Knowledge-Based Systems 189 (2020) 105110

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Assignment of attribute weights with belief distributions for MADM under uncertainties☆

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ARTICLE INFO

Article history:
Received 5 March 2019
Received in revised form 8 October 2019
Accepted 9 October 2019
Available online 14 October 2019

Keywords:
Evidential reasoning
Belief distribution
Entropy weight assignment method
Interval value
Interval belief degree
Incompleteness

ABSTRACT

Multiple attribute decision making (MADM) problems often consist of various types of quantitative and qualitative attributes. Quantitative attributes can be assessed by accurate numerical values, interval values or fuzzy numbers, while qualitative attributes can be evaluated by belief distributions, linguistic variables or intuitionistic fuzzy sets. However, the determination of attribute weights is still an open issue in MADM problems until now. In the traditional objective weight assignment method, attributes are usually assessed by accurate values. In this paper, an entropy weight assignment method is proposed to dealing with the situation where the assessment of attributes can contain uncertainties, e.g., interval values, or contain both uncertainties and incompleteness, e.g., belief distributions. The advantage of the proposed method lies in that uncertainties and incompleteness contained in the interval numerical values or belief distributions can be preserved in the generated weights. Specifically, several pairs of programming models to generate the weights of attributes are constructed in three different circumstances: (1) quantitative attribute expressed by interval values; (2) incomplete belief distribution with accurate belief degrees; and (3) belief distribution constituted by interval belief degrees. The evidential reasoning approach is then utilized to aggregate the distributions of attributes based on the generated attribute weights. The normalized interval weight vector is defined, and the characteristics of the weight assignment method are discussed. The proposed method has been experimented with real data to illustrate its advantages and the potential in supporting MADM with uncertain and incomplete information.

1. Introduction

Multiple attribute decision making (MADM) with various types of attributes is common in practice [1]. Quantitative and qualitative attributes which can be assessed by numerical values and subjective judgments respectively are always included in a MADM problem. The purpose of a MADM problem is to make a ranking order of several selected alternatives, or to choose the best one of them. When qualitative attributes are expressed by belief distributions (BDs) [2–7] or probabilistic linguistic term set (PLTS) [8], the evidential reasoning (ER) approach [1,9–15] can provide a probabilistic aggregation process where uncertainty, ignorance and ambiguity are well coped with. In recent years, the ER approach has been applied in many domains such as fault diagnosis [4,16–18], life cycle assessment [19,20], belief rule based inference [21–25], urban bus transit network assessment [26], data classification [27], medical quality assessment [28], optimal power system dispatch [29], consumer preference prediction [30], sensor data fusion [31,32] and so on. But how to create a set of relatively rational and appropriate weights of attributes is still an open issue.

As we all know, there are three categories of weight assignment methods (WAMs) considering the information used: subjective, objective and hybrid [33–35]. Subjective WAMs such as direct rating method [36–38], weighted least square method [39], AHP [40,41] and Delphi method [42,43] are based on the preferences provided by decision makers (DMs) through questionnaire survey, discussion or brainstorming. When there is no sufficient information available or the DM is lack of knowledge and expertise, subjective methods may not be applied effectively. Moreover, in group decision making (GDM) [8,44–51], DMs

☆ No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.knosys.2019.105110.

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https://doi.org/10.1016/j.knosys.2019.105110
0950-7051/

with diverse background and preference may provide inconsistent attribute weights which need to be aggregated or reach a consensus. Objective WAMs generate attribute weights from the intrinsic information contained in the assessment values. Traditional representative methods include entropy method [7, 14,52–58], standard deviation (SD) [10,59,60], criteria importance through intercriteria correlation (CRITIC) [59–61] and maximizing deviation method [62,63]. In recent years, some new methods such as correlation coefficient and standard deviation integrated (CCSD) method [33], deviation and decision incompatibility based method [10] and combined discriminating power method [2] were proposed to coping with more complex situations. They are all based on one or both of the following two dimensions of information included in the decision matrix. One is the discrepancies of the values that different alternatives be assessed on a specific attribute, typical terms such as contrast intensity [59], incompatibility [10], discriminating power [2] or discrepancy [14] are all characterizations of this type of information. The other one reflects the conflict among the value vectors of different attributes, or in an opposite concept, denotes the correlation or interdependency between each pair of attributes. Hybrid WAMs [35,53,64] utilize both the preferences derived from DMs and the acquired assessment values when these two types of information are entirely or partially available.

Since different sets of attribute weights may generate different solutions to a MADM problem [2], how to calculate weight effectively is particularly important. In the life cycle assessment (LCA) of a complex product, the quantitative data obtained in the environmental and economic dimensions may contain uncertainties and ignorance due to various reasons; while the qualitative judgments acquired from investigation to assessing the social dimension also include ambiguities and incompleteness because of the subjectivity of DMs. Traditional objective or hybrid methods assume that the values been assessed to attributes are in the form of numerical values, e.g. accurate value [33,52,53,59,60,62] and fuzzy number [65–68]. But when the numerical values contain both uncertainties and incompleteness, how to elicit objective attribute weights is still an unanswered question. For example, incomplete interval value proposed in [26] allows ignorance and unreliability be contained in interval value assessments. The method to measure the discrepancy among interval values assessed to all alternatives respect to a specific attribute or the interdependency between each pair of attributes is pivotal for obtaining the weights. Meanwhile, the incompleteness should be reflected in the generated weights, e.g. a final reliability of attribute weights can be computed derived from the incompleteness. Two ways can be adopted to generate attribute weights in this situation. One way is to directly use one or both of the above mentioned discrepancy and interdependency measure for interval-valued assessments to calculate weights objectively. The other way is to employ optimization models to generate the uncertain range of attribute weights. In this paper, the second way is adopted for the purpose that the uncertainties and incompleteness contained in interval-valued assessments can be preserved in the generated attribute weights.

In the development of the ER approach, researches mainly focus on the issue that how to construct a reasonable aggregation rule compared with other evidence combination rules (ECRs) such as a set of D-S rules [69–72]. Although the attribute weights have been assumed to be various types, e.g., accurate value [9,15,19, 26,73–75], interval value [5,12,76] and triangular fuzzy number [1,13], the method of generating attribute weights either from subjective or objective method was not fully discussed in the early two decades since ER has been proposed. In recent years, some literatures [2,10,14] have paid attention on the WAMs for the ER based MADM problem when the information obtained is in the form of BDs. But some issues still need to be further discussed. Firstly, the ignorance/incompleteness contained in the BDs should be reflected from the generated weights, which will directly determine the rationality of the final aggregation results. Secondly, interval belief distribution (IBD) [11,76–79] may be a feasible representation in GDM problems because the judgments by DMs may be inconsistent due to their different backgrounds, expertise and preferences. Moreover, the belief degrees given by individual DM himself/herself may also be imprecise on account of some uncertainties, such as the lack of knowledge. In this paper, for the generation of objective weights, optimization models are constructed based on Shannon's information entropy [80] to tackling the situations that assessments are given in the form of accurate or interval belief degrees. It is a generalization of the entropy method for the assignment of attribute weights in that the above two situations have not been fully discussed in the previous studies.

The main contributions of the paper can be summarized as follows:

(1) The entropy weight assignment method (EWAM) is extended to coping with the situation where assessment values are presented in the form of interval values. The definition of normalized interval weight vector is then given.

(2) The entropy weight assignment method with belief distributions (EWAM with BDs) is studied. Optimization models are constructed to generate the intervals of attribute weights which are caused by the uncertainties and incompleteness contained in BDs.

(3) For the case where the decision matrix is represented by IBDs, EWAM based optimization models are constructed to compute the uncertain ranges of attribute weights. The concept of complete IBD matrix is also provided.

The remainder of this paper is organized as follows. Section 2 is a brief introduction about the ER based MADM framework. In Section 3, EWAM with uncertain numerical values is studied, the concept of normalized interval weight vector is then defined. In Section 4, EWAM with incomplete BDs and IBDs are proposed respectively. Some properties are shown and proved. Section 5 presents a case study to illustrate the methods proposed in Sections 3 and 4, and the comparisons are conducted with some existing methods. This paper is concluded in Section 6.

2. Preliminaries

The ER approach which was developed from D-S evidence theory [69,70,72] is one of MADM methods that can be applied in situations where uncertainties, ambiguities and incompleteness are included. The unique characteristic of the ER approach lies in that incompleteness or ignorance involved in the subjective judgments can be dealt with in a systematic and consistent way. Suppose the frame of discernment which contains N linguistic evaluation grades is given as follows:

$$H = \{H_1, H_2, \ldots, H_n, \ldots, H_N\} \tag{1}$$

In general, H_{n+1} is supposed to be preferred to H_n represented by $H_{n+1} \succ H_n (n = 1, 2, \ldots, N-1)$. The utility of H_n is denoted by $u(H_n)$ such that $0 \le u(H_n) < u(H_{n+1}) \le 1$. Let $\bar{\boldsymbol{e}} = \{e_1, e_2, \ldots, e_i, \ldots, e_L\}$ and $\bar{\boldsymbol{a}} = \{a_1, a_2, \ldots, a_l, \ldots, a_S\}$ be the set of basic attributes and selected alternatives respectively. The relative weight of $e_i (i = 1, 2, \ldots, L)$ is denoted by w_i such that $0 \le w_i \le 1$ and $\sum_{i=1}^{L} w_i = 1$. The state of e_i for alternative a_l to grade H_n can be described as the following expectation:

$$S(e_i(a_l)) = \left\{ \left(H_n, \beta_{n,i}(a_l)\right), n = 1, 2, \ldots, N; \left(H, \beta_{H,i}(a_l)\right) \right\} \tag{2}$$

Eq. (2) is called a basic BD, where $\beta_{n,i}(a_l)$ signifies the intensity to which the state of the ith attribute e_i at a_l is confirmed to H_n, or is termed belief degree. $\beta_{H,i}(a_l)$ denotes the degree of uncertainty that a_l is assessed on e_i, also called the degree of global ignorance/incompleteness. BD is assumed to satisfy the rationality assumption [9] in which $0 \leq \beta_{n,i}(a_l) \leq 1$ and $\sum_{n=1}^{N} \beta_{n,i}(a_l) \leq 1$ are commonly satisfied, and $\sum_{n=1}^{N} \beta_{n,i}(a_l) + \beta_{H,i}(a_l) = 1$. The ER approach uses orthogonal sum operation either recursively [15] or analytically [74] to aggregate the L distributions on a specific alternative and generate global BD on the frame of discernment as follows:

$$S(a_l) = \{(H_n, \beta_n(a_l)), n = 1, 2, \ldots, N; (H, \beta_H(a_l))\} \tag{3}$$

In Eq. (3), $\beta_n(a_l)$ denotes the global belief degree that a_l be assessed on H_n, and $\beta_H(a_l)$ represents the degree of belief unassigned to any individual evaluation grade after all the L attributes in $\bar{\boldsymbol{e}}$ have been considered. Different from the D-S rules where the weight of each evidence is assumed to be identical such that $w_i = 1 (i = 1, 2, \ldots, L)$, the ECR in the ER approach considers the difference of evidence weight which is necessary in MADM problems. Apart from the global BD to give a panoramic view about the assessment on a_l, the ER approach also employs utility function decision making (UFDM) approach to rank alternatives straightforwardly. The maximum and minimum utility of a_l can be generated as follows when $\beta_H(a_l)$ is allocated to $u(H_N)$ and $u(H_1)$ respectively.

$$u_l^{Max} = \sum_{n=1}^{N} \beta_n(a_l)\, u(H_n) + \beta_H(a_l)\, u(H_N) \tag{4}$$

$$u_l^{Min} = \sum_{n=1}^{N} \beta_n(a_l)\, u(H_n) + \beta_H(a_l)\, u(H_1) \tag{5}$$

Then a_l can be judged by the interval $\left[u_l^{Min}, u_l^{Max}\right]$ where $0 \leq u_l^{Min} \leq 1$ and $0 \leq u_l^{Max} \leq 1$. Recently, the ER approach has been developed from many aspects, e.g. ER rule with weight and reliability [14,19,81,82], ER with discrete BD [6], ER with contrary support [83] and so on.

3. EWAM with uncertain numerical values

Suppose S alternatives and L attributes are included in a MADM problem which can be represented by a decision matrix shown in Eq. (6). Each column signifies the assessment vector of a specific attribute e_i respect to all the S alternatives, represented by $S\left(e_i\left(\bar{\boldsymbol{a}}\right)\right) (i = 1, 2, \ldots, L)$; While each row indicates the performance vector that an alternative a_l be assessed to all L attributes, represented by $S\left(\bar{\boldsymbol{e}}(a_l)\right) (l = 1, 2, \ldots, S)$. By aggregating the L values in each performance vector $S\left(\bar{\boldsymbol{e}}(a_l)\right)$, the final performance of alternative a_l represented by $S(a_l)$ can be generated, and then the comparison of the S alternatives can be conducted. If $S\left(\bar{\boldsymbol{e}}(a_l)\right)$ is a numerical vector, simple additive weighting (SAW) method [52] can be used to generate $S(a_l)$ which is also a numerical value.

$$[S(e_i(a_l))]_{S\times L} = \begin{array}{c} \\ S(\bar{\boldsymbol{e}}(a_1)) \\ \cdots \\ S(\bar{\boldsymbol{e}}(a_l)) \\ \cdots \\ S(\bar{\boldsymbol{e}}(a_S)) \end{array} \begin{array}{c} \begin{array}{ccccc} S\left(e_1\left(\bar{\boldsymbol{a}}\right)\right) & \cdots & S\left(e_i\left(\bar{\boldsymbol{a}}\right)\right) & \cdots & S\left(e_L\left(\bar{\boldsymbol{a}}\right)\right) \end{array} \\ \begin{bmatrix} S(e_1(a_1)) & \cdots & S(e_i(a_1)) & \cdots & S(e_L(a_1)) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S(e_1(a_l)) & \cdots & S(e_i(a_l)) & \cdots & S(e_L(a_l)) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S(e_1(a_S)) & \cdots & S(e_i(a_S)) & \cdots & S(e_L(a_S)) \end{bmatrix}_{S\times L} \end{array} \tag{6}$$

3.1. EWAM with crisp values

Let x_{li} be the value of $S(e_i(a_l))$ provided that e_i is a quantitative attribute. The EWAM with crisp values [52,54,58] includes the following three steps:

(1) Standardization of original numerical values of performance:

Linear Proportional Transformation or Standard 0–1 Transformation can be applied in the standardization. For a benefit attribute, we have

$$y_{li} = \frac{x_{li}}{max_{1\leq k\leq S}\{x_{ki}\}} \tag{7}$$

or

$$y_{li} = \frac{x_{li} - min_{1\leq k\leq S}\{x_{ki}\}}{max_{1\leq k\leq S}\{x_{ki}\} - min_{1\leq k\leq S}\{x_{ki}\}} \tag{8}$$

While for a cost attribute, we have

$$y_{li} = \frac{min_{1\leq k\leq S}\{x_{ki}\}}{x_{li}} \tag{9}$$

or

$$y_{li} = \frac{max_{1\leq k\leq S}\{x_{ki}\} - x_{li}}{max_{1\leq k\leq S}\{x_{ki}\} - min_{1\leq k\leq S}\{x_{ki}\}} \tag{10}$$

$$(l = 1, 2, \ldots, S; i = 1, 2, \ldots, L)$$

(2) Normalization of the standardized value from $S(e_i(a_l))$:

$$p_{li} = \frac{y_{li}}{\sum_{k=1}^{S} y_{ki}} (l = 1, 2, \ldots, S; i = 1, 2, \ldots, L) \tag{11}$$

where $\sum_{l=1}^{S} p_{li} = 1$.

(3) Generating the weights of attributes

The entropy of e_i derived from $S\left(e_i\left(\bar{\boldsymbol{a}}\right)\right)$ is calculated by

$$E_i = -\frac{1}{ln(S)} \sum_{l=1}^{S} p_{li} \cdot ln(p_{li}) (i = 1, 2, \ldots, L) \tag{12}$$

where the denominator $\frac{1}{ln(S)}$ is to limit E_i within $[0, 1]$. As is known to all, the higher entropy be assigned to an attribute, the more difficult to discriminate different alternatives on the attribute, so the weight of e_i is calculated by

$$w_i = \frac{1 - E_i}{L - \sum_{k=1}^{L} E_k} (i = 1, 2, \ldots, L) \tag{13}$$

where $\sum_{i=1}^{L} w_i = 1$.

The original EWAM assumes that the assessments to all L attributes are given by precise numerical values. When the situation is uncertain and some quantitative attributes are provided in the form of interval values, or the assessments to qualitative attributes are given by BDs denoted by Eq. (2), how could the entropy method be employed to generate reasonable weights is to be analyzed.

3.2. EWAM with interval numerical assessment

Interval theory is one of the most common forms of uncertainty modeling [76]. A lot of studies have been done on MADM based on interval theory, which can be roughly split into three aspects, interval numerical assessment [26,84,85], IBD [11,76–79] and interval weight or reliability [5,12,19,76]. Two processes can be applied to define the similarity measure such as the information entropy when quantitative attributes are represented by interval numerical assessments. The first one is to directly define the entropy measure of interval values assessed to different alternatives on each attribute, followed by the generation of attribute weights. The second one is to make interval values as constraints to conduct optimization models for the generation of entropy measure and attribute weights. Suppose x_{li} is given by interval value as follows:

$$S(e_i(a_l)) = x_{li} \in \left[x_{li}^-, x_{li}^+\right] \tag{14}$$

The minimum and maximum possible value assessed to e_i respect to all S alternatives is denoted by $min_{1\le k\le S}\{x_{ki}^-\}$ and $max_{1\le k\le S}\{x_{ki}^+\}$ respectively. The entropy of the interval assessment vector on e_i derived from $S\left(e_i\left(\overline{\boldsymbol{a}}\right)\right)$ can be calculated by the following pair of optimization models:

$$\langle Model\,1\rangle \text{ Min/Max } E_i = -\frac{1}{ln\,(S)}\sum_{l=1}^{S} p_{li}\cdot ln(p_{li})\ (i = 1, 2, \ldots, L)$$
$$\text{s.t. Eqs. (7) or (8), (9) or (10), (11)}$$
$$x_{li}^- \le x_{li} \le x_{li}^+ i = 1, 2, \ldots, L; l = 1, 2, \ldots, S$$
$$\sum_{l=1}^{S} p_{li} = 1$$

where $max_{1\le k\le S}\{x_{ki}\}$ and $min_{1\le k\le S}\{x_{ki}\}$ in Eqs. (7)–(10) should be replaced by $max_{1\le k\le S}\{x_{ki}^+\}$ and $min_{1\le k\le S}\{x_{ki}^-\}$ respectively. S variables are included in ⟨Model 1⟩ for generating the range of E_i. Let E_i^- and E_i^+ be the optimal values of ⟨Model 1⟩ respectively. Then, we have $E_i \in [E_i^-, E_i^+]$.

Property 1. *The uncertainty of the generated interval entropy in* ⟨Model 1⟩ *changes continuously with the change of* $\left(x_{li}^+ - x_{li}^-\right)$.

To generate the range of weights from decision matrix in Eq. (6) which is comprised of interval numerical values, the following pair of optimization models is designed.

$$\langle Model\,2\rangle \text{ Min/Max } w_i = \frac{1 - E_i}{L - \sum_{k=1}^{L} E_k}(i = 1, 2, \ldots, L)$$
$$\text{s.t. Eqs. (7) or (8), (9) or (10), (11), (12)}$$
$$x_{li}^- \le x_{li} \le x_{li}^+\ i = 1, 2, \ldots, L; l = 1, 2, \ldots, S$$
$$\sum_{l=1}^{S} p_{li} = 1$$

Let $W = (w_1, w_2, \ldots, w_L)^T$ be the weight vector of L attributes, if $w_i \in \left[w_i^-, w_i^+\right]$ where $0 \le w_i^- \le w_i^+ \le 1$ for $i = 1, 2, \ldots, L$ and $\sum_{i=1}^{L} w_i = 1$, W is said to be an interval weight vector. From ⟨Model 2⟩, the minimum and maximum weight of e_i under interval assessment values can be generated as w_i^- and w_i^+ respectively. In Eq. (14), if $\forall i = 1, 2, \ldots, L$ and $l = 1, 2, \ldots, S$, $x_{li}^- = x_{li}^+$, Eq. (6) becomes a matrix only composed of precise numerical values. Then the weights generated by ⟨Model 2⟩ will satisfy $w_i^- = w_i^+$.

Remark 1. If $\sum_{i=1}^{L} w_i^- \le 1$ and $\sum_{i=1}^{L} w_i^+ \ge 1$, then the interval weights are said to be valid. Invalid interval weights cannot be applied in a MADM problem.

Definition 1. Suppose $W = (w_1, w_2, \ldots, w_L)$ is the weight vector of L attributes with $w_i \in \left[w_i^-, w_i^+\right]$ for $i = 1, 2, \ldots, L$. W is said to be a normalized interval weight vector if the following equations are satisfied:

$$\sum_{i=1}^{L} w_i^- + \left(w_i^+ - w_i^-\right) \le 1 \tag{15}$$

$$\sum_{i=1}^{L} w_i^+ - \left(w_i^+ - w_i^-\right) \ge 1 \tag{16}$$

$$\forall i \in \{1, 2, \ldots, L\}$$

Eqs. (15) and (16) can also be represented by Eqs. (17) and (18) as follows:

$$\sum_{i=1}^{L} w_i^- + \max_{1\le i\le L}\left(w_i^+ - w_i^-\right) \le 1 \tag{17}$$

$$\sum_{i=1}^{L} w_i^+ - \max_{1\le i\le L}\left(w_i^+ - w_i^-\right) \ge 1 \tag{18}$$

Remark 2. If the uncertainty of x_{li} in Eq. (14) measured by $\left(x_{li}^+ - x_{li}^-\right)$ for some $S(e_i(a_l))$ in the decision matrix of Eq. (6) are large to some extent, the interval weights generated by ⟨Model 2⟩ may not be normalized although they are valid. Thus the reliability and effectiveness of the final assessment result will be influenced.

Nevertheless, if $\left(w_i^+ - w_i^-\right)$ for some attributes are large enough although the generated weight vector is normalized, the interval weights are still not appropriate to be used for the final decision-making.

4. EWAM with uncertain or interval BDs of assessment

The main difference between BD and probability assignment lies in that the probability can only be employed by individual elements in the frame of discernment, while BD is defined on the power set of the frame of discernment which is all the subsets of H in Eq. (1). There is a special case that the belief degree be given to the set of all N grades in Eq. (1), which refers to the degree of complete/global ignorance. In the ER approach, the BD is defined on all the N individual elements and the set of N grades. Just as Yang proved in [15], the overall BD by the ER approach is incomplete provided that the BD of at least one attribute contains ignorance no matter how the attribute weights are assigned. The attribute weights derived from BDs should also reflect the feature of ignorance contained in the BDs on basic attributes. Moreover, when qualitative attributes are given by IBDs, optimization models should be designed to generate the weights for the purpose of preserving the features of uncertainties included in IBDs.

4.1. The 0–1 Ave-Entropy method with BDs

For a qualitative attribute e_i, $S(e_i(a_l))$ can be given in the form of BD, fuzzy number or other types of representation. When $S\left(\overline{\boldsymbol{e}}(a_l)\right)$ is composed of L BDs, the ER algorithm which is a nonlinear aggregation method can be applied. The weight of attributes is an important factor in MADM because different weights may produce different evaluation results. The Ave-Entropy method [14] is applicable when $S(e_i(a_l))$ in Eq. (6) are all represented by BDs or have been transformed to BDs from numerical values. Three steps are included in the EWAM with BDs as follows:

(1) Transformation of $S(e_i(a_l))$ to utility as follows:

$$u_{li}^{Max} = \sum_{n=1}^{N} \beta_{n,i}(a_l)\, u(H_n) + \beta_{H,i}(a_l)\, u(H_N) \tag{19}$$

$$u_{li}^{Min} = \sum_{n=1}^{N} \beta_{n,i}(a_l)\, u(H_n) + \beta_{H,i}(a_l)\, u(H_1) \tag{20}$$

$$u_{li}^{Ave} = \left(u_{li}^{Max} + u_{li}^{Min}\right)/2 \tag{21}$$

(2) Normalizing u_{li}^{Ave} by $\tilde{u}_{li}^{Ave} = \frac{u_{li}^{Ave}}{\sum_{k=1}^{S} u_{ki}^{Ave}}$ such that $\sum_{l=1}^{S} \tilde{u}_{li}^{Ave} = 1$. The Standard 0–1 Transformation can be used before

Table 1
BDs of attributes on alternatives.

e_1	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0
a_2	0	1	0	0	0
e_2	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0
a_2	0	0	0	0	1

Table 2
BDs of attributes on alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	
a_1	1	0	0	0	0	
a_2	0	1	0	0	0	
e_2	H_1	H_2	H_3	H_4	H_5	H
a_1	0.8	0	0	0	0	0.2
a_2	0	0	0	0	0.8	0.2

the normalization to narrow the discrepancy among the performances of alternatives such that $\tilde{u}_{li}^{Ave} = \frac{\bar{u}_{li}^{Ave}}{\sum_{k=1}^{S}\bar{u}_{ki}^{Ave}}$, $\bar{u}_{li}^{Ave} = \frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}$. Then

$$\begin{aligned}\tilde{u}_{li}^{Ave} &= \frac{\frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}}{\sum_{t=1}^{S}\frac{u_{ti}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}} \\ &= \frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{\sum_{t=1}^{S}\left(u_{ti}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}\right)} \\ &= \frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{\sum_{t=1}^{S}u_{ti}^{Ave} - S\cdot min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}} \\ &= \frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{\sum_{k=1}^{S}u_{ki}^{Ave} - S\cdot min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}\end{aligned} \tag{22}$$

(3) The entropy of e_i is computed by $E_i = -\frac{1}{ln(S)}\sum_{l=1}^{S}\tilde{u}_{li}^{Ave}\cdot ln(\tilde{u}_{li}^{Ave})$, which is then used to generate the weight of e_i by Eq. (13).

If the Standard 0–1 Transformation is used before the normalization process, we call it 0–1 Ave-Entropy method; otherwise, it is called Ave-Entropy method.

Property 2. *If Linear Proportional Transformation is used to replace the Standard 0–1 Transformation in the above EWAM with BDs, the generated weights are identical with Ave-Entropy method.*

The proof of Property 2 is shown in Appendix. When entropy [14,52–57], CRITIC method [59–61] or CCSD [2,33] is used to determine attribute weights, more than two alternatives are required if the Standard 0–1 Transformation is applied for the normalization of quantitative attributes. If only two alternatives are involved in the assessment, the smaller and bigger assessment value of the two alternatives on each attribute will be 0 and 1 after the Standard 0–1 Transformation. When an assessment is only comprised of qualitative attributes, these three methods can be used to generate weights from average utility if the Standard 0–1 Transformation is not applied even though there are only two alternatives involved. However, when the Standard 0–1 Transformation is not applied, some unreasonable results may be generated in some circumstances.

Example 1. Given a frame of discernment $H = \{H_1, H_2, \ldots, H_5\}$. Table 1 shows the BDs of two attributes e_1 and e_2 on two alternatives a_1 and a_2. It is clear that all the four distributions are absolutely certain and complete assessments.

When the objective method is used, e_2 should probably be assigned with a higher weight than e_1 since the dissimilarity between a_1 and a_2 on e_2 is larger than e_1. Suppose that the utilities of the five evaluation grades are set to be risk averse such that $u(H_1) = 0$, $u(H_2) = 0.45$, $u(H_3) = 0.75$, $u(H_4) = 0.9$, $u(H_5) = 1$, the weights of the two attributes generated by the Ave-Entropy method are $w_1 = 0.5$, $w_2 = 0.5$ respectively even though the Standard 0–1 Transformation is not applied. When the utilities of the five evaluation grades are set to be risk taking, the weights of e_1 and e_2 are also 0.5 and 0.5.

Example 2. Similar with Example 1, two attributes are assessed on two alternatives in the form of BDs which are shown in Table 2. Different from Table 1, the distributions of e_2 on a_1 and a_2 here are incomplete.

When the Ave-Entropy method is used, the weights of the two attributes are $w_1 = 0.653$, $w_2 = 0.347$ respectively although the dissimilarity between a_1 and a_2 on e_2 is larger than e_1.

Property 3. *The weights generated by the 0–1 Ave-Entropy method are continuous respect to the belief degrees of assessments.*

The proof of Property 3 is shown in Appendix. It indicates that the weights generated by the 0–1 Ave-Entropy method will not change much provided that the belief degrees vary slightly.

4.2. The 0–1 Ave-Entropy based optimization model for assigning attribute weights

For the purpose of considering the ignorance contained in the BDs, the utility of e_i on a_l represented by u_{li} is assumed to be in the range of $[u_{li}^{Min}, u_{li}^{Max}]$. So the following pair of optimization models is constructed to generate the minimum and maximum value of w_i.

$$\langle Model\,3\rangle \text{ Min/Max } w_i = \frac{1-E_i}{L-\sum_{k=1}^{L}E_k}(i = 1, 2, \ldots, L)$$
$$\text{s.t. } E_i = -\frac{1}{ln(S)}\sum_{l=1}^{S}\tilde{u}_{li}\cdot ln(\tilde{u}_{li})$$
$$\tilde{u}_{li} = \frac{u_{li} - min_{1\le k\le S}\{u_{ki}\}}{\sum_{k=1}^{S}u_{ki} - S\cdot min_{1\le k\le S}\{u_{ki}\}}$$
$$u_{li}^{Min} \le u_{li} \le u_{li}^{Max} \quad i = 1, 2, \ldots, L; l = 1, 2, \ldots, S$$
Eqs. (19), (20)

$S\times L$ variables are contained in ⟨Model 3⟩ which can be solved by Matlab or Excel, and $2\times L$ times of calculation need to be conducted to generate the minimum and maximum values of the weights on all L attributes. ⟨Model 3⟩ is based on the 0–1 Ave-Entropy method. If Ave-Entropy method is used in ⟨Model 3⟩, $\tilde{u}_{li} = \frac{u_{li} - min_{1\le k\le S}\{u_{ki}\}}{\sum_{k=1}^{S}u_{ki} - S\cdot min_{1\le k\le S}\{u_{ki}\}}$ is to be replaced by $\tilde{u}_{li} = \frac{u_{li}}{\sum_{k=1}^{S}u_{ki}}$ in the optimization model. A special case is that $u_{li}^{Min} = u_{li}^{Max}$ for all the BDs which will lead to the weights generated by the model be crisp values. This occurs when all the subjective judgments provided by DM are complete although uncertainties are contained in the BDs. Next, we take a numerical example in [14] to illustrate the above proposed optimization model.

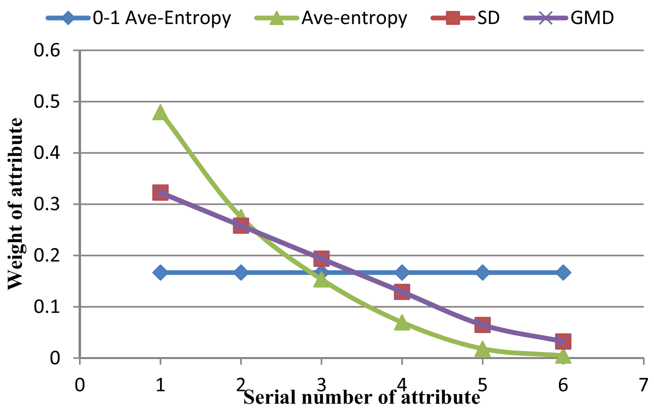

Fig. 1. Attribute weights generated by 0–1 Ave-Entropy, Ave-Entropy, SD and GMD method.

Example 3. Given a frame of discernment $H = \{H_1, H_2, \ldots, H_5\}$. The BDs that 5 alternatives been assessed on 6 attributes are shown in Table 3. From Table 3, we can see that for a specific attribute $e_i(i = 1, 2, \ldots, 6)$, $\beta_{n,i}(a_l) \neq 0(n = l \leq 5)$ and $\beta_{n,i}(a_l) = 0(n \neq l)$, and the ignorance contained in the assessment increases from e_1 to e_6.

According to Eqs. (19)–(21), the average utilities of the 5 alternatives on 6 attributes can be calculated. Fig. 1 shows the generated weights by 0–1 Ave-Entropy, Ave-Entropy, SD method and Gini's mean difference (GMD) method [2,43]. The horizontal and vertical axes represent the serial number and weight of attribute respectively. Taking e_1 and e_2 for example, the average utility vectors of these two attributes on the 5 alternatives are $(0, 0.45, 0.75, 0.9, 1)^T$ and $(0.1, 0.46, 0.7, 0.82, 0.9)^T$ respectively provided that the utilities of the five evaluation grades are set to be equal with Example 1. So the Spearman correlation coefficient between the two vectors is 1. The correlation coefficient between any other two average utility vectors is also 1. Thus the CRITIC method cannot be used in this situation. CCSD method is also not applicable since the correlation coefficient between the utility vector on e_i and the overall utility vector without the consideration of e_i is 1 for $i = 1, 2, \ldots, 6$. The weights generated by 0–1 Ave-Entropy are equal for all the six attributes that we call it the best well-distributed weights. The results need to be discussed because the dissimilarities of BDs on the five alternatives for each of the six attributes are really different. It is equal to the result that the ignorance in each BD is added to the nonzero belief degree. In this situation, all the 6 attributes on each alternative are given the same BD. Comparatively, the weights generated by Ave-Entropy and SD method are easy for us to differentiate the set of attributes which have a dominating role. It is an interesting thing that the weights generated by SD and GMD are the same in this example. Nevertheless, the weights generated by the four methods are crisp values that the incompleteness contained in the BDs cannot be reflected. Taking e_6 for example, the ignorance contained in each of the five alternatives is 0.9 which is the largest incompleteness compared with e_1 to e_5. So it is more rational that the generated weight of $e_i(i = 2, 3, 4, 5, 6)$ is formulated as an uncertain value because ignorance are contained in the BDs respect to these five attributes.

To reflect the ignorance contained in the BDs from the generated weights, $\langle$Model 3$\rangle$ is applied. The utilities of the five evaluation grades are set to be risk averse such that $u(H_1) = 0$, $u(H_2) = 0.45$, $u(H_3) = 0.75$, $u(H_4) = 0.9$ and $u(H_5) = 1$. The weights generated by $\langle$Model 3$\rangle$ based on 0–1 Ave-Entropy are shown in Fig. 2. They are limited to the interval of $\left[w_i^-, w_i^+\right](i = 1, 2, \ldots, 6)$ and satisfies $\sum_{i=1}^{6} w_i = 1$. It can be seen that with the ignorance contained in the BDs on alternatives increased from e_1 to e_6, the difference between w_i^- and w_i^+ becomes larger. If the ignorance contained in one or some of the BDs are too large to some extent, the weights generated by $\langle$Model 3$\rangle$ may be non-normalized. Thus it is necessary to figure out the reason of large ignorance contained in the BD, and decrease the ignorance if possible for a more reliable aggregation result.

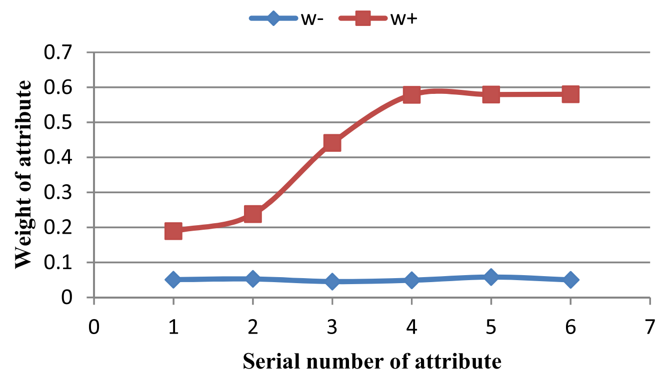

Fig. 2. Interval weights generated by $\langle$Model 3$\rangle$ subject to the 0–1 Ave-Entropy.

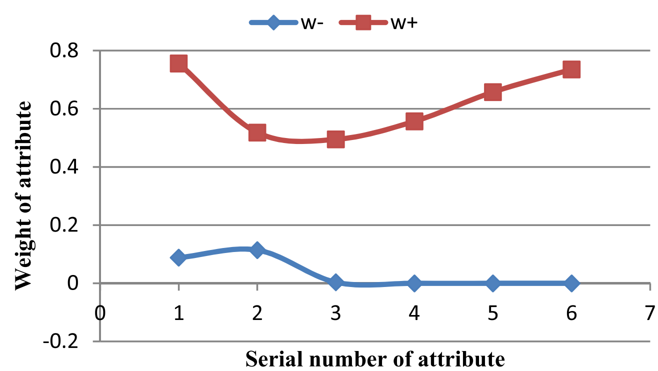

Fig. 3. Interval weights generated by $\langle$Model 3$\rangle$ subject to the Ave-Entropy.

We also generate the interval weights by $\langle$Model 3$\rangle$ based on Ave-Entropy method that are shown in Fig. 3. Although the weights in Fig. 3 are normalized, the difference between the minimum and maximum weight on any one of the six attributes is too large that would make the DM too confusing. From this point of view, the Standard 0–1 Transformation narrows the differences of weights among all the attributes. Meanwhile, it also reduces the uncertainty of interval weights provided that incompleteness are contained in BDs.

4.3. Consideration of interval BDs

$\langle$Model 3$\rangle$ proposed in Section 4.2 assumes that the BDs provided by the DM are comprised of accurate belief degrees. Just as Wang discussed in [11], acquiring precise belief degrees is not easy in some circumstances where IBD is more appropriate to represent DM's subjective judgments. Here, the EWAM with IBDs is proposed.

Suppose the belief degree that a_l be assessed to e_i on evaluation grade H_n is included in an interval such that $\beta_{n,i}(a_l) \in \left[\beta_{n,i}^-(a_l), \beta_{n,i}^+(a_l)\right](n = 1, 2, \ldots, N)$, $\beta_{H,i}(a_l) \in \left[\beta_{H,i}^-(a_l), \beta_{H,i}^+(a_l)\right]$ and $\sum_{n=1}^{N} \beta_{n,i}(a_l) + \beta_{H,i}(a_l) = 1$ [11]. Then Eq. (2) becomes an IBD, which leads Eq. (6) to be an IBD matrix. If $\forall n \in \{1, 2, \ldots, N\}$, $\beta_{n,i}^-(a_l) = \beta_{n,i}^+(a_l)$, the IBD becomes an accurate belief distribution.

Table 3
Belief degrees that 5 alternatives be assessed on 6 attributes.

$\beta_{n,i}(a_l)$	a_1	a_2	a_3	a_4	a_5
e_1	$\{(H_1, 1)\}$	$\{(H_2, 1)\}$	$\{(H_3, 1)\}$	$\{(H_4, 1)\}$	$\{(H_5, 1)\}$
e_2	$\{(H_1, 0.8),(H, 0.2)\}$	$\{(H_2, 0.8),(H, 0.2)\}$	$\{(H_3, 0.8),(H, 0.2)\}$	$\{(H_4, 0.8),(H, 0.2)\}$	$\{(H_5, 0.8),(H, 0.2)\}$
e_3	$\{(H_1, 0.6),(H, 0.4)\}$	$\{(H_2, 0.6),(H, 0.4)\}$	$\{(H_3, 0.6),(H, 0.4)\}$	$\{(H_4, 0.6),(H, 0.4)\}$	$\{(H_5, 0.6),(H, 0.4)\}$
e_4	$\{(H_1, 0.4),(H, 0.6)\}$	$\{(H_2, 0.4),(H, 0.6)\}$	$\{(H_3, 0.4),(H, 0.6)\}$	$\{(H_4, 0.4),(H, 0.6)\}$	$\{(H_5, 0.4),(H, 0.6)\}$
e_5	$\{(H_1, 0.2),(H, 0.8)\}$	$\{(H_2, 0.2),(H, 0.8)\}$	$\{(H_3, 0.2),(H, 0.8)\}$	$\{(H_4, 0.2),(H, 0.8)\}$	$\{(H_5, 0.2),(H, 0.8)\}$
e_6	$\{(H_1, 0.1),(H, 0.9)\}$	$\{(H_2, 0.1),(H, 0.9)\}$	$\{(H_3, 0.1),(H, 0.9)\}$	$\{(H_4, 0.1),(H, 0.9)\}$	$\{(H_5, 0.1),(H, 0.9)\}$

Remark 3 ([5]). Suppose an IBD is denoted as follows:

$$S(e_i(a_l)) = \left\{\left(H_n, [\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)]\right), \quad n = 1, 2, \ldots, N; \left(H, [\beta^-_{H,i}(a_l), \beta^+_{H,i}(a_l)]\right)\right\} \tag{23}$$

where $\beta^-_{H,i}(a_l) = \mathrm{Max}(0, 1 - \sum_{n=1}^{N} \beta^+_{n,i}(a_l))$, $\beta^+_{H,i}(a_l) = 1 - \sum_{n=1}^{N} \beta^-_{n,i}(a_l)$. $S(e_i(a_l))$ is said to be logical if $\sum_{n=1}^{N} \beta^-_{n,i}(a_l) \le 1$. Otherwise, it is illogical.

Illogical IBD cannot be used in the ER algorithm or to generate attribute weights by objective methods. In this case, the subjective judgment from DM should be corrected.

Remark 4 ([11]). Suppose an IBD denoted by Eq. (23) is logical. If we have $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1$ where $\beta_{n,i}(a_l) \in \left[\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)\right]$ $(n = 1, 2, \ldots, N)$, then the IBD is said to be complete. In this situation, $\beta^-_{H,i}(a_l) = \beta^+_{H,i}(a_l) = 0$. Otherwise, it is said to be an incomplete IBD or contain ignorance such that $\sum_{n=1}^{N} \beta_{n,i}(a_l) < 1$ and $\beta^+_{H,i}(a_l) \ge \beta^-_{H,i}(a_l) > 0$.

There may be three situations considering the incompleteness of IBD. (a) If $\sum_{n=1}^{N} \beta^+_{n,i}(a_l) < 1$, $\sum_{n=1}^{N} \beta^-_{n,i}(a_l) < 1$ will always be satisfied, so we have $\beta^-_{H,i}(a_l) > 0$, then $S(e_i(a_l))$ is said to be absolutely incomplete. (b) When $\sum_{n=1}^{N} \beta^+_{n,i}(a_l) \ge 1$ and $\sum_{n=1}^{N} \beta^-_{n,i}(a_l) < 1$, we have $\beta^-_{H,i}(a_l) = 0$ and $\beta^+_{H,i}(a_l) > 0$, then the IBD is either complete or incomplete. (c) When $\sum_{n=1}^{N} \beta^+_{n,i}(a_l) \ge 1$ and $\sum_{n=1}^{N} \beta^-_{n,i}(a_l) = 1$, we have $\beta^-_{H,i}(a_l) = 0$ and $\beta^+_{H,i}(a_l) = 0$, then it is an absolutely complete IBD. In essence, situation c is corresponding to an accurate BD because $\beta_{n,i}(a_l) = \beta^-_{n,i}(a_l)$. From situation (a) to (c), the incompleteness of IBD decreases. It should be mentioned that the situation that both $\sum_{n=1}^{N} \beta^+_{n,i}(a_l) > 1$ and $\sum_{n=1}^{N} \beta^-_{n,i}(a_l) > 1$ be satisfied will not happen because it is illogical.

Definition 2. Given an IBD matrix denoted by Eqs. (6) and (23), if for alternative a_l, $\exists i \in \{1, 2, \ldots, L\}$, $\sum_{n=1}^{N} \beta_{n,i}(a_l) < 1$ where $\beta_{n,i}(a_l) \in \left[\beta^-_{n,i}(a_l), \beta^+_{n,i}(a_l)\right] (n = 1, 2, \ldots, N)$, then the assessment on a_l represented by $S(\overline{\boldsymbol{e}}(a_l))$ is said to be an incomplete IBD vector; Otherwise, the IBD vector on a_l is said to be complete such that $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1 (\forall i \in \{1, 2, \ldots, L\})$.

Definition 3. If $\forall i \in \{1, 2, \ldots, L\}$ and $\forall l \in \{1, 2, \ldots, S\}$, $S(e_i(a_l))$ in Eq. (6) is complete no matter e_i is a quantitative or qualitative attribute, then Eq. (6) is said to be a complete decision matrix; Otherwise, it is an incomplete decision matrix. A special case is that all the L attributes are qualitative attributes represented by IBDs, then Eq. (6) is said to be a complete IBD matrix if all the IBDs are complete.

The definition of incomplete numerical assessment either in the form of accurate or interval value can be referred to [26]. When all the S IBD vectors $(S(\overline{\boldsymbol{e}}(a_l))\, l = 1, 2, \ldots, S)$ are complete, the IBD matrix is complete. According to [15], if the BD of an attribute on a_l is incomplete, the aggregated BD on a_l from all L attributes will also be incomplete. It is also applicable to the IBDs.

Table 4
IBDs assessed on two alternatives.

e_i		H_1	H_2	H_3
a_1	$\beta^-_{n,i}(a_1)$	0.9	0	0
	$\beta^+_{n,i}(a_l)$	1	0.1	0
a_2	$\beta^-_{n,i}(a_2)$	0	0	0.9
	$\beta^+_{n,i}(a_l)$	0	0.1	1

When the subjective judgments are presented in the form of IBDs, the entropy of e_i can be computed by ⟨Model 4⟩ as follows:

$$\langle Model\,4\rangle \ \mathrm{Min/Max}\, E_i = -\frac{1}{ln(S)} \sum_{l=1}^{S} \tilde{u}^{Ave}_{li} \cdot ln(\tilde{u}^{Ave}_{li})$$

$$\text{s.t. } \tilde{u}^{Ave}_{li} = \frac{u^{Ave}_{li}}{\sum_{k=1}^{S} u^{Ave}_{ki}}$$

$$\beta^-_{n,i}(a_l) \le \beta_{n,i}(a_l) \le \beta^+_{n,i}(a_l) \quad (n = 1, 2, \ldots, N; l = 1, 2, \ldots, S)$$

$$\sum_{n=1}^{N} \beta_{n,i}(a_l) \le 1$$

Eqs. (19)–(21)

Here, another example is presented to illustrate the entropy measure of IBDs by ⟨Model 4⟩.

Example 4. Given a frame of discernment $H = \{H_1, H_2, H_3\}$, and the utilities of the three evaluation grades are set to be $u(H_1) = 0.1$, $u(H_2) = 0.6$, $u(H_3) = 0.9$ respectively. The IBDs of e_i on alternative $a_l (l = 1, 2)$ are shown in Table 4.

By applying ⟨Model 4⟩, the entropy measure of the IBDs on $a_l (l = 1, 2)$ respect to e_i is between 0.4690 to 0.6061. E_i reaches the minimum when $S(e_i(a_1)) = \{(H_1, 1), (H_2, 0), (H_3, 0)\}$ and $S(e_i(a_2)) = \{(H_1, 0), (H_2, 0), (H_3, 1)\}$ which corresponding to the situation that the assessments on the two alternatives are highly inconsistent. When $S(e_i(a_1)) = \{(H_1, 0.9), (H_2, 0.1), (H_3, 0)\}$ and $S(e_i(a_2)) = \{(H_1, 0), (H_2, 0), (H_3, 0.9); (H, 0.1)\}$, E_i reaches the maximum. Besides, the utilities of evaluation grades also influence the entropy of e_i to a certain degree.

The following pair of optimization models can then be constructed to generate the interval weight of e_i.

$$\langle Model\,5\rangle \ \mathrm{Min/Max}\, w_i = \frac{1 - E_i}{L - \sum_{k=1}^{L} E_k} (i = 1, 2, \ldots, L)$$

$$\text{s.t. } E_i = -\frac{1}{ln(S)} \sum_{l=1}^{S} \tilde{u}^{Ave}_{li} \cdot ln(\tilde{u}^{Ave}_{li})$$

$$\tilde{u}^{Ave}_{li} = \frac{u^{Ave}_{li}}{\sum_{k=1}^{S} u^{Ave}_{ki}}$$

$$\beta^-_{n,i}(a_l) \le \beta_{n,i}(a_l) \le \beta^+_{n,i}(a_l) \quad (n = 1, 2, \ldots, N; l = 1, 2, \ldots, S)$$

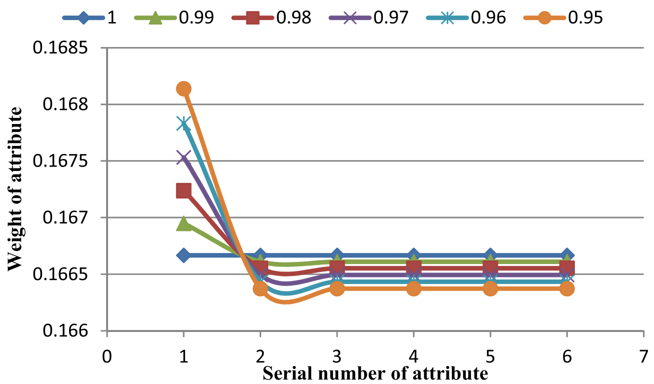

Fig. 4. Changes of attribute weights generated by 0–1 Ave-Entropy with respect to $\beta_{1,1}(a_1)$.

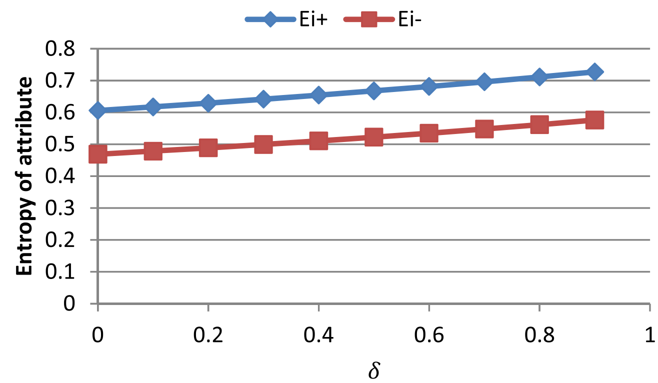

Fig. 5. Changes of interval entropy generated by ⟨Model 4⟩ with respect to δ.

$$\sum_{n=1}^{N}\beta_{n,i}(a_l)\leq 1$$

Eqs. (19)–(21)

In ⟨Model 5⟩, $N \times S \times L$ variables are included, and there are $2 \times L$ times of calculation needed to generate the minimum and maximum values of all L attribute weights.

Additionally, in order to capture the subjective judgments of DMs, preference relations on attribute weights can be incorporated into models 2, 3 and 5. For instance, $w_i \leq w_j (i = 1, 2, \ldots, L; i \neq j)$, $w_i \geq a$, $a \leq \frac{w_i}{w_j} \leq b$ are representative forms of subjective preference on attribute weights. If preference relations are not included in the optimization models, and the generated weights are inconsistent with DM's subjective preferences, the weights should be adjusted.

4.4. Sensitivity analysis

In order to illustrate Property 3, sensitivity analysis is conducted on Example 3 to measure the impact of the change of belief degrees on weights. Here, $\beta_{1,1}(a_1)$ is set to be changed from 1 to 0.95, while other BDs are fixed. Fig. 4 shows the result when the 0–1 Ave-Entropy method is applied. It is clear that the weight of each attribute changes very little when the step of variation for $\beta_{1,1}(a_1)$ is set to be 0.01.

Sensitivity analysis on Example 4 is also conducted to measure the changes of IBDs on the generated interval entropy by ⟨Model 4⟩. The IBD of e_i on a_2 is set to be $S(e_i(a_2)) = \{(H_1, [0, 0]), (H_2, [0 + \delta, 0.1 + \delta]), (H_3, [0.9 - \delta, 1 - \delta])\}$ where $0 \leq \delta \leq 0.9$, while the IBD of e_i on a_1 is fixed. Fig. 5 shows the values of interval entropy generated by ⟨Model 4⟩ when δ increases from 0 to 0.9. It can be seen that the assessments to e_i between a_1 and a_2 become more consistent when δ increases, which leads to the rise of E_i^- and E_i^+ simultaneously.

5. Case study

In this section, a case study is conducted to illustrate the validity and rationality of the proposed approach mentioned in the above section. Several different methods are compared with the proposed method to give a panoramic view of the features of the given models.

5.1. Generating weights from the proposed models

A car selection problem adapted from [9] and [10] is chosen for the illustration. Four qualitative attributes and three quantitative attributes are included in the assessment where six cars are to be compared. The weights in [9] are supposed to be equal such that $w_i = 1/7 (i = 1, 2, \ldots, L)$. The frame of discernment in this case consists of six evaluation grades such as Worst(W), Poor(P), Average(A), Good(G), Excellent(E) and Top(T). The original numerical values assessed to quantitative attributes and grades on qualitative attributes have all been transformed to BDs which are shown in Table 5. For the generation of the overall performance on each car, the ER algorithm is to be used to combine the BDs in Table 5.

From Table 5, we can see that some of the BDs are incomplete. For example, the assessment of Car 5 on e_1 is represented by $S(e_1(a_5)) = \{(G, 0.4), (E, 0.4); (\Omega, 0.2)\}$. It means that Car 5 on 'Acceleration' is assessed to be good and excellent with a belief degree of 0.4, and the degree of ignorance contained in the assessment is 0.2. It will lead to the weights generated by ⟨Model 3⟩ be uncertain, and the overall performance on Car 5 will also be incomplete. Intuitively, the uncertainties on the weight of e_5 will be the largest when ⟨Model 3⟩ is applied because both Car 3 and Car 6 are assessed to be completely ignorant on e_5.

The utilities of the six evaluation grades are set to be risk neutral such that $u(W) = 0$, $u(P) = 0.2$, $u(A) = 0.4$, $u(G) = 0.6$, $u(E) = 0.8$, $u(T) = 1$. It should be mentioned that in a multi-attribute group decision making (MAGDM) problem, different risk preferences of DMs may lead to the discrepancy of utility estimations on evaluation grades due to their diverse background and expertise. Moreover, the risk preference of a DM may change at different decision points due to the changes of external environment [1]. From this point of view, different utility estimation may influence the generated weights and aggregated BD on each alternative. But this is not the focus of our discussion in this paper, and it can be related to [1,13,14]. Fig. 6 shows the weights generated by different methods, i.e. 0–1 Ave-Entropy, Ave-Entropy, SD, CRITIC, CCSD and GMD. The standard deviations of the generated weights by the six methods are shown in Table 6. Obviously, the weights of the seven attributes obtained by Ave-Entropy method have the greatest difference, while the weights generated by the 0–1 Ave-Entropy method have the smallest difference.

Since the BDs on some of the attributes are incomplete, the accurate weights shown in Fig. 6 cannot reflect the intrinsic features of the original information contained in BDs. So ⟨Model 3⟩ is applied to generate interval weights which are shown in Figs. 7 and 8. The dotted lines denote the weights that are not obtained using optimization model. The difference between these two figures lies in that whether the Standard 0–1 Transformation is used in the normalization of average utilities.

In both Figs. 7 and 8, the distance between w_5^- and w_5^+ is the largest because both Car 3 and Car 6 are assessed to

Table 5
BDs that 6 cars be assessed on 7 attributes.

	Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
e_1	Acceleration	{(P,0.2), (A,0.8)}	{(G,0.5), (E,0.5)}	{(E,0.75), (T,0.25)}	{(A,0.4), (G,0.6)}	{(G,0.4), (E,0.4), (Ω,0.2)}	{(G,0.25), (E,0.75)}
e_2	Braking	{(G,1.0)}	{(E,0.333), (T,0.667)}	{(G,0.5), (E,0.5)}	{(P,0.75), (A,0.25)}	{(P,1.0)}	{(E,1.0)}
e_3	Handling	{(A,0.4), (G,0.6)}	{(E,0.6), (T,0.4)}	{(A,0.4), (G,0.4), (Ω,0.2)}	{(A,1.0)}	{(G,1.0)}	{(E,0.5), (T,0.4), (Ω,0.1)}
e_4	Horsepower	{(E,0.333), (T,0.667)}	{(P,0.533), (A,0.467)}	{(G,0.462), (E,0.538)}	{(G,0.385), (E,0.615)}	{(W,0.467), (P,0.533)}	{(A,0.267), (G,0.733)}
e_5	Ride quality	{(G,0.6), (E,0.4)}	{(A,1.0)}	{(Ω,1.0)}	{(G,1.0)}	{(G,1.0)}	{(Ω,1.0)}
e_6	Powertrain	{(A,0.4), (G,0.6)}	{(G,1.0)}	{(E,0.5), (T,0.4), (Ω,0.1)}	{(A,0.4), (G,0.6)}	{(G,0.6), (E,0.4)}	{(E,0.5), (T,0.3), (Ω,0.2)}
e_7	Fuel economy	{(G,1.0)}	{(G,1.0)}	{(E,1.0)}	{(G,1.0)}	{(A,1.0)}	{(G,1.0)}

Table 6
The standard deviations of the attribute weights by different methods.

Methods	0–1 Ave-Entropy	Ave-Entropy	SD	CRITIC	CCSD	GMD
Standard deviation	0.0428	0.1293	0.0561	0.0548	0.0673	0.0589

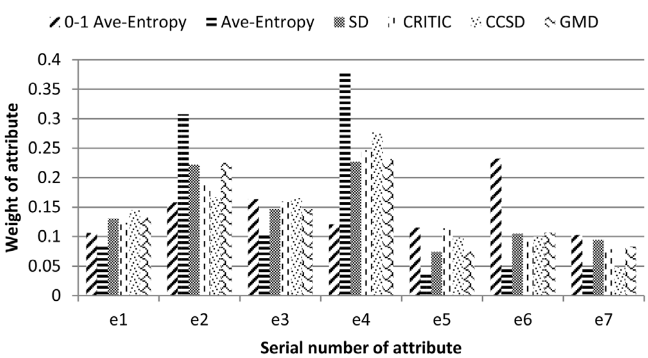

Fig. 6. Weights of the 7 attributes generated by different methods.

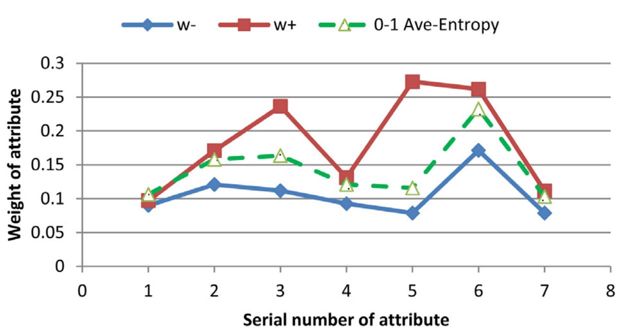

Fig. 7. Interval weights generated by ⟨Model 3⟩ subject to the 0–1 Ave-Entropy method.

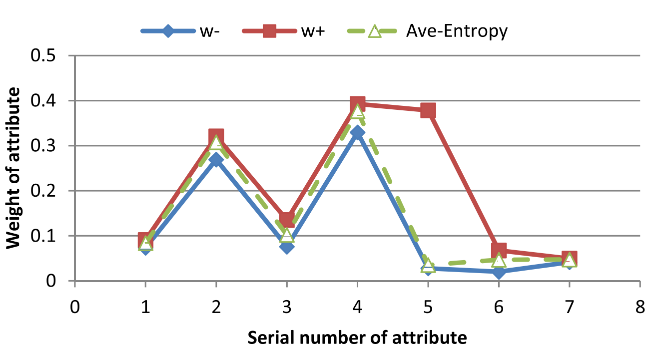

Fig. 8. Interval weights generated by ⟨Model 3⟩ subject to the Ave-Entropy method.

be completely ignorant on 'Ride quality'. But the weights in Fig. 8 are non-normalized interval weights because $\sum_{i=1}^{7} w_i^- + \max_{1\le i\le 7}\left(w_i^+ - w_i^-\right) > 1$. Comparatively, the weights generated in Fig. 7 are normalized interval weights which are more rational.

5.2. Generating weights from some other methods

Here, several objective WAM based optimization models are applied to generate attribute weights for the comparison of these methods against the proposed method.

(1) SD method based on BDs

Suppose the BD matrix in Eq. (6) has been transformed to utility matrix denoted by $[u_{li}]_{S\times L}$, where u_{li} signifies the utility of $S(e_i\,(a_l))$. Let $\tilde{u}_{li}$ be the normalized utility of e_i be evaluated on a_l. $\tilde{u}_{li}$ can be set to be equal to u_{li} or the transformed utility by Standard 0–1 Transformation. Then the standard deviation that the utility of e_i on all the L alternatives can be computed by

$$\sigma_i = \sqrt{\frac{1}{S}\sum_{l=1}^{S}\left(\tilde{u}_{li} - \frac{\sum_{k=1}^{S}\tilde{u}_{ki}}{S}\right)^2},\ i = 1, 2, \ldots, L \tag{24}$$

In Eq. (24), if u_{li}^{Ave} is used as $\tilde{u}_{li}$ to determine σ_i, we call it the Ave-σ method. Then we have

$$w_i = \frac{\sigma_i}{\sum_{t=1}^{L}\sigma_t},\ i = 1, 2, \ldots, L \tag{25}$$

In order to cope with the ignorance contained in the BDs, Eq. (25) can be extended to a pair of optimization models as follows:

$$\langle Model\,6\rangle\ Min/Max\ w_i = \frac{\sigma_i}{\sum_{t=1}^{L}\sigma_t}\ (i = 1, 2, \ldots, L)$$
$$\text{s.t. Eqs. (19), (20), (24)}$$
$$u_{li}^{Min} \le \tilde{u}_{li} \le u_{li}^{Max}$$
$$i = 1, 2, \ldots, L;\ l = 1, 2, \ldots, S$$

Fig. 9 shows the interval weights generated by ⟨Model 6⟩ based on the BDs in Table 5. Like Fig. 7, the curve represented by 'SD' is the results generated by Ave-σ method where the optimization process is not applied.

(2) CCSD based on BDs

Let d_{li} be the overall utility of a_l without the consideration of e_i, then we have

$$d_{li} = \sum_{j=1, j\ne i}^{L} \tilde{u}_{lj} w_j \tag{26}$$

where $\tilde{u}_{lj}$ can be equal to u_{lj} or the transformed utility by Standard 0–1 Transformation. The correlation coefficient between the utility vector of e_i and the overall utility vector without the

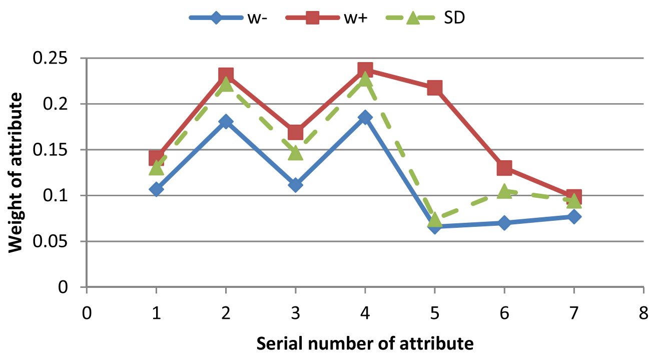

Fig. 9. Interval weights generated by ⟨Model 6⟩.

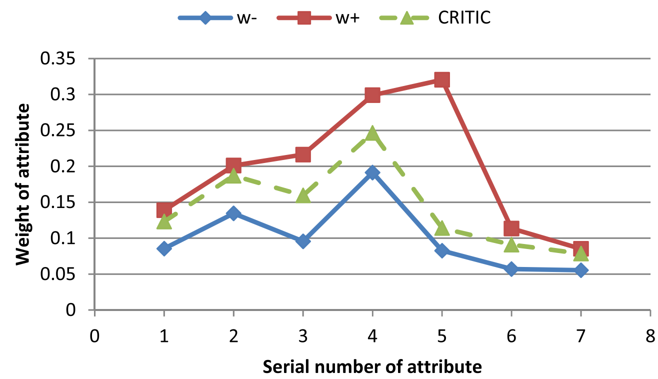

Fig. 10. Interval weights generated by ⟨Model 8⟩.

Table 7
Weights generated by CCSD based optimization model.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
w_i	0.12	0.1555	0.1261	0.222	0.206	0.1029	0.0678

consideration of e_i is computed by

$$R_i=\frac{\sum_{l=1}^{S}\left(\tilde{u}_{li}-\frac{\sum_{k=1}^{S}\tilde{u}_{ki}}{S}\right)\left(d_{li}-\frac{\sum_{k=1}^{S}d_{ki}}{S}\right)}{\sqrt{\sum_{l=1}^{S}\left(\tilde{u}_{li}-\frac{\sum_{k=1}^{S}\tilde{u}_{ki}}{S}\right)^2\left(d_{li}-\frac{\sum_{k=1}^{S}d_{ki}}{S}\right)^2}} \tag{27}$$

Then the following optimization model can be constructed to generate the attribute weights.

$$\langle Model\,7\rangle\ Min\ \sum_{i=1}^{L}\left(w_i-\frac{\sigma_i\sqrt{1-R_i}}{\sum_{k=1}^{L}\sigma_k\sqrt{1-R_k}}\right)^2$$
$$\text{s.t. Eqs. (19), (20), (24), (27)}$$
$$\sum_{i=1}^{L}w_i=1$$
$$0\le w_i\le 1$$
$$u_{li}^{Min}\le\tilde{u}_{li}\le u_{li}^{Max}$$
$$i=1,2,\ldots,L;\ l=1,2,\ldots,S$$

$(S+1)\times L$ variables are included in ⟨Model 7⟩. It should be mentioned that ⟨Model 7⟩ only need to be solved once to create the weights of L attributes, and the weights generated by ⟨Model 7⟩ are crisp values which are shown in Table 7. So the incompleteness contained in the BDs from Table 5 is not reflected although the incompleteness is considered in the constraints of ⟨Model 7⟩.

(3) CRITIC based on BDs

The CRITIC method, known as Criteria Importance Through Intercriteria Correlation, considers both the contrast intensity and conflicting character of the evaluation criteria [59]. It utilizes the standard deviation to measure the contrast intensity among different alternatives respect to a specific attribute, while Spearman correlation coefficient is employed to quantify the conflict between each pair of attributes. The correlation coefficient between e_i and e_j is calculated by

$$r_{ij}=\frac{\sum_{l=1}^{S}\left(\tilde{u}_{li}-\frac{\sum_{k=1}^{S}\tilde{u}_{ki}}{S}\right)\left(\tilde{u}_{lj}-\frac{\sum_{k=1}^{S}\tilde{u}_{kj}}{S}\right)}{\sqrt{\sum_{l=1}^{S}\left(\tilde{u}_{li}-\frac{\sum_{k=1}^{S}\tilde{u}_{ki}}{S}\right)^2\left(\tilde{u}_{lj}-\frac{\sum_{k=1}^{S}\tilde{u}_{kj}}{S}\right)^2}} \tag{28}$$

Then the weights of attributes are generated by

$$w_i=\frac{\sigma_i\cdot\sum_{j=1}^{L}(1-r_{ij})}{\sum_{k=1}^{L}\sigma_k\cdot\sum_{j=1}^{L}(1-r_{kj})}(i=1,2,\ldots,L) \tag{29}$$

If average utility is used as $\tilde{u}_{li}$ in Eqs. (28) and (29), it is called Ave-CRITIC method. In consideration of the ignorance included in BDs, Ave-CRITIC method is extended to the following pair of optimization models:

$$\langle Model\,8\rangle\ Min/Max\ w_i=\frac{\sigma_i\cdot\sum_{j=1}^{L}(1-r_{ij})}{\sum_{k=1}^{L}\sigma_k\cdot\sum_{j=1}^{L}(1-r_{kj})}$$
$$(i=1,2,\ldots,L)$$
$$\text{s.t. Eqs. (19), (20), (24), (28)}$$
$$u_{li}^{Min}\le\tilde{u}_{li}\le u_{li}^{Max}$$
$$i=1,2,\ldots,L;\ l=1,2,\ldots,S$$

By applying ⟨Model 8⟩, the interval weights can be generated and shown in Fig. 10, and the curve of 'CRITIC' is the result by applying Ave-CRITIC method. The weights generated by ⟨Model 8⟩ are also normalized interval weights according to Definition 1.

(4) GMD method based on BDs

The GMD of e_i considering the utilities of BDs on all S alternatives is computed by

$$G(e_i)=\frac{1}{S(S-1)}\sum_{l=1}^{S}\sum_{h=1}^{S}|u_{li}-u_{hi}| \tag{30}$$

Then the weight of e_i can be generated as

$$w_i=\frac{G(e_i)}{\sum_{k=1}^{L}G(e_k)} \tag{31}$$

To capture the incompleteness of assessment in the original BDs, the following GMD based optimization models are constructed.

$$\langle Model\,9\rangle\ Min/Max\ w_i=\frac{G(e_i)}{\sum_{k=1}^{L}G(e_k)}\ (i=1,2,\ldots,L)$$
$$\text{s.t. Eqs. (19), (20), (30)}$$
$$u_{li}^{Min}\le u_{li}\le u_{li}^{Max}$$
$$i=1,2,\ldots,L;\ l=1,2,\ldots,S$$

The interval weights generated by ⟨Model 9⟩ are shown in Fig. 11. The curve of 'GMD' refers to the generated weights by applying Eqs. (30) and (31) where u_{li} is determined by u_{li}^{Ave}. From Figs. 9 and 11, it can be seen that the interval weights generated by the GMD based model are similar with that of the SD based model.

Table 8
The maximum, average and standard deviation values of w_i^{DIS}.

	0–1 Ave-Entropy	Ave-Entropy	SD	CRITIC	GMD
$\max_{1\le i\le 7} w_i^{\mathrm{DIS}}$	0.1941	0.3506	0.1517	0.2379	0.158
$\frac{1}{7}\sum_{i=1}^{7} w_i^{\mathrm{DIS}}$	0.0767	0.0854	0.061	0.0961	0.0621
$\sqrt{\frac{1}{7}\sum_{i=1}^{7}\left(w_i^{\mathrm{DIS}}-\frac{\sum_{i=1}^{7} w_i^{\mathrm{DIS}}}{7}\right)^2}$	0.06	0.11	0.0391	0.0649	0.0413

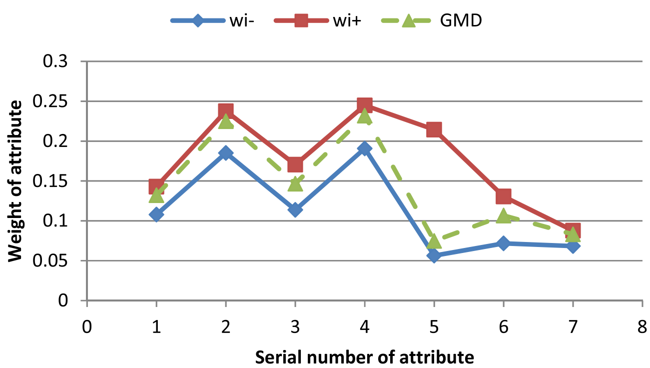

Fig. 11. Interval weights generated by ⟨Model 9⟩.

5.3. Comparative analysis

(1) Comparison with other objective WAMs

Let the distance between the minimum and maximum values of w_i be denoted by $w_i^{\mathrm{DIS}} = w_i^+ - w_i^-$. From Figs. 7 to 11, the features of each model can be summarized as follows. All the models generate normalized interval weights except the Ave-Entropy based optimization model. The reason lies in that $w_i^{\mathrm{DIS}}(i = 1, 2, 3, 4, 6)$ in Fig. 8 is too small, while $w_5^{\mathrm{DIS}} = 0.3506$ is very large comparatively. From Eq. (17), we can see that when the Ave-Entropy based optimization model is applied, the first part $\sum_{i=1}^{L} w_i^-$ is not small enough and the second part $\max_{1\le i\le L}\left(w_i^+ - w_i^-\right)$ is too large which results in the non-normalized interval weights. The maximum, average and standard deviation values of w_i^{DIS} by the five models are shown in the 2nd to 4th row of Table 8. From Table 8, it is clear that the Ave-Entropy based optimization model creates the maximum value of $\max_{1\le i\le 7} w_i^{\mathrm{DIS}}$. Although the ignorance contained in the BDs of e_5 respect to the 6 cars is the largest, the value of w_5^{DIS} is still too large compared with other four methods which will lead to a relatively uncertain aggregation result when the ER algorithm is used. This just reflects that the 0–1 Ave-Entropy based optimization model decreases the uncertainties of interval weights compared with the Ave-Entropy model. The last row of Table 8 shows that the standard deviation of w_i^{DIS} on the 7 attributes by the Ave-Entropy based model is the largest. Thus the generated weights cannot reflect the ignorance in the original BDs effectively because the incompleteness is enlarged on w_5 and narrowed on w_1, w_3 and w_6.

In Table 5, it can be seen that the extent of incompleteness contained in the BDs of the 7 attributes is $e_5 > e_3 = e_6 > e_1 > e_2 = e_4 = e_7$. Specifically, e_5 is assessed to be completely ignorant on both Car 3 and Car 6, so the additive ignorance on e_5 is $\beta_{H,5}(a_3) + \beta_{H,5}(a_6) = 2$. e_3 and e_6 are assessed to be partly ignorant on Car 3 and Car 6, and the additive ignorance on these two attributes are equal such that $\beta_{H,3}(a_3) + \beta_{H,3}(a_6) = \beta_{H,6}(a_3) + \beta_{H,6}(a_6) = 0.3$. For e_1, the additive ignorance is $\beta_{H,1}(a_5) = 0.2$ because only Car 5 is assessed to be partly ignorant. The additive ignorance on e_2, e_4 and e_7 are 0 since the BDs of each of the three attributes on all the 7 cars do not contain ignorance. Intuitively, w_5^{DIS} should be the largest among the 7 attributes, while w_2^{DIS}, w_4^{DIS} and w_7^{DIS} are the smallest. w_3^{DIS} and w_6^{DIS} ought to be smaller than w_5^{DIS} and larger than w_1^{DIS} which is bigger than the smallest three attributes. The weights generated by the SD and GMD based models in Figs. 9 and 11 show that the values of w_2^{DIS}, w_3^{DIS}, w_4^{DIS} and w_6^{DIS} are close with each other that is inconsistent with our intuition. Fig. 10 which shows the weights generated by the CRITIC based model presents similar values on w_3^{DIS} and w_4^{DIS}. It is also irrational according to the above discussion. The weights generated by the 0–1 Ave-Entropy based model shown in Fig. 7 are relatively reasonable. In Fig. 7, w_5^{DIS} is the largest, the value of w_3^{DIS} is similar with that of w_6^{DIS}, which is smaller than w_5^{DIS}. The values of w_2^{DIS}, w_4^{DIS} and w_7^{DIS} are close with each other and smaller than w_3^{DIS} and w_6^{DIS}. It just accords with our intuition. The comparisons of these different WAMs are shown in Table 9.

The idea of the incompatibility-based method proposed in [10] is similar with the CCSD method. Since the original SD, CCSD, CRITIC and GMD methods assume that all attributes are represented by numerical values, these methods are a purely objective weighting process. Comparatively, the proposed WAM in this paper contains qualitative or quantitative attributes which are denoted by BDs. So the DM can directly express his/her subjective judgments instead of providing numerical data even on qualitative attributes passively. Although the weights are not provided directly by the DM, the generated weights have a certain degree of subjectivity because the BD is the subjective judgment provided by the DM. Besides, preference relations on attribute weights can be incorporated into both the proposed models and the incompatibility-based method to capture the subjective judgments of DMs. From this point of view, the proposed method can be seen as a hybrid WAM.

The proposed method allows that the BDs provided by DMs contain local or global ignorance. As a result, more practical problems can be dealt with because DMs often express their judgments in incomplete manners, thus ⟨Model 3⟩ which is an optimization model is constructed just based on this condition. In comparison, Ave-entropy, SD, CCSD, CRITIC and GMD in the previous studies all require the assessment values to be crisp and complete. So these methods have limitations because the assessment either in the form of numerical value or subjective judgment like BD may be unreliable to a certain degree. For example, data acquisition equipment may be unstable in some complex environment in the LCA, so the unreliability of the equipment is interpreted as the incompleteness involved in the acquired data. Moreover, the data from stable equipment may also change under different working conditions, so incomplete interval value [26] is preferable for the representation of attribute assessment. For this reason, the WAMs involving interval numerical values and IBDs are proposed in ⟨Model 2⟩ and ⟨Model 5⟩ respectively, whereas the other six methods do not consider these two aspects in the previous studies. Although the method in [10] considered the incompatibility of BDs, IBDs is not discussed, and the incompleteness of the BDs on basic attributes is also not addressed.

In the proposed method, the uncertainty of the generated attribute weights is not too sensitive to the ignorance of BDs,

Table 9
Comparison of several WAMs.

Property	Method						
	Proposed	Ave-entropy	SD	CCSD	CRITIC	GMD	Incompatibility-based method
Normalized weights	✓	Sometimes not	✓	✓	✓	✓	✓
Sensitivity to the ignorance of BD	Moderate	Relatively high					
Considering the discrepancy on the risk preference of DMs		✓					✓
Optimization models included	✓			✓			✓
The extent of subjectivity	To a certain degree	To a certain degree	No	No	No	No	To a certain degree
Interval numerical value permitted	✓						
Assessment in the form of crisp numerical value	✓		✓	✓	✓	✓	
Assessment represented by BD	✓	✓					✓
Taking into account the ignorance in the original data	✓						
Consideration of IBD	✓						

which leads to the normalized interval weights as depicted in Fig. 7. The 2nd column of Table 8 also reflects this feature of the proposed WAM. Comparatively, the Ave-entropy method which does not consists of a standard 0–1 transformation is more sensitive to the incompleteness of BDs. So it sometimes generates a non-normalized interval weight vector as shown in Fig. 8 and the 3rd column of Table 8. From the above discussion, we can draw a conclusion that different forms of assessment such as crisp numerical value, complete and incomplete interval value, BD with accurate and interval belief degrees can all be dealt with to create attribute weights by the proposed models. This allows us to deal with a broader range of decision-making problems.

(2) Comparison with some methods for GDM

In addition to the Ave-entropy method [14] and incompatibility-based method [10], none of the above methods takes into account the discrepancy on the risk preference of DMs because GDM [50,51] is not the focus of the discussion. In recent years, the influence-guided GDM in social network has attracted the attention of many researchers on two aspects. One is opinion evolution and feedback mechanism for consensus researching process (CRP) [44,46,47,49–51], the other one is incomplete preference estimation [45,47,48,86] and opinion aggregation [8]. The comparisons between the proposed method and these GDM methods are specified as follows:

① Incomplete information estimation

In [47,48,86] and our proposed method, a common issue lies in that the information or preferences derived from experts include incomplete or missing preferences, although they are in different forms. Both [47] and [48] proposed the estimation of incomplete preferences based on trust relationships. Comparatively, we do not propose an estimation process because incomplete information is regarded as the ignorance, e.g. the local and global ignorance presented in Table 5. The BDs in Table 5 can also be seen as the aggregated opinion from a group of DMs. The estimation of incomplete preferences in [48] is derived from the trust relationships based on social network. Here, the social influence commonly existed in some GDM scenarios is not considered because individual decision-making is the focus of this paper. How to estimate incomplete preferences in a rational way for individual decision-making problems where trust relationships cannot be obtained is an interesting issue. Furthermore, in some GDM problems, experts do not interact with each other. A typical case is the selection of a president where there are a large number of voters who do not know each other.

② Opinion evolution and aggregation

Just as the above mentioned, the BD can be seen as the subjective judgment by an individual or the aggregated preference from a group of DMs. So the proposed EWAM focuses primarily on the attribute aggregation process in the context of MADM. In this situation, the opinion evolution and feedback mechanism in a social network group decision making (SNGDM) process is not stressed here. As such, it is a relatively static process compared with the dynamic GDM process [44–49,51]. The difference of opinion aggregation process between [46–49,51] and the proposed method lies in the operator and object of aggregation. Simple additive weighting (SAW) is used to construct a collective decision matrix, and the aggregation of experts' opinions is stressed in [46–49], while the ER algorithm which represents a nonlinear operator is used on the aggregation of attribute values in this paper. The method in [19] presents the ER rule to aggregate the assessments of multiple attributes and multiple experts where the weights and reliabilities of both experts and attributes are considered. But the opinion evolution and how to generate the weights and reliabilities are still open issues in the ER based MADM approaches.

③ Dissimilarity measure

In the EWAM with BDs, one point is similar to the 'consensus measure' in a CRP. In a CRP, the distances between each pair of individual preference relations are employed to measure the similarity degree among DMs. Specifically, the in-degree centrality index is utilized to determine the importance degree of a DM in the social network. The in-degree centrality index is generated from the weighted adjacent matrix where each element denotes the trust strength from a DM to another one. In fact, the trust strength reflects the correlation between each pair of DMs to some extent. In the proposed method, the entropy measure is used to quantify the dissimilarity of BDs among all the alternatives on each attribute. The entropy of an attribute which is the measure of dissonance can also be seen as the 'average distance' between each pair of alternatives respect to a designated attribute.

④ Preference representation

In the above mentioned literatures, the DM's preferences are represented in different ways using distinct structures, such as interval-valued intuitionistic fuzzy preference relations (IV-IFPRs) [46], fuzzy preference relations (FPR) [47,48], PLTS [8] and distributed linguistic trust [49]. Our future research may expand to the situation of MAGDM, where a group of DMs present their judgments in the form of BDs on some attributes, together with some different types of preference representations on other attributes. The CRP could then be discussed under this circumstance. Moreover, the preference estimation is also to be included in the EWAM.

Fig. 12 shows the EWAM proposed in this paper and the whole decision-making procedure. Three steps are specified as follows:

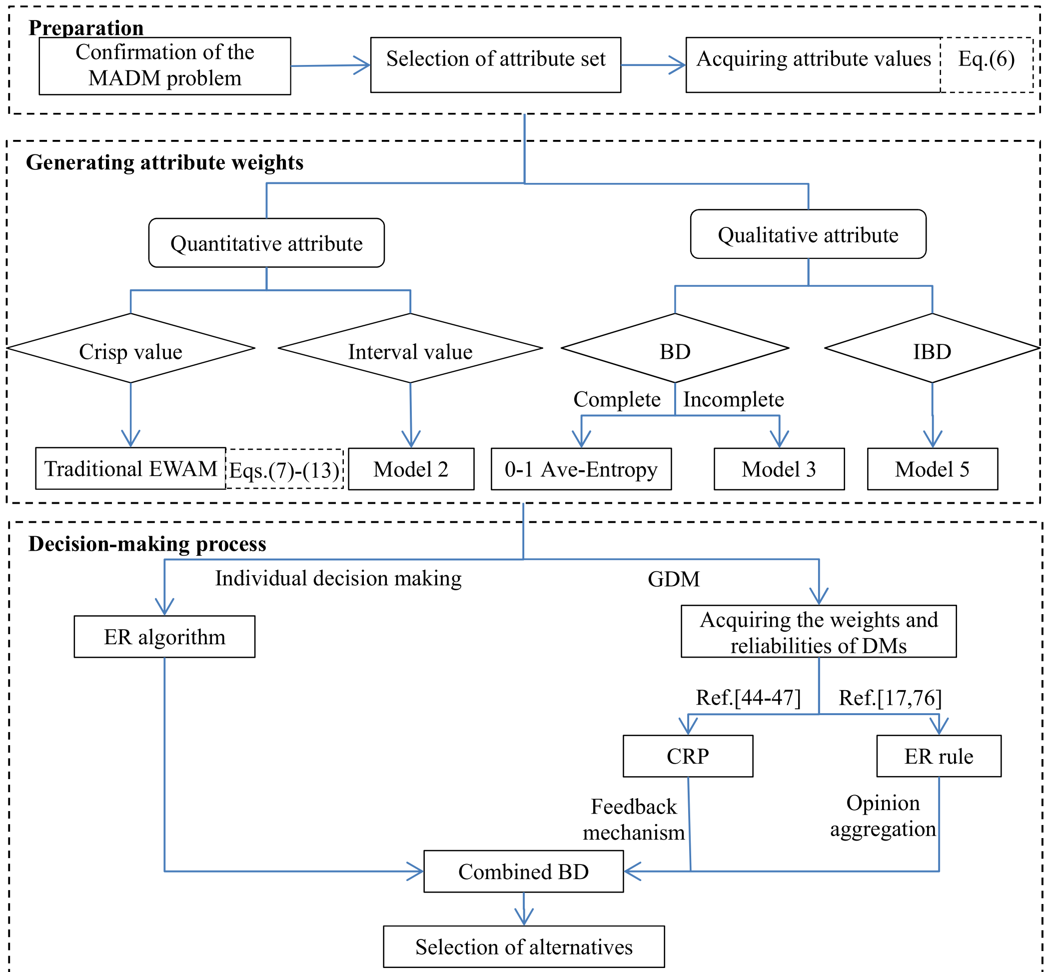

Fig. 12. EWAM and the decision-making process.

① The first step is the preparation of the MADM problem. Three major tasks are to be done: (1) Confirmation of the MADM problem, which refers to the identification of DMs and a set of alternatives. (2) Then the attribute set should be constructed according to specific standard or elicited from an individual or a group of DMs. (3) With the generated set of attributes, the values on both quantitative and qualitative attributes are to be acquired. The values of quantitative attributes can be obtained from equipment, statistical data, investigation, etc., while qualitative attributes are usually provided by DMs subjectively.

② The second step is to generate attribute weights from the values acquired. Four different situations may arise as follows: (1) If only quantitative attributes represented by accurate numerical values are included in the MADM problem, traditional EWAM shown in Eqs. (7)–(13) can be used to calculate attribute weights. (2) When some of the quantitative attributes are measured by interval values, ⟨Model 2⟩ is applied to generate the minimum and maximum weights of attributes. (3) If qualitative attributes represented by BDs are included, 0–1 Ave-Entropy method is used provided that the BDs are all complete assessments. (4) Otherwise, ⟨Model 3⟩ is to be utilized to generate interval weights when some of the BDs are incomplete. (5) The last is the most uncertain situation that IBDs are included, which leads to ⟨Model 5⟩ be applied.

③ The third step is the decision-making process. If only an individual is involved in the MADM problem, the ER approach [1, 9–15] can be directly employed to aggregate the BDs on qualitative attributes together with the numerical values on quantitative attributes. When a group of DMs are involved, the weights and reliabilities of DMs should be firstly generated in an appropriate way [19]. Then CRP [44–49,51] and opinion aggregation can be implemented. Since weights and reliabilities of DMs and attributes should all be tackled in a rational way, ER rule [19,81,83] is suitable to be applied in the aggregation. Finally, a combined BD with acceptable adjustment cost and consensus level is generated, following by a comparison or selection of alternatives.

6. Conclusions

Uncertainty is ubiquitous in practical MADM problems due to the complexity of decision-making circumstances such as unreliability of data sources, subjectivity of judgment by individual person or different backgrounds and expertise of DMs. How to elicit attribute weights in a rational and appropriate way from various kinds of uncertain available information is significant. In this paper, EWAM is proposed to tackling with the situations where attributes are assessed by interval numerical values, BDs with accurate belief degrees and IBDs. Several pairs of entropy based optimization models are constructed to generate attribute weights in an objective way under these circumstances. The properties of the proposed models are discussed. The advantage of EWAM lies in that the uncertainties and incompleteness contained in the original assessment information are preserved in each of the three situations from the generated weights. Some comparisons with other WAMs are conducted to illustrate the effectiveness of the proposed models. The ER approach can thus be used to cope with MADM problems which consists of precise numerical values, interval values, BDs and IBDs on quantitative and qualitative attributes provided that attribute weights are difficult to be generated subjectively. Just as the above mentioned, future researches would be extended to GDM in social network, and a combination of several objective and subjective WAMs is also to be implemented in an appropriate way for a more flexible process to deal with complex situations.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under the Grant No. 71601060 and 71571166, NSFC-Zhejiang Joint Fund for the Integration of Industrialization and Informatization under the Grant No. U1709215, Innovative Research Groups of the National Natural Science Foundation of China under the Grant No. 71521001, Natural Science Foundation of Anhui province, China under the Grant No. 1908085MG223.

Appendix

A.1. Proof of Property 2

If Linear Proportional Transformation is used before the normalization, we have

$$\bar{u}_{li}^{Ave} = \frac{u_{li}^{Ave}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}$$

$$\tilde{u}_{li}^{Ave} = \frac{\bar{u}_{li}^{Ave}}{\sum_{t=1}^{S}\bar{u}_{ti}^{Ave}} = \frac{\frac{u_{li}^{Ave}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}}{\sum_{t=1}^{S}\frac{u_{ti}^{Ave}}{max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}} = \frac{u_{li}^{Ave}}{\sum_{t=1}^{S}u_{ti}^{Ave}} = \frac{u_{li}^{Ave}}{\sum_{k=1}^{S}u_{ki}^{Ave}}$$

Obviously, $\tilde{u}_{li}^{Ave}$ is identical with that Linear Proportional Transformation or Standard 0–1 Transformation is not used. So the generated weights are the same with the Ave-Entropy method.

A.2. The calculation process of weights in Example 1

According to Eqs. (19) and (20), the maximum and minimum utilities of a_1 on e_1 are calculated as follows:

$$\begin{aligned} u_{11}^{Max} &= \sum_{n=1}^{5}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_5) \\ &= (1\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0\times 1 = 0 \end{aligned}$$

$$\begin{aligned} u_{11}^{Min} &= \sum_{n=1}^{N}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_1) \\ &= (1\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0\times 0 = 0 \end{aligned}$$

Then the average utility of a_1 on e_1 can be computed by Eq. (21) as follows:

$$u_{11}^{Ave} = \left(u_{11}^{Max} + u_{11}^{Min}\right)/2 = 0$$

Similarly, the maximum, minimum and average utilities of a_2 on e_1 are computed by

$$\begin{aligned} u_{21}^{Max} &= \sum_{n=1}^{5}\beta_{n,1}(a_2)\,u(H_n) + \beta_{H,1}(a_2)\,u(H_5) \\ &= (0\times 0 + 1\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0\times 1 = 0.45 \end{aligned}$$

$$\begin{aligned} u_{21}^{Min} &= \sum_{n=1}^{N}\beta_{n,1}(a_2)\,u(H_n) + \beta_{H,1}(a_2)\,u(H_1) \\ &= (0\times 0 + 1\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0\times 0 = 0.45 \end{aligned}$$

$$u_{21}^{Ave} = \left(u_{21}^{Max} + u_{21}^{Min}\right)/2 = 0.45$$

According to the Ave-Entropy method, the normalization process of the utility is conducted as follows:

$$\tilde{u}_{11}^{Ave} = \frac{u_{11}^{Ave}}{u_{11}^{Ave} + u_{21}^{Ave}} = \frac{0}{0 + 0.45} = 0$$

$$\tilde{u}_{21}^{Ave} = \frac{u_{21}^{Ave}}{u_{11}^{Ave} + u_{21}^{Ave}} = \frac{0.45}{0 + 0.45} = 1$$

Then the entropy of e_1 is computed by

$$E_1 = -\frac{1}{ln(2)}\left(\tilde{u}_{11}^{Ave} ln\tilde{u}_{11}^{Ave} + \tilde{u}_{21}^{Ave} ln\tilde{u}_{21}^{Ave}\right) = 0$$

The entropy of e_2 can be calculated according to the above procedure such that $E_2 = 0$. So the weight of e_1 and e_2 are generated by Eq. (13) as follows:

$$w_1 = \frac{1 - E_1}{2 - (E_1 + E_2)} = 0.5, \quad w_2 = \frac{1 - E_2}{2 - (E_1 + E_2)} = 0.5$$

A.3. The calculation process of weights in Example 2

In Example 2, the BDs of a_1 and a_2 on e_1 is identical with Example 1. So we have $E_1 = 0$. According to Eqs. (19)–(21), the maximum, minimum and average utilities of a_1 on e_2 are computed as follows:

$$\begin{aligned} u_{12}^{Max} &= \sum_{n=1}^{5}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_5) \\ &= (0.8\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0.2\times 1 = 0.2 \end{aligned}$$

$$\begin{aligned} u_{12}^{Min} &= \sum_{n=1}^{N}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_1) \\ &= (0.8\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0\times 1) \\ &\quad + 0.2\times 0 = 0 \end{aligned}$$

$$u_{12}^{Ave} = \left(u_{11}^{Max} + u_{11}^{Min}\right)/2 = 0.1$$

The maximum, minimum and average utilities of a_2 on e_2 are computed as follows:

$$\begin{aligned} u_{22}^{Max} &= \sum_{n=1}^{5}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_5) \\ &= (0\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0.8\times 1) \\ &\quad + 0.2\times 1 = 1 \end{aligned}$$

$$\begin{aligned} u_{22}^{Min} &= \sum_{n=1}^{N}\beta_{n,1}(a_1)\,u(H_n) + \beta_{H,1}(a_1)\,u(H_1) \\ &= (0\times 0 + 0\times 0.45 + 0\times 0.75 + 0\times 0.9 + 0.8\times 1) \\ &\quad + 0.2\times 0 = 0.8 \end{aligned}$$

$$u_{22}^{Ave} = \left(u_{11}^{Max} + u_{11}^{Min}\right)/2 = 0.9$$

Then the entropy of e_2 is computed by

$$E_2 = -\frac{1}{ln(2)}\left(\tilde{u}_{12}^{Ave} ln\tilde{u}_{12}^{Ave} + \tilde{u}_{22}^{Ave} ln\tilde{u}_{22}^{Ave}\right) = 0.469$$

So the weight of e_1 and e_2 are generated by Eq. (13) as follows:

$$w_1 = \frac{1 - E_1}{2 - (E_1 + E_2)} = 0.653, \quad w_2 = \frac{1 - E_2}{2 - (E_1 + E_2)} = 0.347$$

$$\begin{aligned}
\tilde{u}'^{Ave}_{li} &= \frac{u'^{Ave}_{li} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u'^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}} = \frac{u'^{Ave}_{li} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{l-1} u'^{Ave}_{ki} + \sum_{k=l+1}^{S} u'^{Ave}_{ki} + u'^{Ave}_{li} - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}\\
&= \frac{u^{Ave}_{li} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right) - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{l-1} u^{Ave}_{ki} + \sum_{k=l+1}^{S} u^{Ave}_{ki} + u^{Ave}_{li} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right) - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}\\
&= \frac{u^{Ave}_{li} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}\\
&= \frac{u^{Ave}_{li} - min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)} \qquad (A.1)
\end{aligned}$$

$$\begin{aligned}
\tilde{u}'^{Ave}_{ti} &= \frac{u'^{Ave}_{ti} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u'^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}} = \frac{u^{Ave}_{ti} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{l-1} u'^{Ave}_{ki} + \sum_{k=l+1}^{S} u'^{Ave}_{ki} + u'^{Ave}_{li} - S\cdot min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}\\
&= \frac{u^{Ave}_{ti} - min_{1\le k\le S}\left\{u'^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S} u'^{Ave}_{ki} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}\\
&= \frac{u^{Ave}_{ti} - min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\}}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)} \qquad (A.2)\\
&\quad (t = 1,\ldots,l-1,l+1,\ldots,S)
\end{aligned}$$

Box I.

A.4. Proof of Property 3

Suppose $\beta_{n,i}(a_l)\,(n \in \{1,2,\ldots,N\}, i \in \{1,2,\ldots,L\}, l \in \{1,2,\ldots,S\})$ changes to $\beta'_{n,i}(a_l) = \beta_{n,i}(a_l) + \delta$ where $|\delta|$ is very small. Since $\sum_{n=1}^{N}\beta_{n,i}(a_l) + \beta_{H,i}(a_l) = 1$ and $\sum_{k=1}^{n-1}\beta_{k,i}(a_l) + \left(\beta_{n,i}(a_l)+\delta\right) + \sum_{k=n+1}^{N}\beta_{k,i}(a_l) + \beta'_{H,i}(a_l) = 1$, we have $\sum_{n=1}^{N}\beta_{n,i}(a_l) + \delta + \beta'_{H,i}(a_l) = 1$. Then $\beta'_{H,i}(a_l) = \beta_{H,i}(a_l) - \delta$. From Eqs. (19)–(21), we have

$$\begin{aligned}
u^{Ave}_{li} &= \frac{\left(u^{Max}_{li} + u^{Min}_{li}\right)}{2} = \sum_{n=1}^{N}\beta_{n,i}(a_l)\,u(H_n)\\
&\quad + \beta_{H,i}(a_l)\frac{u\left(H_1\right)+u(H_N)}{2}\\
u'^{Ave}_{li} &= \frac{\left(u'^{Max}_{li} + u'^{Min}_{li}\right)}{2} = \sum_{k=1}^{n-1}\beta_{k,i}(a_l)\,u(H_k) + \sum_{k=n+1}^{N}\beta_{k,i}(a_l)\,u(H_k)\\
&\quad + \left(\beta_{n,i}(a_l)+\delta\right)u\left(H_n\right) + \left(\beta_{H,i}(a_l)-\delta\right)\frac{u\left(H_1\right)+u(H_N)}{2}\\
&= \sum_{k=1}^{N}\beta_{k,i}(a_l)\,u(H_k) + \delta u\left(H_n\right)\\
&\quad + \beta_{H,i}(a_l)\frac{u\left(H_1\right)+u(H_N)}{2} - \delta\frac{u\left(H_1\right)+u(H_N)}{2}\\
&= u^{Ave}_{li} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)
\end{aligned}$$

Here, for attribute e_i, it is obvious that $u'^{Ave}_{ti} = u^{Ave}_{ti}$ for $t = 1,\ldots,l-1,l+1,\ldots,S$, and $\lim_{|\delta|\to 0}|u'^{Ave}_{li} - u^{Ave}_{li}| = 0$. From Eq. (22), we have Eqs. (A.1) and (A.2) as given in Box I.

There may be three situations which are shown as follows:

(1) If $u^{Ave}_{li} \ne min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}$, and $u'^{Ave}_{li} \ne min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\}$, then we have

$$min\{min_{1\le k\le S,k\ne l}\left\{u^{Ave}_{ki}\right\}, u'^{Ave}_{li}\} = min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}$$

In this situation, Eqs. (A.1) and (A.2) can be represented by

$$\begin{aligned}
&\tilde{u}'^{Ave}_{li}\\
&= \frac{u^{Ave}_{li} - min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\} + \delta\left(u\left(H_n\right) - \frac{u(H_1)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\} + \delta\left(u\left(H_n\right) - \frac{u(H_1)+u(H_N)}{2}\right)}\\
&\tilde{u}'^{Ave}_{ti}\\
&= \frac{u^{Ave}_{ti} - min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\} + \delta\left(u\left(H_n\right) - \frac{u(H_1)+u(H_N)}{2}\right)}\\
&\quad (t = 1,\ldots,l-1,l+1,\ldots,S)
\end{aligned}$$

Considering Eq. (22), $\tilde{u}^{Ave}_{li} = \frac{u^{Ave}_{li} - min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}}$, $\tilde{u}^{Ave}_{ti} = \frac{u^{Ave}_{ti} - min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}}{\sum_{k=1}^{S} u^{Ave}_{ki} - S\cdot min_{1\le k\le S}\left\{u^{Ave}_{ki}\right\}}$, so we have $\lim_{|\delta|\to 0}|\tilde{u}'^{Ave}_{li} - \tilde{u}^{Ave}_{li}| = 0$ and $\lim_{|\delta|\to 0}|\tilde{u}'^{Ave}_{ti} - \tilde{u}^{Ave}_{ti}| = 0$ for $t = 1,\ldots,l-1,l+1,\ldots,S$. Then

$$\begin{aligned}
E'_i &= -\frac{1}{ln\left(S\right)}\sum_{k=1}^{S}\tilde{u}'^{Ave}_{ki} ln\tilde{u}'^{Ave}_{ki}\\
&= -\frac{1}{ln\left(S\right)}\left(\sum_{t=1}^{l-1}\tilde{u}'^{Ave}_{ti} ln\tilde{u}'^{Ave}_{ti} + \sum_{t=l-1}^{S}\tilde{u}'^{Ave}_{ti} ln\tilde{u}'^{Ave}_{ti} + \tilde{u}'^{Ave}_{li} ln\tilde{u}'^{Ave}_{li}\right)
\end{aligned} \qquad (A.3)$$

$$\tilde{u}_{li}^{\prime Ave} = \frac{u_{li}^{Ave} - u_{li}^{\prime Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot u_{li}^{\prime Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}$$
$$= \frac{u_{li}^{Ave} - \left(u_{li}^{Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)\right) + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot \left(u_{li}^{Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)\right) + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}$$
$$= \frac{u_{li}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - (S-1)\cdot\delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)} \tag{A.4}$$

Box II.

$$\tilde{u}_{ti}^{\prime Ave} = \frac{u_{ti}^{Ave} - u_{li}^{\prime Ave}}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot u_{li}^{\prime Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}$$
$$= \frac{u_{ti}^{Ave} - \left(u_{li}^{Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)\right)}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot \left(u_{li}^{Ave} + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)\right) + \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}$$
$$= \frac{u_{ti}^{Ave} - u_{li}^{Ave} - \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot u_{li}^{Ave} - (S-1)\cdot\delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}$$
$$= \frac{u_{ti}^{Ave} - min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - \delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)}{\sum_{k=1}^{S} u_{ki}^{Ave} - S \cdot min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\} - (S-1)\cdot\delta\left(u\left(H_n\right) - \frac{u\left(H_1\right)+u(H_N)}{2}\right)} \quad (t = 1, \ldots, l-1, l+1, \ldots, S) \tag{A.5}$$

Box III.

From Eq. (13), we can see that the change of w_i is little provided that $|\delta|$ is small enough.

(2) If $u_{li}^{Ave} = min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$, $u_{li}^{\prime Ave} = min\{min_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\}$, then we have $\overline{u}_{li}^{\prime Ave} = \overline{u}_{li}^{Ave} = 0$, and $max\{max_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\} = max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$. In this situation, Eqs. (A.1) and (A.2) can be represented by Eq. (A.4) in Box II.

Since $\tilde{u}_{li}^{Ave} = \frac{\overline{u}_{li}^{Ave}}{\sum_{k=1}^{S}\overline{u}_{ki}^{Ave}}$, then $\tilde{u}_{li}^{\prime Ave} = \tilde{u}_{li}^{Ave} = 0$. See Eq. (A.5) in Box III.

Similarly, we have $\lim_{|\delta|\to 0}|\tilde{u}_{ti}^{\prime Ave} - \tilde{u}_{ti}^{Ave}| = 0$ for $t = 1, \ldots, l-1, l+1, \ldots, S$. Given Eqs. (13) and (A.3), we conclude that the change of w_i is little provided that $|\delta|$ is small enough.

(3) If $u_{li}^{Ave} = min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$, $u_{li}^{\prime Ave} \ne min\{min_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\}$, then we have $u_{li}^{\prime Ave} > u_{li}^{Ave}$, and $\overline{u}_{li}^{Ave} = 0$, $\overline{u}_{li}^{\prime Ave} > 0$. If $|\delta|$ is small enough, there at least exists another alternative a_s such that $u_{si}^{Ave} = u_{li}^{Ave}$. So $min\{min_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\} = min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$ although $u_{li}^{\prime Ave}$ is not the minimum utility anymore. For attribute e_i, if $\forall t, s \in \{1, 2, \ldots, S\}$, $\left|u_{ti}^{Ave} - u_{si}^{Ave}\right| < \varepsilon$ where ε is small enough, we could not clearly discriminate the S alternatives respect to e_i, and e_i should better be excluded from the specific assessment problem. Otherwise, $max\{max_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\} = max_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$. So it is similar with the first situation.

It should be mentioned that when $|\delta|$ is small enough, the situation that $u_{li}^{Ave} \ne min_{1\le k\le S}\left\{u_{ki}^{Ave}\right\}$ and $u_{li}^{\prime Ave} = min\{min_{1\le k\le S,k\ne l}\left\{u_{ki}^{Ave}\right\}, u_{li}^{\prime Ave}\}$ will not happen.

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