u_{li}^{Min}$, $u_{li} \in [u_{li}^{Min}, u_{li}^{Max}]$. Then the method for measuring the entropy of interval utility will lead to the rationality of the generated weights.

In [62], minimal satisfaction is used to deal with the incompleteness contained in the subjective assessment. According to the method, let V_{li} be the minimal satisfaction of e_i on a_l compared with other $S - 1$ alternatives such that $V_{li} = u_{li}^{Min} - \max_{s \neq l} \{u_{si}^{Max}\}$ and $-1 \leq V_{li} \leq 1$. The process of generating the weight of e_i using this method can be summarized as follows:

$$\begin{aligned}
 & \begin{bmatrix} S(e_i(a_1)) \\ \dots \\ S(e_i(a_l)) \\ \dots \\ S(e_i(a_S)) \end{bmatrix} \xrightarrow[\text{Eq. (29)}]{\text{Eq. (28)}} \begin{bmatrix} [u_{li}^{Min}, u_{li}^{Max}] \\ \dots \\ [u_{li}^{Min}, u_{li}^{Max}] \\ \dots \\ [u_{Si}^{Min}, u_{Si}^{Max}] \end{bmatrix} \\
 & \rightarrow \begin{bmatrix} V_{li} \\ \dots \\ V_{li} \\ \dots \\ V_{Si} \end{bmatrix} \rightarrow \begin{bmatrix} \bar{V}_{li} \\ \dots \\ \bar{V}_{li} \\ \dots \\ \bar{V}_{Si} \end{bmatrix} \\
 & \rightarrow \begin{bmatrix} \tilde{V}_{li} \\ \dots \\ \tilde{V}_{li} \\ \dots \\ \tilde{V}_{Si} \end{bmatrix} \rightarrow E_i \rightarrow w_i \tag{31}
 \end{aligned}$$

where $\bar{V}_{li} = \frac{V_{li}-(-1)}{1-(-1)}$, and $\tilde{V}_{li} = \frac{\bar{V}_{li}}{\sum_{l=1}^S \bar{V}_{li}}$ such that $\sum_{l=1}^S \tilde{V}_{li} = 1$.

Compared with Eq. (31), we propose that u_{li}^{Ave} be used to calculate the entropy of e_i . We call it average utility based entropy weight assignment (ave-entropy) method. The specific steps of the process are shown as follows.

(1) The first step is to transform $S(e_i(a_l))$ to u_{li}^{Ave} for e_i on all S alternatives by Eqs. (28)–(30).

(2) Secondly, u_{li}^{Ave} is normalized by $\tilde{u}_{li}^{Ave} = \frac{u_{li}^{Ave}}{\sum_{l=1}^S u_{li}^{Ave}}$ to satisfy the condition that $\sum_{l=1}^S \tilde{u}_{li}^{Ave} = 1$. If the standard 0–1 transformation is conducted before the normalization, $\tilde{u}_{li}^{Ave} = \frac{\bar{u}_{li}^{Ave}}{\sum_{l=1}^S \bar{u}_{li}^{Ave}}$, where $\bar{u}_{li}^{Ave} = \frac{u_{li}^{Ave} - \min_{l=1, \dots, S} u_{li}^{Ave}}{\max_{l=1, \dots, S} u_{li}^{Ave} - \min_{l=1, \dots, S} u_{li}^{Ave}}$.

(3) Thirdly, the normalized average utility \tilde{u}_{li}^{Ave} is used to calculate the entropy of e_i by Eq. (26) as follows: $E_i = -\frac{1}{\ln(S)} \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{u}_{li}^{Ave}$, which is then applied to generate the weight of e_i by Eq. (27).

The process of the method is given in Eq. (32) as follow.

$$\begin{aligned}
 & \begin{bmatrix} S(e_i(a_1)) \\ \dots \\ S(e_i(a_l)) \\ \dots \\ S(e_i(a_S)) \end{bmatrix} \xrightarrow[\text{Eq. (30)}]{\text{Eq. (28)}} \begin{bmatrix} u_{1i}^{Ave} \\ \dots \\ u_{li}^{Ave} \\ \dots \\ u_{Si}^{Ave} \end{bmatrix} \\
 & \times \left(\begin{array}{c} \text{Standard 0 – 1} \\ \text{transformation} \\ \rightarrow \\ \begin{bmatrix} \bar{u}_{1i}^{Ave} \\ \dots \\ \bar{u}_{li}^{Ave} \\ \dots \\ \bar{u}_{Si}^{Ave} \end{bmatrix} \end{array} \right) \\
 & \xrightarrow{\text{Normalize}} \begin{bmatrix} \tilde{u}_{1i}^{Ave} \\ \dots \\ \tilde{u}_{li}^{Ave} \\ \dots \\ \tilde{u}_{Si}^{Ave} \end{bmatrix} \xrightarrow{\text{Eq. (26)}} E_i \xrightarrow{\text{Eq. (27)}} w_i \tag{32}
 \end{aligned}$$

Property 1 (Inoperative). When the belief distributions assigned to an attribute on all S alternatives are the same, the weight of the attribute generated by Eq. (32) is 0.

Property 1 is obvious because when $S(e_i(a_l)) = S(e_i(a_k)) (l \neq k; l, k = 1, 2, \dots, S)$, we have $u_{li}^{Ave} = u_{ki}^{Ave}$ from Eqs. (28)–(30). Since $\tilde{u}_{li}^{Ave} = \frac{u_{li}^{Ave}}{\sum_{l=1}^S u_{li}^{Ave}}$, we have $\tilde{u}_{li}^{Ave} = \frac{1}{S} (l = 1, 2, \dots, S)$. Then from Eqs. (26) and (27), we have $E_i = -\frac{1}{\ln(S)} \sum_{l=1}^S \frac{1}{S} \ln \frac{1}{S} = 1$, $w_i = 0$. In this case, the pignistic probability distance of $S(e_i(a_l))$ and $S(e_i(a_k))$ is 0. So the attribute can be deleted from the assessment which will lead to the simplification of the complex assessment model. Thus, entropy can be interpreted as the similarity of different alternatives on an attribute. Furthermore, if the belief distributions to an attribute on all alternatives are very close with each other, the attribute can also be deleted because the importance of it is not obvious in the assessment. So a ‘threshold’ value can be set in a real life MADM problem. An attribute will only be involved in an assessment process if the dissimilarity value of the attribute from all alternatives extends the ‘threshold’ value. Otherwise, it will be deleted in the specific assessment problem. A special case is that we could not differentiate any discrepancy from the assessment to all alternatives for each attribute. It will lead to the weights of all L attributes be the same although it rarely happens in real MADM problems.

Here, we take a numerical example that contains 5 alternatives and 6 attributes to illustrate the method by Eqs. (31) and (32). The belief degrees are shown in Table 1. From Table 1, we can see that for a specific attribute $e_i (i = 1, 2, \dots, 6)$, $\beta_{n,i}(a_l) \neq 0 (n = l \leq 5)$ and $\beta_{n,i}(a_l) = 0 (n \neq l)$, and the ignorance contained in the assessment increases from e_1 to e_6 . Take e_1 for example, the belief degree that a_1 be assessed to e_1 on evaluation grade H_1 is $\beta_{1,1}(a_1) = 1$, while $\beta_{n,1}(a_1) = 0 (n = 2, 3, 4, 5)$. And the incompleteness of a_1 being assessed to e_1 is $\beta_{H,1}(a_1) = 0$. Intuitively, e_1 should be assigned with the largest weight because the discrepancies among a_1 to a_5 for e_1 are considerably large since the consistency between a_l and $a_k (l, k = 1, 2, \dots, 5; l \neq k)$ for e_1 is zero. In other words, the dissimilarity measure [61] between the distributions of each pair of alternatives for e_1 represented by $D(e_1(a_{lk})) (l, k = 1, 2, \dots, 5; l \neq k)$ is 1. While the assessments to e_6 from a_1 to a_5 are more consistent because the ignorance contained in the assessment is 0.9 that will lead to its smallest weight.

The utilities of the five evaluation grades are set to be risk aversion such that $u(H_1) = 0, u(H_2) = 0.45, u(H_3) = 0.75, u(H_4) = 0.9$ and $u(H_5) = 1$. Figs. 7 to 12 show the generated weights of the 6 attributes by Eqs. (32), (31) and the standard deviation (SD) method [39] when $u(H_1)$ is given a very little change from 0, 0.001, 0.01, 0.02, 0.05 to 0.1 provided that the utilities of other four evaluation grades remain the same. From Figs. 7 to 12, the horizontal axis represents the belief degree of $\beta_{n,i}(a_l) (n = l \leq 5, i = 1, 2, \dots, 6)$, i.e. $\beta_{n,1}(a_1) = 1 (n = l \leq 5)$ denotes e_1 , $\beta_{n,2}(a_1) = 0.8 (n = l \leq 5)$ denotes e_2 , $\beta_{n,3}(a_1) = 0.6 (n = l \leq 5)$ denotes e_3 , and so on; while the vertical axis represents the weights generated by Eq. (32) (the curve of ‘Ave’), Eq. (31) (the curve of ‘Min Max’) and SD method respectively.

From the curve of ‘Min Max’ in Fig. 7, we can see that when the minimal satisfaction approach is used, the generated weights of the 6 attributes are equal if we set $u(H_1) = 0$, which is $e_i = 0.167 (i = 1, 2, \dots, 6)$. Although the utility of H_1 increases steadily from Figs. 7 to 12, the difference among the relative weights of the 6 attributes is not obvious until $u(H_1) = 0.05$ (Fig. 11) when Eq. (31) is applied. When the utilities of the five evaluation grades are set to be risk taking, i.e. $u(H_1) = 0, u(H_2) = 0.1, u(H_3) = 0.25, u(H_4) = 0.55$ and $u(H_5) = 1$, the generated weights of the 6 attributes by Eq. (31) are also equal. It seems unreliable that different risk

Table 1
Belief degrees that 5 alternatives be assessed to 6 attributes.

$\beta_{n,i}(a_l)$	a_1	a_2	a_3	a_4	a_5
e_1	$(H_1, 1)$	$(H_2, 1)$	$(H_3, 1)$	$(H_4, 1)$	$(H_5, 1)$
e_2	$(H_1, 0.8;H, 0.2)$	$(H_2, 0.8;H, 0.2)$	$(H_3, 0.8;H, 0.2)$	$(H_4, 0.8;H, 0.2)$	$(H_5, 0.8;H, 0.2)$
e_3	$(H_1, 0.6;H, 0.4)$	$(H_2, 0.6;H, 0.4)$	$(H_3, 0.6;H, 0.4)$	$(H_4, 0.6;H, 0.4)$	$(H_5, 0.6;H, 0.4)$
e_4	$(H_1, 0.4;H, 0.6)$	$(H_2, 0.4;H, 0.6)$	$(H_3, 0.4;H, 0.6)$	$(H_4, 0.4;H, 0.6)$	$(H_5, 0.4;H, 0.6)$
e_5	$(H_1, 0.2;H, 0.8)$	$(H_2, 0.2;H, 0.8)$	$(H_3, 0.2;H, 0.8)$	$(H_4, 0.2;H, 0.8)$	$(H_5, 0.2;H, 0.8)$
e_6	$(H_1, 0.1;H, 0.9)$	$(H_2, 0.1;H, 0.9)$	$(H_3, 0.1;H, 0.9)$	$(H_4, 0.1;H, 0.9)$	$(H_5, 0.1;H, 0.9)$

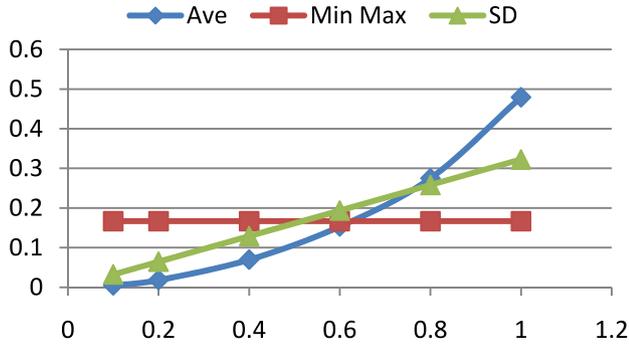


Fig. 7. Attribute weights when $u(H_1) = 0$.

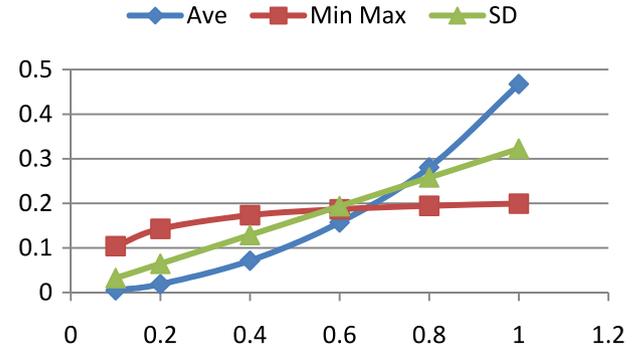


Fig. 10. Attribute weights when $u(H_1) = 0.02$.

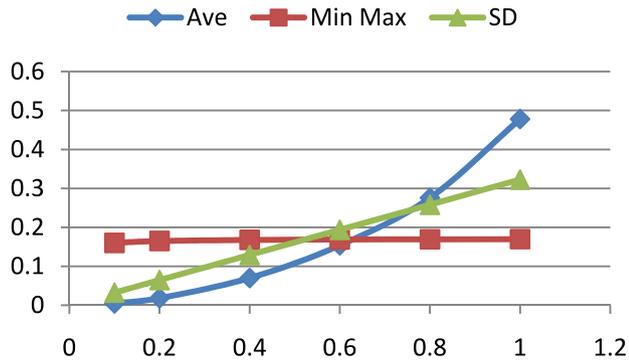


Fig. 8. Attribute weights when $u(H_1) = 0.001$.

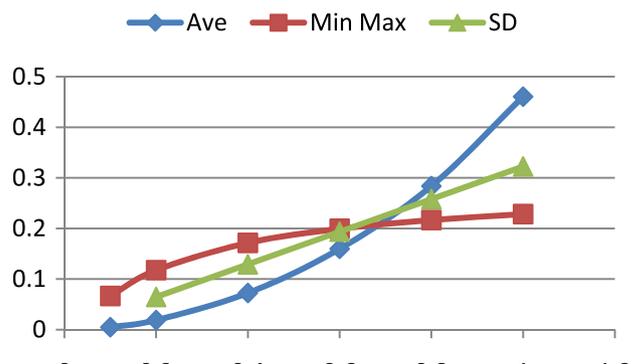


Fig. 11. Attribute weights when $u(H_1) = 0.05$.

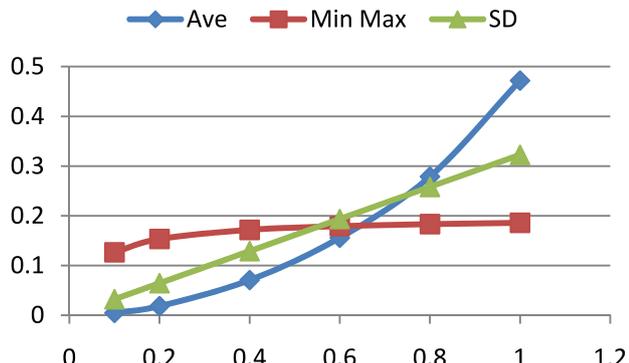


Fig. 9. Attribute weights when $u(H_1) = 0.01$.

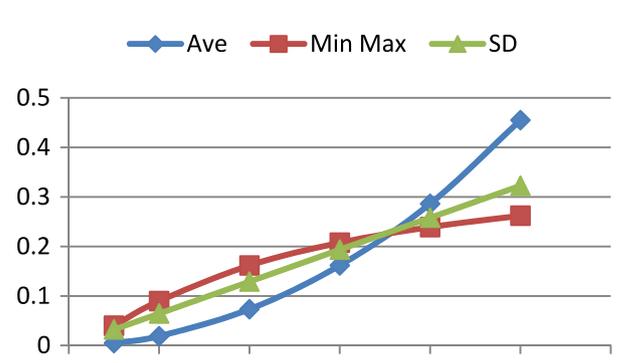


Fig. 12. Attribute weights when $u(H_1) = 0.1$.

preferences of DMs do not affect the attribute weights. When the SD method is used, the weights of the six attributes remain the same from Figs. 7 to 12, which imply that the utility of evaluation grade has no influence on the generated weights. It is also not convincing.

In contrast, when Eq. (32) is used to calculate the entropy of $S(e_i(a_l))$, the difference among the weights (the curves of 'Ave' in Figs. 7–12) of e_1 to e_6 is obvious. Specifically, from Figs. 7 to 12, e_1 which is given the belief degree of $\beta_{n,1}(a_l) = 1(n = l \leq 5)$ and

$\beta_{n,1}(a_l) = 0(n \neq l)$ is always assigned with the largest weight, while the generated weight of e_6 which is given the belief degree of $\beta_{n,6}(a_l) = 0.1(n = l \leq 5)$ and $\beta_{n,6}(a_l) = 0(n \neq l)$ is the smallest. From Figs. 7 to 12, the standard deviations of the six attribute weights are shown in Table 2.

It can be seen that with the value of $u(H_1)$ increases, the standard deviation decreases from 'Ave', which means the difference among the weights of the six attributes becomes small. Since the

Table 2
Standard deviations of the six attribute weights.

$u(H_1)$	0	0.001	0.01	0.02	0.05	0.1
Ave	0.1671	0.1667	0.1649	0.1638	0.1618	0.1604
Min Max	0	0.0034	0.0209	0.0336	0.0575	0.0796
SD	0.1027	0.1027	0.1027	0.1027	0.1027	0.1027

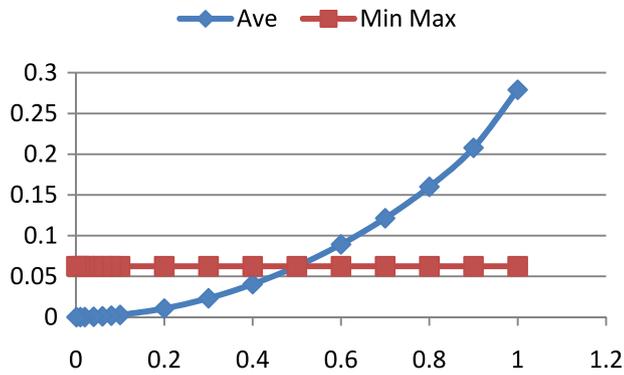


Fig. 13. Attribute weights using Eqs. (31) or (32) when $u(H_1) = 0$.

generated weights of the 6 attributes change steadily from Figs. 7 to 12, the following property of ave-entropy process is presented.

Property 2 (Continuous). The generated weights of attributes change steadily with the change of the utility of evaluation grade when the process of Eq. (32) is applied.

The proof of Property 2 is shown in the Appendix. Property 2 gives us a further insight into the ave-entropy weight assignment process. If the generated attribute weights change obviously when the utility of an evaluation grade is given a very little change, the method will not be reliable. Here, another example which contains 5 alternatives and 16 attributes is given to illustrate Property 2. The belief degrees are shown in Table 3.

The generated weights based on the data in Table 3 when we set $u(H_1) = 0, u(H_2) = 0.45, u(H_3) = 0.75, u(H_4) = 0.9$ and $u(H_5) = 1$ are shown in Fig. 13. The meanings of the horizontal and vertical axes in Fig. 13 are the same with Figs. 7 to 12. We also calculate the weights of the 16 attributes shown in Fig. 14 when we set $u(H_1) = 0.001$ provided that the utilities of other four evaluation grades remain the same. The results in Figs. 13 and 14 show that although the utility of H_1 changes very little, the generated weights change a lot when Eq. (31) (the curve of 'Min Max') is used. The ave-entropy method (the curve of 'Ave' in Figs. 13 and 14) seems more rational because it gives us a continuous change related to the utility of evaluation grade.

Fig. 15 shows the sensitivity analysis conducted on the change of number of alternatives for ave-entropy based on Table 1. Here, '2 alternatives' refers to the weights are generated by only a_1 and a_2 in Table 1. '3 alternatives' means the weights are calculated by a_1, a_2 and a_3 , while '4 alternatives' represents a_5 is not considered in the calculation process. From Fig. 15, we can see that with the number of alternatives increases from 2 to 5, the differences among the weights of the 6 attributes decrease. When only a_1 and a_2 are considered, i.e. the 2nd and 3rd column in Table 1, the difference between the weight of e_1 and e_2 is 0.454 which is relatively large although the dissimilarity between the belief distribution of a_1 and a_2 for e_1 and e_2 has not such obvious difference. According to [61], the dissimilarity measure between a_1 and a_2 for e_1 represented by $D(e_1(a_{12}))$ is 0.45 provided that the DM is risk aversion as the above defined, while $D(e_2(a_{12})) = 0.288$. So with the number of alternatives increase, the method seems more rational.

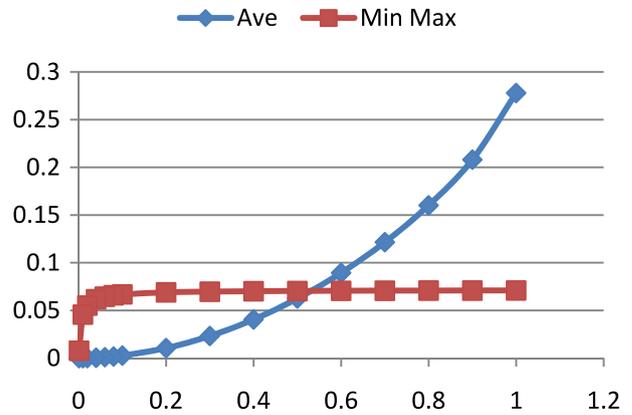


Fig. 14. Attribute weights using Eqs. (31) or (32) when $u(H_1) = 0.001$.

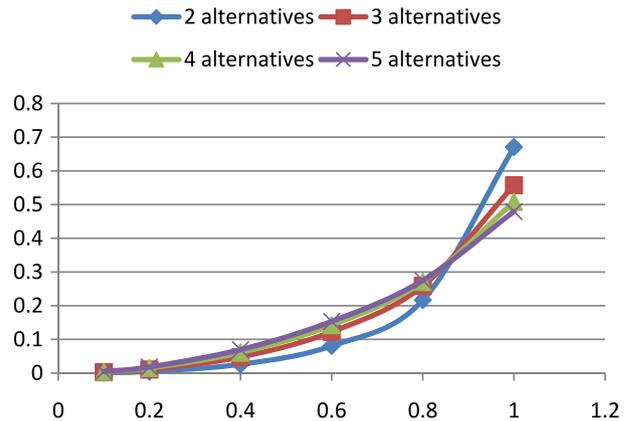


Fig. 15. Sensitivity analysis on the number of alternatives.

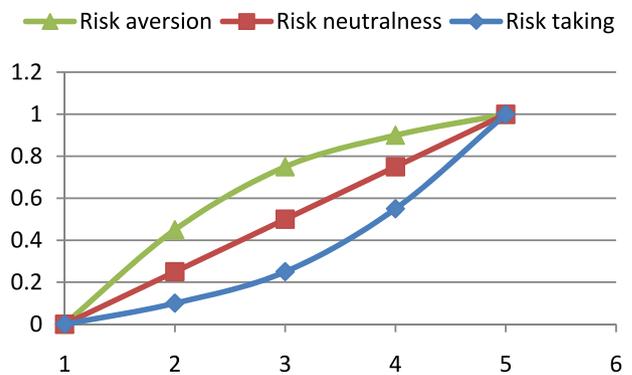


Fig. 16. Three different types of utility function depending on the preference of DM towards risk.

4.4. Consideration of risk preference of DM

In group decision making, different utility functions may be estimated by DMs depending on their different preferences which would derive from their discrepancy on backgrounds or value judgments [4,36]. And it may be also changed with the time variable. Three different types of utility function depending on the preference of DM towards risk are shown in Fig. 16. In Fig. 16, the horizontal axis represents the 5 evaluation grades which are H_1, H_2, H_3, H_4, H_5 , while the vertical axis indicates the utility of evaluation grade such that $0 \leq u(H_n) \leq 1 (n = 1, 2, \dots, 5)$.

A unique utility function which only represents one kind of risk preference will lead to the irrationality of the generated weights.

Table 3
Belief degrees that 5 alternatives be assessed to 16 attributes.

	a_1	a_2	a_3	a_4	a_5
e_1	(H ₁ , 1)	(H ₂ , 1)	(H ₃ , 1)	(H ₄ , 1)	(H ₅ , 1)
e_2	(H ₁ , 0.9;H, 0.1)	(H ₂ , 0.9;H, 0.1)	(H ₃ , 0.9;H, 0.1)	(H ₄ , 0.9;H, 0.1)	(H ₅ , 0.9;H, 0.1)
e_3	(H ₁ , 0.8;H, 0.2)	(H ₂ , 0.8;H, 0.2)	(H ₃ , 0.8;H, 0.2)	(H ₄ , 0.8;H, 0.2)	(H ₅ , 0.8;H, 0.2)
e_4	(H ₁ , 0.7;H, 0.3)	(H ₂ , 0.7;H, 0.3)	(H ₃ , 0.7;H, 0.3)	(H ₄ , 0.7;H, 0.3)	(H ₅ , 0.7;H, 0.3)
e_5	(H ₁ , 0.6;H, 0.4)	(H ₂ , 0.6;H, 0.4)	(H ₃ , 0.6;H, 0.4)	(H ₄ , 0.6;H, 0.4)	(H ₅ , 0.6;H, 0.4)
e_6	(H ₁ , 0.5;H, 0.5)	(H ₂ , 0.5;H, 0.5)	(H ₃ , 0.5;H, 0.5)	(H ₄ , 0.5;H, 0.5)	(H ₅ , 0.5;H, 0.5)
e_7	(H ₁ , 0.4;H, 0.6)	(H ₂ , 0.4;H, 0.6)	(H ₃ , 0.4;H, 0.6)	(H ₄ , 0.4;H, 0.6)	(H ₅ , 0.4;H, 0.6)
e_8	(H ₁ , 0.3;H, 0.7)	(H ₂ , 0.3;H, 0.7)	(H ₃ , 0.3;H, 0.7)	(H ₄ , 0.3;H, 0.7)	(H ₅ , 0.3;H, 0.7)
e_9	(H ₁ , 0.2;H, 0.8)	(H ₂ , 0.2;H, 0.8)	(H ₃ , 0.2;H, 0.8)	(H ₄ , 0.2;H, 0.8)	(H ₅ , 0.2;H, 0.8)
e_{10}	(H ₁ , 0.1;H, 0.9)	(H ₂ , 0.1;H, 0.9)	(H ₃ , 0.1;H, 0.9)	(H ₄ , 0.1;H, 0.9)	(H ₅ , 0.1;H, 0.9)
e_{11}	(H ₁ , 0.08;H, 0.92)	(H ₂ , 0.08;H, 0.92)	(H ₃ , 0.08;H, 0.92)	(H ₄ , 0.08;H, 0.92)	(H ₅ , 0.08;H, 0.92)
e_{12}	(H ₁ , 0.06;H, 0.94)	(H ₂ , 0.06;H, 0.94)	(H ₃ , 0.06;H, 0.94)	(H ₄ , 0.06;H, 0.94)	(H ₅ , 0.06;H, 0.94)
e_{13}	(H ₁ , 0.04;H, 0.96)	(H ₂ , 0.04;H, 0.96)	(H ₃ , 0.04;H, 0.96)	(H ₄ , 0.04;H, 0.96)	(H ₅ , 0.04;H, 0.96)
e_{14}	(H ₁ , 0.02;H, 0.98)	(H ₂ , 0.02;H, 0.98)	(H ₃ , 0.02;H, 0.98)	(H ₄ , 0.02;H, 0.98)	(H ₅ , 0.02;H, 0.98)
e_{15}	(H ₁ , 0.01;H, 0.99)	(H ₂ , 0.01;H, 0.99)	(H ₃ , 0.01;H, 0.99)	(H ₄ , 0.01;H, 0.99)	(H ₅ , 0.01;H, 0.99)
e_{16}	(H ₁ ,0.001;H,0.999)	(H ₂ ,0.001;H,0.999)	(H ₃ ,0.001;H,0.999)	(H ₄ ,0.001;H,0.999)	(H ₅ ,0.001;H,0.999)

For this reason, the generated objective weights are better to be assigned with interval values provided that different types of utility function are considered and assumed to be constraints. To capture the difference on risk preference of DM, the utility of H_n is assumed to be interval values [4,36] as follows:

$$u(H_n) \in [u_n^-, u_n^+] \quad (n = 1, 2, \dots, N) \tag{33}$$

Therefore, to handle the risk preference of different DMs, the following optimization model is constructed to calculate the maximum and minimum value of w_i :

$$\langle Model 1 \rangle \max/\min w_i = \frac{1 - E_i}{L - \sum_{i=1}^L E_i} \quad (i = 1, 2, \dots, L)$$

$$s. t. E_i = -\frac{1}{\ln(S)} \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{m}_{li}^{Ave}$$

$$\tilde{u}_{li}^{Ave} = \frac{u_{li}^{Ave}}{\sum_{l=1}^S u_{li}^{Ave}} \quad \left(\text{or } \tilde{u}_{li}^{Ave} = \frac{\bar{u}_{li}^{Ave}}{\sum_{l=1}^S \bar{u}_{li}^{Ave}} \right)$$

$$u_n^- \leq u(H_n) \leq u_n^+ \quad (n = 1, 2, \dots, N)$$

Eqs. (28), (29), (30)

The above programming model that could be computed by Excel contains N variables which are $u(H_n)(n = 1, 2, \dots, N)$. Let w_i^+ and w_i^- be the optimized values of the objective function in $\langle Model 1 \rangle$. So the right and left extensions of w_i are obtained considering the preference discrepancy of different DMs.

Take the case in Table 1 for example, the utility intervals are assumed to be between risk taking and risk aversion such that $0 \leq u(H_1) \leq 0.1, 0.1 \leq u(H_2) \leq 0.45, 0.25 \leq u(H_3) \leq 0.75, 0.55 \leq u(H_4) \leq 0.9, 0.9 \leq u(H_5) \leq 1$. The generated interval weights from $\langle Model 1 \rangle$ are shown in Table 4 and Fig. 17.

Given that the weights are calculated by the ave-entropy method and the utility estimation may be fuzzy, the combined utility assessed on a_i could be generated by the following two programming models according to Wang [31], Guo [47] and Zhou [36]'s models.

$$\langle Model 2 \rangle \text{Max} \sum_{n=1}^N \beta_n(a_i) u(H_n) + \beta_H(a_i) u(H_N)$$

$$s. t. u_n^- \leq u(H_n) \leq u_n^+ \quad (n = 1, 2, \dots, N)$$

$$w_i = \frac{1 + \frac{1}{\ln(S)} \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{m}_{li}^{Ave}}{L + \frac{1}{\ln(S)} \sum_{i=1}^L \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{m}_{li}^{Ave}} \quad (i = 1, 2, \dots, L)$$

$$\langle Model 3 \rangle \text{Min} \sum_{n=1}^N \beta_n(a_i) u(H_n) + \beta_H(a_i) u(H_1)$$

$$s. t. u_n^- \leq u(H_n) \leq u_n^+ \quad (n = 1, 2, \dots, N)$$

Table 4
Interval weights considering the risk preference of DMs.

	$\beta_{n,i}(a_i)$ ($n = l \leq 5$)	w_i^-	w_i^+
e_1	1	0.469	0.626
e_2	0.8	0.202	0.278
e_3	0.6	0.11	0.157
e_4	0.4	0.045	0.073
e_5	0.2	0.01	0.019
e_6	0.1	0.002	0.005

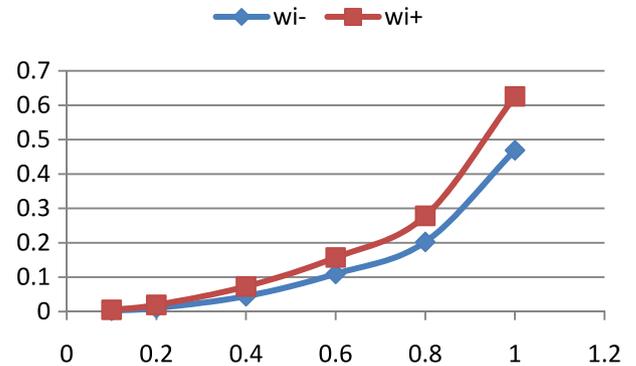


Fig. 17. Interval weights considering the risk preference of DMs.

$$w_i = \frac{1 + \frac{1}{\ln(S)} \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{m}_{li}^{Ave}}{L + \frac{1}{\ln(S)} \sum_{i=1}^L \sum_{l=1}^S \tilde{u}_{li}^{Ave} \ln \tilde{m}_{li}^{Ave}} \quad (i = 1, 2, \dots, L)$$

The analytical algorithm to generate $\beta_n(a_i)$ can be related to [6]. From $\langle Model 2 \rangle$ and $\langle Model 3 \rangle$, the overall maximum and minimum expected utilities can be generated as u_i^{Max} and u_i^{Min} . Then the utility interval assessed on a_i can be denoted by $u_i \in [u_i^{Min}, u_i^{Max}]$.

In summary, the ER approach with multiple kinds of attributes and objective weight assignment process is shown in Fig. 18.

Fig. 18 provides us a complete process to tackling with a MADM problem including both quantitative and qualitative attributes. Three steps are involved in the decision making process. Step 1 is the acquirement of assessment values on both quantitative and qualitative attributes, followed by the transformation of assessment values to the belief degrees on the general frame of discernment. For a quantitative BA or CA with certain value or a qualitative attribute assigned with belief distribution, the transformation rules in [2] can be used. If a quantitative attribute is assessed by interval value, we can apply the method in [23] or [31] to transform the interval value to belief distribution. Otherwise, transformation rules proposed in Section 3 can be used for FIA and DIA. Step 2

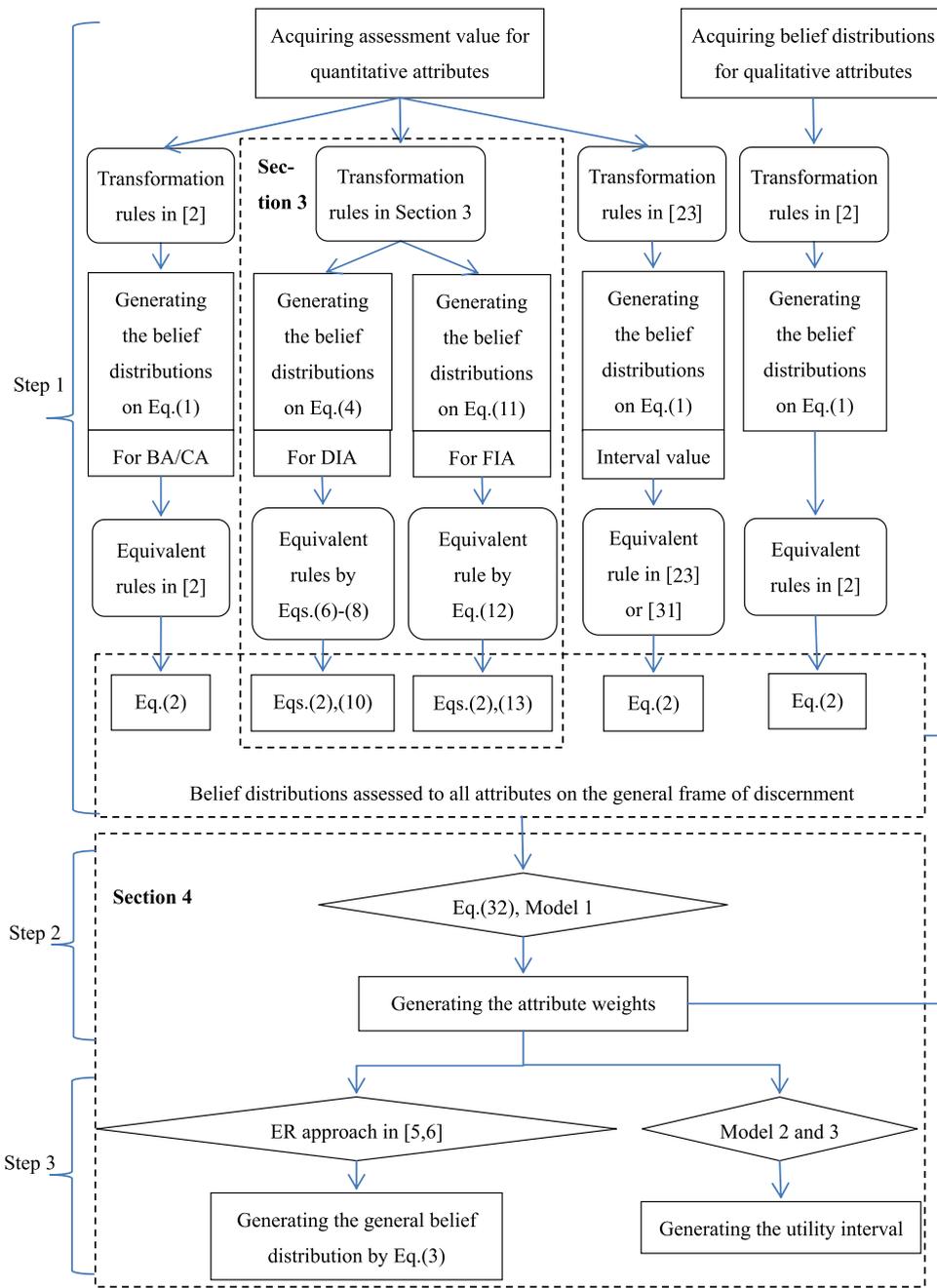


Fig. 18. ER approach with multiple kinds of attributes and objective weight assignment.

is the generation of attribute weights from the assessment values of both quantitative and qualitative attributes which is discussed in Section 4.3. Step 3 is the aggregation of attribute values from the transformed distributions in step 1 and the generated attribute weights in step 2 by the ER approach or Models 1 to 3 provided in Section 4.4. Next, we will give two case studies to illustrate the whole decision making process.

5. Case study

In this section, two examples are provided to illustrate the approaches proposed in this paper. In the first case, FIA and DIA are contained in the assessment process, while weight assignment method is included in the second case.

Table 5
Attribute structure and assessed values on the two houses.

Type of attribute	Symbol	Attribute	House A	House B
CA	e_1	Price (CNY)	1,050,000	880,000
Qualitative	e_2	Position of building	G	A
Qualitative	e_3	Safety	B	A
FIA	e_4	Area of housing (square meter)	133	110
Qualitative	e_5	Reputation of developer	A	G
FIA	e_6	Storey of house	5th	12th
Qualitative	e_7	School district	A	G
Qualitative	e_8	Suitability of living	B	G
Qualitative	e_9	Brand of elevator	G	G
DIA	e_{10}	Distance to gas station or high voltage transmission line (km)	0.9	1.5

Table 6
Attribute values of quantitative attributes corresponding to the four evaluation grades.

	B(H ₄)	G(H ₃)	A(H ₂)	W(H ₁)
Price (CNY)	No more than 800,000	900,000	1,000,000	More than 1,100,000
Area of housing	120–125 square meters	110 square meters or 130 square meters	100 square meters or 135 square meters	Less than 90 or more than 144 square meters
Storey of house	9–10th floor	13th floor or 7th floor	15th floor or 3th floor	1st or 18th floor
Distance to gas station or high voltage transmission line	More than 1 km	0.7	0.5 km	Less than 0.3 km

Table 7
Belief degrees of attributes after transformation.

Type of attribute	Attribute	House A	House B
CA	Price (CNY)	(W,0.5; A,0.5)	(G,0.8; B,0.2)
Qualitative	Position of building	(G,1.0)	(A,1.0)
Qualitative	Safety	(B,1.0)	(A,1.0)
FIA	Area of housing (square meter)	(A,0.6; G,0.4)	(G,1.0)
Qualitative	Reputation of developer	(A,1.0)	(G,1.0)
FIA	Storey of house	(A,0.5;G,0.5)	(G,0.67;B,0.33)
Qualitative	School district	(A,1.0)	(G,1.0)
Qualitative	Suitability of living	(B,1.0)	(G,1.0)
Qualitative	Brand of elevator	(G,1.0)	(G,1.0)
DIA	Distance to gas station or high voltage transmission line (km)	(G,0.33;B,0.67)	(B,1.0)

5.1. House selection problem including DIA and FIA

The first case is a selection process of buying a house. A person is going to buy a house in a or his family in China. There are many factors to be considered, including price, position of building, safety, area of housing, reputation of developer, storey of house, school district, suitability of living, brand of elevator, distance to gas station or high voltage transmission line. Since quality of house is highly related with reputation of developer, it is not listed again. In Table 5, 10 attributes are shown. Among these attributes, ‘area of housing’ and ‘storey of house’ are two FIAs to the person. Since the house is bought for his family including his wife, child and him, he feels that 120 to 125 square meters is the best size because if area is more than 144 square meters, price will be higher and tax will be doubled to 4% in China. On the other hand, if area is too small, it is not suitable to live for a family of three people although it is cheaper. ‘Storey of house’ is another FIA for him because he likes the 9–10th floor the most. If floor is too high, he will feel nervous because he has acrophobia, while there may not be good sunshine if floor is too low. So a house will be less satisfied no matter whether floor is more or less than the 9–10th floor. ‘Distance to gas station or high voltage transmission line’ is a DIA because gas station is a potential hazard and high voltage transmission line has the risk of radiation. It is safer to be far away from them no matter to what direction considering the longitude and latitude. The houses he is going to choose are from two 18-floor buildings in different locations of the same city. Four evaluation grades are included in the assessment, which are Best (H₄), Good (H₃), Average (H₂), Worst (H₁). The assessed values about the two houses are shown in Table 5.

It is clear that e₁, e₄, e₆ and e₁₀ are four quantitative attributes in the attribute structure. The assessment criteria which reflect the values of the four quantitative attributes corresponding to the four evaluation grades are shown in Table 6.

From Table 6, the frames of discernment of the four quantitative attributes are constructed. For instance, the frame of discernment for ‘area of housing’ is shown in Fig. 19. Since e₄ is a FIA, utility between every two adjacent evaluation grades are linear increase when the value is between 90 and 120 square meters, while linear decrease when the value is more than 125 square meters. So we can transform any value to the belief degrees on its evaluation grades in Fig. 19.

Table 8
Combined belief degrees assigned to the two houses.

	B(H ₄)	G(H ₃)	A(H ₂)	W(H ₁)
House A	25.91%	32.61%	37.08%	4.41%
House B	12.74%	70.43%	16.83%	0

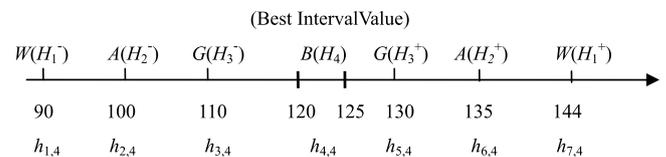


Fig. 19. Relationship between values of e₄ and its frame of discernment.

Based on Table 6, the assessed values of the four quantitative attributes could be transformed to belief degrees on the four evaluation grades which are shown in Table 7.

The weights of attributes in this example are supposed to be equal. Then the aggregated belief degrees for the two houses can be generated by applying the ER algorithm, which are shown in Table 8. Suppose the utility of the four evaluation grades are u(H₁) = 0, u(H₂) = 0.3, u(H₃) = 0.7, u(H₄) = 1, then the utilities of the two houses are 0.5986 and 0.6709 respectively.

5.2. SRDPA problem with ave-entropy weight assignment

In this subsection, a strategic R&D project assessment (SRDPA) problem adapted from [3] is solved by the ave-entropy method to illustrate its validity. The assessment is in the period that the R&D process has been finished and already in market considering the life cycle of a project. It is hoped that the final generated score to a product on the general level could reflect the performance of completion.

5.2.1. Description of the SRDPA problem

SRDPA problem is in essence a MADM problem which is characterized by many qualitative and quantitative attributes whose values may be precise, fuzzy or incomplete. In this SRDPA problem, the performance of four R&D projects denoted by a_i (i = 1, 2, 3, 4) for a car manufacturer is assessed on 3 general attributes, which are decomposed into 8 s level attributes and 17 third level attributes

Table 9
Attribute framework and belief distributions of four projects in the SRDPA problem.

General attributes	Criteria in the second level	Factors in the lowest level (contents of assessment)	Type of project			
			Light trailer (a_1)	Heavy trailer (a_2)	MPV (a_3)	SRV (a_4)
Quality of production E_1	Scale and importance of project E_{11}	<i>Workload e_{11}^1</i>	<i>D(1.0)</i>	<i>D(1.0)</i>	<i>A(0.33)</i> <i>B(0.67)</i>	<i>A(1.0)</i>
		<i>Origin of person e_{11}^2</i>	<i>B(0.15)</i> <i>C(0.70)</i> <i>D(0.15)</i>	<i>B(0.15)</i> <i>C(0.70)</i> <i>D(0.15)</i>	<i>B(0.15)</i> <i>C(0.70)</i> <i>D(0.15)</i>	<i>B(0.15)</i> <i>C(0.70)</i> <i>D(0.15)</i>
		<i>Importance of project e_{11}^3</i>	<i>A(0.45)</i> <i>B(0.165)</i> <i>C(0.165)</i> <i>E(0.22)</i>	<i>A(0.83)</i> <i>B(0.085)</i> <i>C(0.085)</i>	<i>A(0.2)</i> <i>B(0.4)</i> <i>C(0.4)</i>	<i>A(0.70)</i> <i>B(0.15)</i> <i>C(0.15)</i>
	Content of technique E_{12}	<i>Advance of critical techniques e_{12}^1</i>	<i>B(0.67)</i> <i>C(0.33)</i>	<i>A(0.67)</i> <i>B(0.33)</i>	<i>A(0.4)</i> <i>B(0.5)</i> <i>C(0.1)</i>	<i>A(0.6)</i> <i>B(0.3)</i> <i>C(0.1)</i>
		<i>Ratio between quality and price e_{12}^2</i>	<i>A(0.56)</i> <i>B(0.22)</i> <i>C(0.22)</i>	<i>A(0.25)</i> <i>B(0.67)</i> <i>C(0.08)</i>	<i>A(0.3)</i> <i>B(0.7)</i>	<i>A(0.1)</i> <i>B(0.4)</i> <i>C(0.5)</i>
		<i>Reliability of product e_{12}^3</i>	<i>H(1.0)</i>	<i>H(1.0)</i>	<i>H(1.0)</i>	<i>H(1.0)</i>
		<i>Economy e_{12}^4</i>	<i>A(1.0)</i>	<i>B(1.0)</i>	<i>A(0.4)</i> <i>B(0.6)</i>	<i>B(0.7)</i> <i>C(0.3)</i>
	Theoretical value and level of innovation E_{13}	<i>Theoretical standard of project e_{13}^1</i>	<i>A(0.11)</i> <i>B(0.45)</i> <i>C(0.33)</i> <i>D(0.11)</i>	<i>A(0.42)</i> <i>B(0.50)</i> <i>D(0.08)</i>	<i>A(0.5)</i> <i>B(0.5)</i>	<i>A(0.8)</i> <i>B(0.2)</i>
		<i>Degree of innovation e_{13}^2</i>	<i>B(0.05)</i> <i>C(0.275)</i> <i>D(0.225)</i> <i>E(0.45)</i>	<i>B(0.125)</i> <i>C(0.415)</i> <i>D(0.29)</i> <i>E(0.17)</i>	<i>A(0.1)</i> <i>B(0.1)</i> <i>C(0.35)</i> <i>D(0.25)</i> <i>E(0.2)</i>	<i>B(0.1)</i> <i>C(0.2)</i> <i>D(0.1)</i> <i>E(0.6)</i>
		<i>Ratio of individual design e_{13}^3</i>	<i>B(1.0)</i>	<i>B(1.0)</i>	<i>B(1.0)</i>	<i>B(1.0)</i>
Process control E_2	Quality of project E_{21}	<i>A(0.44)</i> <i>B(0.56)</i>	<i>A(0.83)</i> <i>B(0.17)</i>	<i>A(0.5)</i> <i>B(0.4)</i> <i>C(0.1)</i>	<i>A(0.5)</i> <i>B(0.5)</i>	
		<i>Completion time for a project E_{22}</i>	<i>B(1.0)</i>	<i>A(1.0)</i>	<i>A(1.0)</i>	<i>A(1.0)</i>
	<i>Investment E_{23}</i>	<i>A(1.0)</i>	<i>A(1.0)</i>	<i>A(1.0)</i>	<i>A(1.0)</i>	
Added value by project E_3	Project team E_{31}	<i>Documents of rules and regulations established about project group e_{31}^1</i>	<i>A(0.22)</i> <i>B(0.11)</i> <i>C(0.335)</i> <i>D(0.225)</i> <i>E(0.11)</i>	<i>A(0.33)</i> <i>B(0.21)</i> <i>C(0.335)</i> <i>D(0.125)</i>	<i>A(0.2)</i> <i>B(0.2)</i> <i>C(0.4)</i> <i>D(0.2)</i>	<i>A(0.2)</i> <i>B(0.2)</i> <i>C(0.4)</i>
		<i>Routine operational management documents e_{31}^2</i>	<i>A(0.11)</i> <i>B(0.33)</i> <i>C(0.56)</i>	<i>A(0.25)</i> <i>B(0.5)</i> <i>C(0.17)</i>	<i>A(0.3)</i> <i>B(0.5)</i> <i>C(0.2)</i>	<i>A(0.1)</i> <i>B(0.4)</i> <i>C(0.4)</i>
		<i>Management documents about R&D process of products e_{31}^3</i>	<i>A(0.22)</i> <i>B(0.11)</i> <i>C(0.56)</i> <i>D(0.11)</i>	<i>A(0.33)</i> <i>B(0.58)</i> <i>C(0.09)</i>	<i>A(0.6)</i> <i>B(0.3)</i> <i>C(0.1)</i>	<i>A(0.4)</i> <i>B(0.4)</i> <i>C(0.2)</i>
	Continuity of technique E_{32}	<i>Accumulation and continuity of technique</i>	<i>A(0.67)</i> <i>C(0.22)</i> <i>E(0.11)</i>	<i>A(0.92)</i> <i>E(0.08)</i>	<i>A(0.7)</i> <i>C(0.2)</i> <i>E(0.1)</i>	<i>A(0.7)</i> <i>C(0.3)</i>

*Quantitative attributes are in italic type.

that are shown in the first three columns of Table 9. Among the 17 basic attributes, some reflect the content of life cycle sustainability assessment (LCSA) to certain degree. For example, “economy (e_{12}^4)” is the cost spent on designing, manufacturing and using a product [3]. It almost covers the stages of life cycle cost (LCC) for a product in the LCSA framework except the raw material acquisition and waste management phases. “Quality of production (E_1)” is the overall satisfaction generated from the stated characteristics, which means how the consumer’s demand could be satisfied by the features of a product [3]. It accords with the stakeholder of consumer in the social life cycle assessment (SLCA) structure associated with LCSA framework.

Five evaluation grades are defined to assess the four projects as follows:

$$H = \{Worst, Poor, Average, Good, Best\} \tag{34}$$

For illustration purpose, *Best* is represented by *A*, *Good* by *B*, *Average* by *C*, *Poor* by *D* and *Worst* by *E*. Then

$$H^{General} = \{E, D, C, B, A\} \tag{35}$$

In Table 9, 17 basic attributes in the assessment framework are split into two parts: 10 qualitative attributes and 7 quantitative attributes which are in italic type. Eqs. (34) or (35) is just the frame of discernment on the general level. For each qualitative attribute, the number of evaluation grades are not all consistent with Eq. (35) for the convenience of original data collection. The

Table 10
Generated attribute weights by Eq. (32) and several other methods.

Attributes in three levels			Serial number	Attribute weights				
				Ave-entropy	GAHP	SD	CRITIC	CCSD
E1	E ₁₁	e ₁₁ ¹	1	0.1177	0.0257	0.2468	0.268	0.3057
		e ₁₁ ²	2	0	0.0278	0	0	0
		e ₁₁ ³	3	0.08	0.0607	0.0869	0.0758	0.0967
	E ₁₂	e ₁₂ ¹	4	0.118	0.0201	0.0678	0.0738	0.0124
		e ₁₂ ²	5	0.0493	0.041	0.0541	0.0773	0.0786
		e ₁₂ ³	6	0	0.0575	0	0	0
		e ₁₂ ⁴	7	0.0732	0.0336	0.0887	0.1576	0.132
	E ₁₃	e ₁₃ ¹	8	0.0528	0.0316	0.0834	0.0676	0.0306
		e ₁₃ ²	9	0.0812	0.0524	0.066	0.0557	0.075
		e ₁₃ ³	10	0	0.0453	0	0	0
E2	E ₂₁	E ₂₁	11	0.1145	0.1779	0.031	0.0298	0.0416
	E ₂₂	E ₂₂	12	0.0483	0.0787	0.079	0.0527	0.0506
	E ₂₃	E ₂₃	13	0	0.1051	0	0	0
E3	E ₃₁	e ₃₁ ¹	14	0.0607	0.0331	0.0413	0.0292	0.0448
		e ₃₁ ²	15	0.0684	0.0399	0.0435	0.0305	0.0427
		e ₃₁ ³	16	0.0502	0.0434	0.0721	0.0487	0.0429
	E ₃₂	E ₃₂	17	0.0856	0.1263	0.0394	0.0332	0.0465

original assessment information consists of both numerical values on the 7 quantitative attributes and belief distributions on the 10 qualitative attributes whose evaluation grades may be more or less than 5. In the last 4 columns of Table 9, belief distributions that each project be assessed to all the 17 attributes are presented. They are already the transformed belief distributions from original information provided by experts according to specific rules. For example, we can see that “Ratio between quality and price (e₁₂²)” associated with “Content of technique (E₁₂)” for light trailer is evaluated to be *Best* with a belief degree of 0.56, to be *Good* with a belief degree of 0.22, to be *Average* with a belief degree of 0.22. The above statement could be represented by the following expectation:

$$S(e_{12}^2(a_1)) = \{(Best, 0.56), (Good, 0.22), (Average, 0.22)\}$$

Note that the total belief degree for the statement of e₁₂² on light trailer sums to 1, which means that the assessment is complete. But it is not always the case. The assessments to “Reliability of product (e₁₂³)” on all the four projects are completely ignorant because the original information provided by experts is ‘unknown’ [3]. So both complete and incomplete assessments are included in Table 9. The DM wants to obtain a rank order of the four projects by aggregating the distributed assessments on 17 attributes.

5.2.2. Generating weights using different methods

The ave-entropy method is used to generate the weights of attributes. In the first situation, the utilities of the five evaluation grades are assumed to be risk neutrality, such that u(E) = 0, u(D) = 0.25, u(C) = 0.5, u(B) = 0.75 and u(A) = 1. Since the assessment to e₁₁², e₁₂³, e₁₃³ and E₂₃ on the four projects are the same, these four attributes have no effect in the ranking and comparison of the four projects. So we set the weights of these four attributes to be 0 in the SRDPA problem. The belief distributions that each attribute be assessed on all the four projects should be transformed to average utilities by Eqs. (28)–(30) first. Take e₁₂² for example, u_{1,122}^{Ave} = 0.835, u_{2,122}^{Ave} = 0.7925, u_{3,122}^{Ave} = 0.825, u_{4,122}^{Ave} = 0.65.

Secondly, average utilities are tackled with standard 0–1 transformation method by the equation that $\bar{u}_{ii}^{Ave} = \frac{u_{ii}^{Ave} - \min_{l=1,2,3,4} u_{ii}^{Ave}}{\max_{l=1,2,3,4} u_{ii}^{Ave} - \min_{l=1,2,3,4} u_{ii}^{Ave}}$. So the maximal and minimal value of an attribute be assessed on all the four projects will be given the value of 1 and 0 respectively. Take e₁₂² for example, $\bar{u}_{1,122}^{Ave} = 1$, $\bar{u}_{2,122}^{Ave} = 0.77027$, $\bar{u}_{3,122}^{Ave} = 0.94595$, $\bar{u}_{4,122}^{Ave} = 0$. Then the

normalized average utilities can be get by $\tilde{u}_{ii}^{Ave} = \frac{\bar{u}_{ii}^{Ave}}{\sum_{l=1}^4 \bar{u}_{ii}^{Ave}}$. For instance, $\tilde{u}_{1,122}^{Ave} = 0.36816$, $\tilde{u}_{2,122}^{Ave} = 0.28358$, $\tilde{u}_{3,122}^{Ave} = 0.34826$, $\tilde{u}_{4,122}^{Ave} = 0$.

Thirdly, Eq. (26) is used to generate the entropy of each attribute from the normalized average utilities of each attribute on all the four projects. Take e₁₂² for example, $E_{122} = -\frac{1}{\ln(4)} \sum_{l=1}^4 \tilde{u}_{ii}^{Ave} \ln \tilde{u}_{ii}^{Ave} = 0.7882$.

Finally, the attribute weights could be obtained by Eq. (27) that are shown in the 5th column of Table 10. We also calculate the weights of the 17 attributes by SD, CRITIC and CCSD which are shown in the last three columns of Table 10. u_{ii}^{Ave} is directly used to generate weights with no normalization for SD, CRITIC and CCSD because 0 ≤ u_{ii}^{Ave} ≤ 1. The 6th column in Table 10 presents the weights generated by GAHP method in [3].

5.2.3. Comparison of several methods

Fig. 20 shows the weights generated by ave-entropy, GAHP in [3], SD, CRITIC and CCSD. The horizontal and vertical axes represent the number of the 17 attributes and the weights of the 17 attributes respectively. From the SD, CRITIC and CCSD methods, e₁₁¹ is assigned with the weight of 0.2468, 0.268 and 0.3057 respectively which are much larger than the other 16 attributes. The reason lies in that the three methods are all based on standard deviation which is determined by the divergence of distributions on projects. Since the distributions that e₁₁¹ be assessed on the four projects are quite different compared with other 16 attributes, e₁₁¹ is given more importance. Comparatively, the ave-entropy method and GAHP creates relatively well-distributed weights. e₁₁², e₁₂³, e₁₃³ and E₂₃ are assigned with the weight of 0 by ave-entropy, SD, CRITIC and CCSD because the distributions of each of the 4 attributes on these 4 projects are the same. So these four attributes are not included in generating the weights by the four methods. It should be mentioned that e₁₁², e₁₂³, e₁₃³ and E₂₃ may not always be given the weight of 0 if some other projects are assessed because there perhaps exist some differences among other projects on any of the four attributes. And when some other projects are assessed, an attribute except e₁₁², e₁₂³, e₁₃³ and E₂₃ may be given the same distribution. So the attribute weights may change when different projects are assessed.

The comparisons of the five methods are shown in Table 11. Abnormal weights means that the weight of at least one attribute exceeds the rational times of other attributes. On the contrary, well-distributed weight means the difference between each pair of attributes is not extremely large. A special case is that all

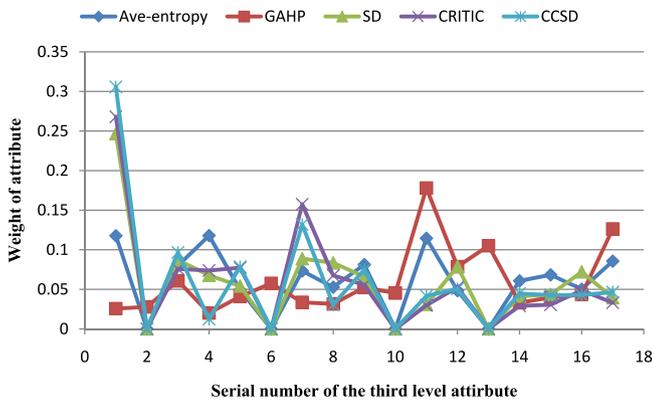


Fig. 20. Weights generated by Ave-entropy, GAHP, SD, CRITIC and CCSD.

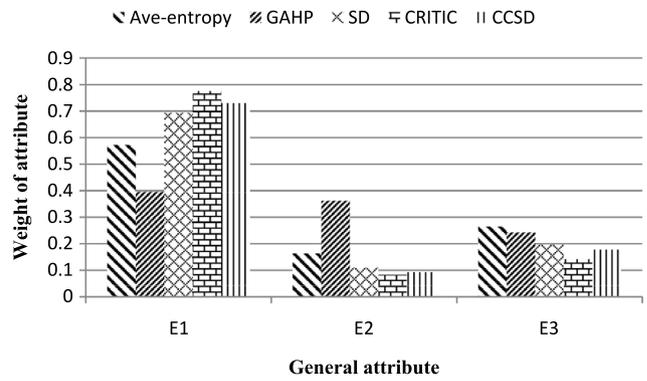


Fig. 21. Comparison of the attribute weights on the general level among the five methods.

Table 11 Comparison of several methods.

Property	Method				
	Ave-entropy	GAHP	SD	CRITIC	CCSD
Abnormal weights on e_{11}^1			✓	✓	✓
Consideration of the utilities of evaluation grades	✓		✓	✓	✓
Well-distributed weights	Relatively	Relatively			
Considering the risk preference of DM	✓	✓			
Including the optimization process					✓
The extent of subjectivity	Little	High	Little	Little	Little
The extent of sensitivity to discrepancies	Moderate	None	High	High	High
Practicability in consideration of alternative number	≥ 2	≥ 0	≥ 2	≥ 2	≥ 2

attributes have the same weight which we call 'the best well-distributed weight'. However, the best well-distributed weight for 17 attributes which is 1/17 is not reasonable because we can at least distinguish one attribute which is more important than the other ones. Among these five methods, only the CCSD method includes an optimization method. Since in the GAHP method [3,4,36], a group of DMs are required to provide subjective judgments on the comparison between every two attributes, the generated weights involve a high extent of subjectivity even if there is no qualitative attribute in the attribute structure. Compared with the GAHP method, the subjectivity contained in each of the other four methods is much lower because only the utilities of evaluation grades should be estimated by DMs. In both the GAHP and the ave-entropy method, the risk preference of DM is considered which lead to several constructed programming models. So these two methods are particularly applicable and effective when a group of DMs are involved in a decision making problem.

For a more comprehensive view of the results, the comparisons of the attribute weights in the general level from the five methods are also presented by Fig. 21. Compared with the weights by GAHP in [3], we can see that the dissimilarities of the attribute weights in the general level are more obvious by the other four methods. From this point of view, the weight of general level attribute is sensitive to the discrepancies of basic attribute on different alternatives when ave-entropy, SD, CRITIC or CCSD is used. Considering that Ave-entropy, SD, CRITIC and CCSD are based on the discrepancies on different alternatives, these four methods are applicable when at least two alternatives are contained. Comparatively, GAHP can be used even if no alternative is involved in.

In the second situation, suppose that $0 \leq u(H_1) \leq 0.1 \leq u(H_2) \leq 0.45, 0.25 \leq u(H_3) \leq 0.75, 0.55 \leq u(H_4) \leq 0.9,$

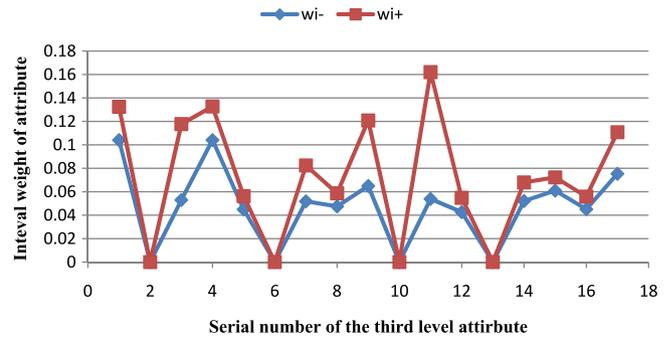


Fig. 22. Interval weights generated considering the risk preferences of DMs.

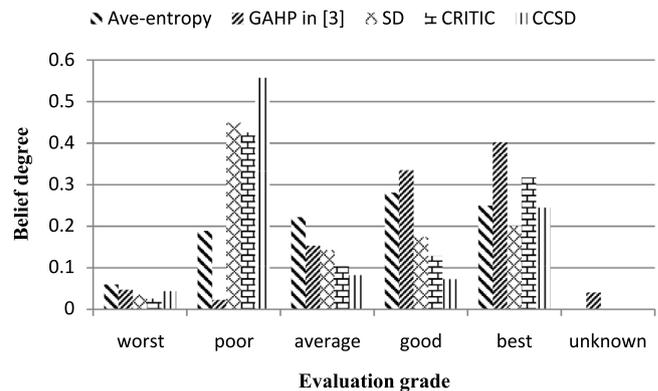


Fig. 23. The general distributions on light trailer from five different generated weights.

$1 \leq u(H_5) \leq 1$. So nonlinear programming (Model 1) is applied to calculate the interval weights of the 17 attributes. The generated interval weights represented by $[w_i^-, w_i^+](i = 1, 2, \dots, 17)$ are shown in Fig. 22. From Fig. 22, we can see that different risk preferences do influence the attribute weights to some extent.

5.2.4. ER modeling framework for SRDPA

Fig. 23 shows the general distributions from the weights calculated by the five methods on light trailer. The horizontal and vertical axes represent the evaluation grade and belief degree respectively.

The average utilities and rank orders of the four R&D projects on the overall performance and the three first level attributes are generated and shown in Table 12.

Table 12
Average utilities and rank order of the four R&D projects on major attributes.

Attribute level	Methods	Average utility and rank							
		Light trailer		Heavy trailer		MPV		SRV	
		Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank
General assessment	Ave-entropy	0.6180	4	0.7412	3	0.7950	2	0.8082	1
	GAHP	0.7556	4	0.8556	1	0.8221	2	0.8221	2
	SD	0.5137	4	0.5796	3	0.8219	2	0.8817	1
	CRITIC	0.5714	4	0.5730	3	0.8222	2	0.8549	1
	CCSD	0.4798	4	0.4883	3	0.8052	2	0.8575	1
Quality of production	Ave-entropy	0.5673	4	0.6570	3	0.7710	2	0.7961	1
	GAHP	0.6439	4	0.6878	3	0.6969	1	0.6898	2
	SD	0.4816	4	0.5119	3	0.8038	2	0.8785	1
	CRITIC	0.5580	3	0.5377	4	0.8120	2	0.8488	1
	CCSD	0.4500	3	0.4318	4	0.7942	2	0.8541	1
Process control	Ave-entropy	0.8278	4	0.9722	1	0.8920	3	0.9100	2
	GAHP	0.8796	4	0.9820	1	0.9289	3	0.9408	2
	SD	0.7623	4	0.9956	1	0.9829	3	0.9858	2
	CRITIC	0.7715	4	0.9924	1	0.9700	3	0.9750	2
	CCSD	0.7848	4	0.9878	1	0.9514	3	0.9595	2
Added results	Ave-entropy	0.6340	4	0.8081	1	0.7759	2	0.7198	3
	GAHP	0.6997	4	0.8707	1	0.8169	2	0.7927	3
	SD	0.6121	4	0.7982	2	0.8067	1	0.7374	3
	CRITIC	0.6176	4	0.8030	2	0.8060	1	0.7393	3
	CCSD	0.6177	4	0.7958	1	0.7751	2	0.7156	3

From Table 12, it is clear that the four projects were ranked in the original assessment by GAHP as follows: Heavy Trailer > MPV ~ SRV > Light Trailer; while they are ranked with the ave-entropy weights as: SRV > MPV > Heavy Trailer > Light Trailer. Compared with the rank in [3], the difference on the general level lies mainly in “Quality of production” whose weight is 0.5723 from the ave-entropy method. So although heavy trailer ranked the first on “Process control” and “Added results”, the average utility of it on “Quality of production” is much less than MPV and SRV which leads to its lower general result. From Table 9, we can see that the higher weight of “Quality of production” compared with 0.4 in [3] results from the high discrepancy of the distribution to e_{11}^1 on the four projects. Fig. 23 shows that the combined ‘unknown’ belief degree is 0 except GAHP. This is mainly caused by the fact that the only distribution which contains ignorance (e_{12}^3) is given the weight of 0 by ave-entropy, SD, CRITIC and CCSD considering that the distributions of e_{12}^3 on the four projects are the same. From Table 10 in [3], it revealed that the assessment to e_{12}^3 is completely unknown which leads to its unknown belief degree of 1 on the four projects. Although Table 12 shows that the rank orders in the general level from ave-entropy, SD, CRITIC, CCSD are the same, the difference on the combined belief degrees is obvious which is depicted in Fig. 23. For example, when assessing light trailer, the belief degree on ‘poor’ is extremely high by SD, CRITIC and CCSD. Comparatively, ave-entropy and GAHP provide a relatively distributed belief degrees on all the evaluation grades. Besides, the combined belief degrees can also be generated from the interval weights in Fig. 22 based on programming models in [47]. It needs to be mentioned that if the value of \bar{u}_{ii}^{Ave} is used to generate weights by SD, CRITIC and CCSD, the weights are well-distributed, especially for SD that e_{31}^1 is given the minimum weight of 0.0672 and e_{11}^1 is given the maximum weight of 0.0856. In this case, it seems a little unreasonable since all the attributes are given the similar weights by SD. By CRITIC and CCSD, the maximum and minimum weights are $w_{12}^4 = 0.1317$, $w_{31}^1 = 0.0496$ and $w_{12}^4 = 0.1295$, $w_{31}^1 = 0.0181$ respectively which seems more reasonable. Since \bar{u}_{ii}^{Ave} is already the dimensionless value, whether it should be transferred to \bar{u}_{ii}^{Ave} still needs to be discussed.

In real SRDPA problems, the generated performances from subjective and objective weights which are generated by the judgments of DMs and the discrepancy of assessment on different projects respectively should be considered simultaneously for a

more rational result. Nevertheless, the method in Section 4 provides us a way to generate attribute weights when the assessments are given by belief distributions, together with the consideration of risk preferences of DMs.

6. Conclusions

In this paper, the ER approach was firstly extended for solving DIA and FIA. The frames of discernment and equivalent rules of DIA and FIA were presented. Then, we studied the transformation rules for DIA from both accurate and uncertain values to belief degrees on evaluation grades in the frame of discernment. Secondly, an ave-entropy based weight assignment process is shown to dealing with MADM problems which contain uncertain and incomplete subjective judgments assessed on qualitative attributes. The risk preference of DMs is considered to construct entropy method based programming models. It is hoped that the extension of the ER approach in this paper can contribute to widening its applications in real life problems. It should be noted that the difference between $h_{n,i}$ and $h_{n+1,i}$ is not necessarily the same as the difference between $h_{2N-n-1,i}$ and $h_{2N-n,i}$, so non-symmetrical preferences between the worst interval value for DIA can be modeled instead of symmetrical preferences. FIA can also be modeled non-symmetrical preferences for the same reason. Furthermore, utility function may be non-linear between every two adjacent evaluation grades. So how to transform assessment value in these two situations will be studied in the future. Just as Diakoulaki and Deng said, there is no single method that can guarantee a more accurate set of attribute weights than others [39,67]. So the ave-entropy method is absolutely not suitable for any MADM problems, such as the case of interval belief degree or interval value assessment which will be further studied.

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Appendix

Proof of Property 2. Suppose the utility of $H_n(n \in \{1, 2, \dots, N\})$ increases a value δ which is very small. Then $u'(H_n) = u(H_n) + \delta$. The utilities of the N evaluation grades will change to $\{u(H_1), u(H_2), \dots, u'(H_n), \dots, u(H_N)\}$. There are three possible scenarios.

(1) When $n = 1$, from Eqs. (28)–(30), we have

$$\begin{aligned} u_{li}^{Ave} &= \frac{(u_{li}^{Max} + u_{li}^{Min})}{2} \\ &= \sum_{n=1}^N \beta_{n,i}(a_i) u(H_n) + \beta_{H,i}(a_i) \frac{u(H_1) + u(H_N)}{2} \\ u_{li}'^{Ave} &= \frac{(u_{li}'^{Max} + u_{li}'^{Min})}{2} \\ &= \beta_{1,i}(a_i) u'(H_1) + \sum_{n=2}^N \beta_{n,i}(a_i) u(H_n) \\ &\quad + \beta_{H,i}(a_i) \frac{u'(H_1) + u(H_N)}{2} \\ &= \beta_{1,i}(a_i) (u(H_n) + \delta) + \sum_{n=2}^N \beta_{n,i}(a_i) u(H_n) \\ &\quad + \beta_{H,i}(a_i) \frac{(u(H_n) + \delta) + u(H_N)}{2} \\ &= \beta_{1,i}(a_i) u(H_n) + \beta_{1,i}(a_i) \cdot \delta + \sum_{n=2}^N \beta_{n,i}(a_i) u(H_n) \\ &\quad + \frac{1}{2} \beta_{H,i}(a_i) \cdot \delta + \beta_{H,i}(a_i) \frac{u(H_n) + u(H_N)}{2} \\ &= \sum_{n=1}^N \beta_{n,i}(a_i) u(H_n) + \beta_{H,i}(a_i) \frac{u(H_1) + u(H_N)}{2} \\ &\quad + \beta_{1,i}(a_i) \cdot \delta + \frac{1}{2} \beta_{H,i}(a_i) \cdot \delta \\ &= u_{li}^{Ave} + \left(\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i) \right) \cdot \delta \end{aligned}$$

It is clear that when δ is very small, $\lim_{\delta \rightarrow 0} u_{li}'^{Ave} - u_{li}^{Ave} = 0$. Then

$$\begin{aligned} \tilde{u}_{li}'^{Ave} &= \frac{u_{li}'^{Ave}}{\sum_{l=1}^S u_{li}'^{Ave}} = \frac{u_{li}^{Ave} + (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i)) \cdot \delta}{\sum_{l=1}^S \{u_{li}^{Ave} + (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i)) \cdot \delta\}} \\ &= \frac{u_{li}^{Ave} + (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i)) \cdot \delta}{\sum_{l=1}^S u_{li}^{Ave} + \delta \cdot \sum_{l=1}^S (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i))} \end{aligned}$$

From Eq. (26), we have

$$\begin{aligned} E_i' &= -\frac{1}{\ln(S)} \sum_{l=1}^S \tilde{u}_{li}'^{Ave} \ln \tilde{u}_{li}'^{Ave} \\ &= -\frac{1}{\ln(S)} \sum_{l=1}^S \frac{u_{li}^{Ave} + (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i)) \cdot \delta}{\sum_{l=1}^S u_{li}^{Ave} + \delta \cdot \sum_{l=1}^S (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i))} \\ &\quad \times \ln \frac{u_{li}^{Ave} + (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i)) \cdot \delta}{\sum_{l=1}^S u_{li}^{Ave} + \delta \cdot \sum_{l=1}^S (\beta_{1,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i))} \end{aligned}$$

From Eq. (27), we can conclude that the change of w_i is little provided that δ is small enough.

(2) When $n \in 2, \dots, N - 1$, from Eqs. (28)–(30), we have

$$u_{li}^{Ave} = \beta_{n,i}(a_i) u'(H_n) + \sum_{m=1}^{n-1} \beta_{m,i}(a_i) u(H_m)$$

$$\begin{aligned} &+ \sum_{m=n+1}^N \beta_{m,i}(a_i) u(H_m) + \beta_{H,i}(a_i) \frac{u(H_1) + u(H_N)}{2} \\ &= \beta_{n,i}(a_i) (u(H_n) + \delta) + \sum_{m=1}^{n-1} \beta_{m,i}(a_i) u(H_m) \\ &\quad + \sum_{m=n+1}^N \beta_{m,i}(a_i) u(H_m) + \beta_{H,i}(a_i) \frac{u(H_1) + u(H_N)}{2} \\ &= u_{li}^{Ave} + \beta_{n,i}(a_i) \cdot \delta \end{aligned}$$

Similarly, when δ is very small, $\lim_{\delta \rightarrow 0} u_{li}'^{Ave} - u_{li}^{Ave} = 0$ which will lead to the very little change of w_i .

(3) When $n = N$, we have

$$u_{li}'^{Ave} = u_{li}^{Ave} + \left(\beta_{N,i}(a_i) + \frac{1}{2} \beta_{H,i}(a_i) \right) \cdot \delta$$

It is similar with the first scenario.

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