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Weight assignment method for multiple attribute decision making with dissimilarity and conflict of belief distributions

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ABSTRACT

Belief distribution (BD) is the scheme of representing uncertain and imprecise subjective assessment in the evidential reasoning methodology. In a multiple attribute decision making (MADM) problem, how to elicit attribute weights rationally from subjective assessments is an open issue. Moreover, the support degree of assessment for the final decision is critically important because it has a direct implication on the likelihood of making a right decision. The aim of this paper is firstly to identify the intrinsic information carried by different attributes in the form of BDs for generating attribute weights in a MADM problem. Thus, we present the concept of conflict measure between two attributes on both the alternative and evaluation grade level. A novel weight assignment method is further proposed based on the conflict measure between attributes and the divergence of different BDs. Secondly, the paper puts forward the external divergence and internal indeterminacy to measure the support degree of the final aggregated results for decision making. They are determined by the defined concept of dissimilarity and uncertainty measures on alternatives. A series of properties and comparative analysis are given to demonstrate the rationality and effectiveness of the proposed methods.

1. Introduction

Multiple attribute decision making (MADM) problems (Dong, Liu, Chiclana, Kou, & Herrera-Viedma, 2019; Liang, Gong, Dong, & Ding, 2018; Liu, Zhang, Wu, & Dong, 2019) are usually comprised of a set of attributes, which can be either quantitative or qualitative. Qualitative attributes can be expressed by various forms, such as linguistic variable (Gou, Xu, Liao, & Herrera, 2018; Wu, Chang, Cao, & Liang, 2019), belief distribution (BD) (Dubois & Prade, 1988; Fu, Xu, & Xue, 2018), intuitionistic fuzzy set (Chen, 2014; Li & Deng, 2019; Milošević, Petrović, & Jeremić, 2017); hesitant fuzzy sets (Xue, Xu, Wang, & Ren, 2019); and so on. The evidential reasoning (ER) approach (Akhoundi & Nazif, 2018; Wang, Yang, & Xu, 2006; Xiao, 2019; Xu et al., 2016, 2017; Yang and Xu, 2002, 2013; Yang, 2001; Zhou et al., 2018, 2019) provides a probabilistic aggregation process for MADM problems where assessments are represented by BDs, and uncertainty, incompleteness and fuzziness can all be dealt with in a consistent way. Since attribute weight is an important factor in the aggregation process of a MADM problem, how to generate a set of rational and valid weights is significant for the ER approach or other evidence combination methods (Xiao, 2019; Yin, Deng, & Deng, 2019). Weight assignment methods can be broadly classified into three categories: subjective, objective and hybrid (Fu & Wang, 2015; Wang & Luo, 2010). Subjective method includes direct rating method (Bottomley & Doyle, 2013), weighted least square method (Chu, Kalaba, & Spingarn, 1979), Delphi (Hwang & Lin, 1987) and so on. This kind of method extracts the attribute weights directly from the decision maker (DM) through interview, discussion or questionnaire. Objective method generates attribute weights from the intrinsic information of the assessment values. It can be classified into two sub-categories. One is based on the divergence of values from the assessment of different alternatives on each attribute. Representative methods include the entropy weight assignment method (EWAM) (Song, Zhu, Peng, & Santibanez Gonzalez, 2017; Zhou, Liu, Yang, & Chen, 2019), standard deviation (SD) (Chin, Fu, & Wang, 2015; Diakoulaki, Mavrotas, & Papayannakis, 1995), maximizing deviation method (Qian & Luan, 2017) and discriminating power method (Fu et al., 2018). The other sub-category not only depends on the dimension of information extracted in the first category, but also the correlation between each pair of attributes that reflects interdependency or conflict of attributes. Typical methods include criteria importance through intercriteria correlation (CRITIC) (Diakoulaki et al., 1995), correlation coefficient and standard deviation integrated

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(CCSD) method (Wang & Luo, 2010), deviation and decision incompatibility based method (Chin et al., 2015), and so on. Hybrid method (Yang, Yang, Xu, & Khoveyni, 2017) is applied when both subjective judgment and numerical evaluation values can be obtained.

Although a great number of methods for assigning attribute weights have been proposed in the last few decades, how to elicit appropriate weights from subjective assessments on attributes still remains an open issue, especially if there is no prior knowledge. The above mentioned objective weight assignment methods such as EWAM, SD method, CRITIC, CCSD and maximizing deviation method all assume that each attribute is assessed by a numerical value no matter it is in a quantitative or qualitative nature. In real decision-making problems, qualitative attributes are usually expressed by subjective judgments, e.g., BD, hesitant fuzzy sets. If there is no prior knowledge on the importance of attributes, the method of deriving attribute weights from the subjective assessments together with numerical values on quantitative attributes needs to be studied comprehensively. In recent years, some research has been devoted on generating attribute weights provided that qualitative attributes are denoted by BDs, such as the above mentioned discriminating power method. But they only considered one dimension of information which reveals the discrepancy among the BDs of different alternatives associated with a specific attribute. The other dimension which measures the interdependency between the BDs of each pair of attributes was not considered in the previous studies. In either the CRITIC or CCSD, the interdependency is quantified by the Spearman correlation coefficient among attributes which are all represented by numerical values. When the values of attributes are represented by BDs, the conflict or dissimilarity measure between BDs is the basis to measure the interdependency between attributes. Up to now, many metrics have been proposed on the conflict or dissimilarity measure such as Tessem's distance (Tessem, 1993), combined dissimilarity measure (Liu, Dezert, Pan, & Mercier, 2011); Jousselme's distance (Jousselme, Grenier, & Bossé, 2001); cosine similarity (Wen, Wang, & Xu, 2008), correlation coefficient (Jiang, 2018), Liu's distance (Liu, 2006). But none of them is perfect to tackling with all circumstances which will be detailed in Section 3.3. If BDs are transformed to numerical values such as utilities, some information contained in BDs may not be preserved. So how to measure the interdependency between two attributes represented by BDs is still an unanswered question. Inspired by the concept of dissimilarity measure between the BDs of DMs (Fu, Yang, & Yang, 2015) and alternatives (Fu et al., 2018) defined by Fu et al., this paper proposes the dissimilarity measure between the BDs of two attributes on an alternative. Then the conflict measure between two attributes is defined on both the alternative and evaluation grade level in order to quantify the interdependency. As a result, a novel weight assignment method which considers both the above two dimensions derived from BDs is developed in this paper. Optimization models are also constructed for the consideration of incompleteness included in BDs.

When the subjective judgments from multiple sources are aggregated to an overall assessment, how to measure the support degree of assessment for a decision is a critical issue because it may affect the likelihood of making a right decision on two aspects. One is the external divergence which can be quantified by the dissimilarity of different attributes. The other one is the internal indeterminacy which is correlated with the uncertainty measure of original subjective judgments. The uncertainty measure of belief distributions has widely accepted solutions (Deng, 2016). In the past decades, a lot of uncertainty measures have been proposed, e.g., Klir & Ramer's discord (Klir & Ramer, 1990); Deng entropy (Deng, 2016); Radim & Prakash's total uncertainty (Jiroušek & Shenoy, 2018); Jousselme's ambiguity measure (Jousselme, Liu, Grenier, & Bosse, 2006), Yager's interval-valued entropy (Ronald & Yager, 2018); Yang's total uncertainty measure (Yang & Han, 2016). But they are all defined in the context of Dempster-Shafer's evidence theory. In a decision-making problem, the utility of each focal element should be considered in the definition of the uncertain degree in a BD. For example, A is assessed to be 'Excellent' and 'Average' with the belief degree of 0.5 and 0.5, while B is assumed to be 'Good' and 'Average' with the belief degree of 0.5 and 0.5. According to any one of the above mentioned uncertainty measures on mass function, the result is identical for A and B although it may be different by each method. It is not reasonable because the difference between the utility of 'Excellent' and 'Average' is larger than 'Good' and 'Average', which leads to the divergence of opinion for A is more than B in either a GDM situation or individual decision circumstance. So the uncertainty measure on mass function should be improved to considering the difference among the utilities of discrete focal elements. In this paper, the concepts of dissimilarity measure and uncertainty measure on alternatives are proposed to generate the support degree of assessment for decision making. The uncertainty measure on the BDs of all attributes and the aggregated BD are presented, followed by the definition of incompatibility measure among the BDs of attributes for the purpose of quantifying the discrepancy of the BDs on different attributes.

The main contributions of the paper are summarized as follows:

(1) The conflict measures between two attributes on both the alternative and evaluation grade level are defined provided that subjective judgements are represented in the form of BDs. The advantages of the measurement are analyzed compared with existing conflict or distance metrics.
(2) A novel weight assignment method is developed from two dimensions extracted from the subjective judgments of attributes. This enables us to determine the weights of qualitative attributes if we are confronted with lack of prior knowledge. Comparative analysis is conducted to show the effectiveness and applicability of the proposed method.
(3) In order to facilitate the DM to measure the support degree of assessment for decision making in a MADM problem, the dissimilarity measure on alternatives and the uncertainty measure of BD are proposed. The concepts of average and global uncertainty measure are defined, which induce the total incompatibility measure among the BDs of all attributes for an alternative. This enables us to utilize the aggregated result in a more rational way instead of just depending on the ranking order of different alternatives.

The reminder of the paper is organized as follows. Section 2 is a brief presentation of the ER approach. In Section 3, conflict measure on the alternative level and evaluation grade level are firstly given, followed by the description of a comprehensive weight assignment method based on BDs. Sections 4 and 5 are both used to discuss the support degree of assessment for decision making. More specifically, Section 4 provides a method to quantify the external dissimilarity measure on alternatives, while Section 5 proposes a measurement of internal indeterminacy or uncertainty on belief structures. Section 6 presents a case study and the comparative analysis with existing methodological methods are conducted. This paper is concluded in Section 7.

2. Preliminaries

The ER approach uses the belief structure to represent uncertain subjective judgment on qualitative attribute based on the framework of Dempster-Shafer (D-S) evidence theory (Jiroušek & Shenoy, 2018; Klir & Ramer, 1990; Ronald & Yager, 2018) and decision-making theory. A set of evaluation grades for the assessment of an attribute on an alternative constitute the frame of discernment which is profiled as follows:

$$H = \{H_1, H_2, \cdots H_n, \cdots, H_N\} \tag{1}$$

where $H_n (n = 1, 2, \cdots, N)$ each denotes an evaluation grade, and H_{n+1} is supposed to be preferred to $H_n (n = 1, 2, \cdots, N-1)$. They are collectively exhaustive and mutually exclusive. The utility of H_n is represented by $u(H_n)$, and $u(H_{n+1}) > u(H_n)$. In a MADM problem, subjective judgments may be used to evaluate one alternative against others on either qualitative or quantitative attributes. Let $\boldsymbol{A} = \{a_1, a_2, \cdots, a_l, \cdots, a_S\}$ and $\boldsymbol{E} = \{e_1, e_2, \cdots, e_i, \cdots, e_L\}$ be the alternative vector and the basic attribute vector in a MADM problem respectively, where S and L denotes the number of alternatives and attributes. The belief degree that a_l be evaluated on e_i to the nth evaluation grade H_n is profiled by $\beta_{n,i}(a_l)$ with $0 \le \beta_{n,i}(a_l) \le 1$ and $\sum_{n=1}^{N}\beta_{n,i}(a_l) \le 1$.

Definition 1. Suppose $\beta_{n,i}(a_l)$ is the intensity to which the state of a single attribute e_i at alternative a_l be assessed to an evaluation grade H_n. Then the belief distribution (abbreviated as BD) that a_l be assessed on e_i is profiled as follows:

$$S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)),\ n = 1, 2, \cdots, N;\ (H, \beta_{H,i}(a_l))\} \tag{2}$$

where $(H_n, \beta_{n,i}(a_l))$ is an element of $S(e_i(a_l))$, representing that the evidence points to the proposition H_n to the degree of $\beta_{n,i}(a_l)$.

Eq. (2) stands for the state of a basic attribute evaluated for an alternative to all the N grades. H_n is referred to as a focal element of $S(e_i(a_l))$ if $\beta_{n,i}(a_l) > 0$. $\beta_{H,i}(a_l)$ is interpreted as the global ignorance included in the assessment of a_l on e_i, or called the degree of total incompleteness. $\sum_{n=1}^{N}\beta_{n,i}(a_l) + \beta_{H,i}(a_l) = 1$ is a basic condition which means the sum of the uncertain belief degree on all the$N$ grades and the degree of ignorance is one. $S(e_i(a_l))$ is said to be an incomplete BD if $\beta_{H,i}(a_l) > 0$; otherwise, it is complete when $\beta_{H,i}(a_l) = 0$ (or $\sum_{n=1}^{N}\beta_{n,i}(a_l) = 1$).

Example 1. In a car selection problem, some quantitative and qualitative attributes are included in the assessment where six alternative cars are to be compared. The attributes include Acceleration (e_1), Braking (e_2), Handling (e_3), Horsepower (e_4), Ride quality (e_5), Powertrain (e_6) and Fuel economy (e_7). The frame of discernment consists of six evaluation grades such that $H = \{$Worst (H_1), Poor (H_2), Average (H_3), Good (H_4), Excellent (H_5) and Top (H_6)$\}$.

In the assessment of Car 6, for example, the assessor is (1) 25% sure that its "Acceleration" is *good*, and 75% sure that it is *excellent*; (2) 50% sure that its "Handling" is *excellent* and 40% sure that it is *top*; (3) 100% unaware of the assessment on "Ride quality". Then the assessment on "Acceleration", "Handling" and "Ride quality" can be represented by BDs as follows:

$$S(e_1(a_6)) = \{\{(G, 0.25), (E, 0.75)\}\}$$

$$S(e_3(a_6)) = \{(E, 0.5), (T, 0.4), (H, 0.1)\}$$

$$S(e_5(a_6)) = \{(H, 1.0)\}$$

$S(e_1(a_6))$ and $S(e_3(a_6))$ are both uncertain assessment because more than one evaluation grades are picked up, or in other words, there are more than one focal elements in these two BDs. Note that $S(e_1(a_6))$ is a complete assessment because the total belief degree for the statement sums up to 1 such that $\beta_{4,1}(a_6) + \beta_{5,1}(a_6) = 1$, which means that the information provided by the assessor is complete. The missing 0.1 in $S(e_3(a_6))$ represents the degree of ignorance or incompleteness. $S(e_5(a_6))$ is said to be completely ignorant because the global ignorance represented by $\beta_{H,5}(a_6)$ is 1.

In a MADM problem, a decision matrix can then be represented as follows:

$$[S(e_i(a_l))]_{S\times L} = \begin{array}{c} \\ S(\boldsymbol{E}(a_1)) \\ \cdots \\ S(\boldsymbol{E}(a_l)) \\ \cdots \\ S(\boldsymbol{E}(a_S)) \end{array} \begin{array}{c} S(e_1(\boldsymbol{A}))\cdots S(e_i(\boldsymbol{A}))\cdots S(e_j(\boldsymbol{A}))\cdots S(e_L(\boldsymbol{A})) \\ \begin{bmatrix} S(e_1(a_1)) & \cdots & S(e_i(a_1))\cdots & S(e_j(a_1))\cdots & S(e_L(a_1)) \\ \cdots & & \cdots & \cdots & \cdots \\ S(e_1(a_l)) & \cdots & S(e_i(a_l))\cdots & S(e_j(a_l))\cdots & S(e_L(a_l)) \\ \cdots & & \cdots & \cdots & \cdots \\ S(e_1(a_S)) & \cdots & S(e_i(a_S))\cdots & S(e_j(a_S))\cdots & S(e_L(a_S)) \end{bmatrix}_{S\times L} \end{array} \tag{3}$$

In Eq. (3), each row represents the BDs that an alternative a_l be evaluated on all the L attributes denoted by a vector $S(\boldsymbol{E}(a_l)) = (S(e_1(a_l)), \cdots, S(e_i(a_l)), \cdots, S(e_L(a_l)))$, while each column denotes the vector that all the S alternatives be assessed on an attribute e_i such that $S(e_i(\boldsymbol{A})) = (S(e_i(a_1)), \cdots, S(e_i(a_l)), \cdots, S(e_i(a_S)))^T$. The relative weight of the ith attribute is represented by w_i such that $0 \le w_i \le 1$ and $\sum_{i=1}^{L} w_i = 1$. Either the recursive/analytical ER algorithm (Wang et al., 2006; Yang & Xu, 2002) or the ER rule (Yang & Xu, 2013; Zhou, Liu, Chen, & Yang, 2018) can be used as the aggregation approach to generate a general assessment from $S(\boldsymbol{E}(a_l))$. The generated BD on a_l from $S(\boldsymbol{E}(a_l))$ is denoted as follows:

$$S(a_l) = \{(H_n, \beta_n(a_l)),\ n = 1, 2, \cdots, N;\ (H, \beta_H(a_l))\} \tag{4}$$

where $\beta_n(a_l)$ represents the combined BD that a_l be evaluated on H_n, and $\beta_H(a_l)$ denotes the global ignorance assessed on a_l. The general BD on each alternative can then be compared using the average utility, maximum utility and minimum utility.

3. Assignment of attribute weight based on BDs

The dissimilarity measure of assessments is a critical issue to derive attribute weights in an objective way for MADM problems. When the assessments are presented in the form of numerical values, geometric models such as Euclidean distance, Manhattan distance, Chebyshev distance can be applied. Besides, Hamming distance, correlation coefficient, information entropy can also be employed to quantify the dissimilarity between objects. If the numerical values are not sufficiently informative, fuzzy sets can be introduced to represent the assessment information. In recent years, many researchers have devoted to measuring the similarity/dissimilarity of fuzzy sets (Gou et al., 2018; Milošević et al., 2017). When the assessments are represented by BDs, how to calculate the similarity/dissimilarity of BDs is an open issue. In (Fu et al., 2018) and (Fu et al., 2015), dissimilarity measure between the BDs of alternatives and DMs are defined respectively. Suppose the BDs of attribute e_i on two alternatives a_l and a_m are represented by Eq. (2), then the distributed dissimilarity vector between a_l and a_m respect to e_i is denoted as follows:

$$GD(e_i(a_{lm})) = ((H_1, \beta_{1,i}(a_{lm})), (H_2, \beta_{2,i}(a_{lm})), \cdots, (H_N, \beta_{N,i}(a_{lm}))) \tag{5}$$

where

$$\beta_{n,i}(a_{lm}) = |\beta_{n,i}(a_l) - \beta_{n,i}(a_m)|(n = 1, 2, \cdots, N) \tag{6}$$

Let $D(e_i(a_{lm}))$ be the dissimilarity measure between a_l and a_m with respect to attribute e_i. Then the dissimilarity measure between the BDs of a_l and a_m on e_i is defined as follows (Fu et al., 2018):

$$D(e_i(a_{lm})) = \sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta_{n,i}(a_{lm})\beta_{s,i}(a_{lm})u(H_{|s-n|}) \tag{7}$$

where

$$u(H_{|s-n|}) = u(H_s) - u(H_n) \tag{8}$$

The dissimilarity measure defined in Eq. (7) quantifies the divergence between the BDs of two alternatives on a specific attribute. The purpose of the dissimilarity measure defined in (Fu et al., 2018) is to compute the divergence among the BDs of all alternatives on one attribute, named as discriminating power, which is used to generate attribute weight for a MADM problem. Here, the dissimilarity is defined to measure the discrepancy between the BDs of two attributes respect to an alternative. The purpose of the dissimilarity measure defined in this paper is to calculate the subsequently defined conflict measure between two attributes, which is then applied to compute attribute weight together with the discriminating power. Intuitively, if the conflict between the BD of one attribute and all other attributes is large, it will be allocated with higher weight, and vice versa. So both dimensions are considered to be relevant to attribute weight generation in the background of BD. Although some objective methods such as the CRITIC and CCSD considered these two dimensions, they were designed to tackling with the situation where the data are presented in the form of accurate numerical values. Next, we will present the conflict measure between two attributes when subjective judgments are presented in the form of BDs.

3.1. Conflict measure on the alternative level

Given the BDs of two attributes e_i and e_j on a_l represented by Eq. (2), the *belief degree of dissimilarity* that a_l be assessed on H_n between e_i and e_j is computed by

$$\beta_{n,|i-j|}(a_l) = |\beta_{n,i}(a_l) - \beta_{n,j}(a_l)| \tag{9}$$

Definition 2. Suppose the belief degree of dissimilarity is calculated by Eq. (9), the *distributed dissimilarity vector* between $S(e_i(a_l))$ and $S(e_j(a_l))$ denoted by $GD(e_{ij}(a_l))$ is defined as follows:

$$\begin{aligned} &GD(e_{ij}(a_l)) \\ &\quad = ((H_1, \beta_{1,|i-j|}(a_l)), (H_2, \beta_{2,|i-j|}(a_l)), \cdots, (H_N, \beta_{N,|i-j|}(a_l))) \\ &\qquad (i, j = 1, 2, \cdots, L) \end{aligned} \tag{10}$$

The distributed dissimilarity vector in Def.2 provides us a panoramic view of the discrepancy between two attributes on a specific alternative. It is obvious that when $i = j$, $\beta_{n,|i-j|}(a_l) = 0 (n = 1, 2, \cdots, N)$ which leads to $GD(e_{ij}(a_l)) = (0, 0, \cdots, 0)$. Based on Eq. (10), the *distributed dissimilarity matrix* between $S(e_i(\boldsymbol{A}))$ and $S(e_j(\boldsymbol{A}))$ can be denoted by

$$\boldsymbol{GD} = [GD(e_{ij}(\boldsymbol{A}))]_{S\times N} = (GD(e_{ij}(a_1)), GD(e_{ij}(a_2)), \cdots, GD(e_{ij}(a_S)))^T \tag{11}$$

Definition 3. Given the distributed dissimilarity vector defined in Eq. (10), an *improved dissimilarity measure* between $S(e_i(a_l))$ and $S(e_j(a_l))$ is proposed as follows:

$$D(e_{ij}(a_l)) = \frac{1}{u(H_N) - u(H_1)} \sum_{n=1}^{N-1} \sum_{s=n+1}^{N} \beta_{n,|i-j|}(a_l)\beta_{s,|i-j|}(a_l)u(H_{|s-n|}) \tag{12}$$

Def. 3 defines the dissimilarity measure between two attributes associated with an alternative. Here, a multiplier $\frac{1}{u(H_N)-u(H_1)}$ is added in the dissimilarity measure for the purpose that the maximum value of $D(e_{ij}(a_l))$ attains to 1. For example, if the belief degree of e_i assessed on a_l to grade H_1 is 1 such that $\beta_{1,i}(a_l) = 1$, and the belief degree that e_j be evaluated on a_l to grade H_N is 1 such that $\beta_{N,j}(a_l) = 1$, then from Eq. (9) we have $\beta_{1,|i-j|}(a_l) = 1$, $\beta_{N,|i-j|}(a_l) = 1$ and $\beta_{n,|i-j|}(a_l) = 0 (n = 2, 3, \cdots, N-1)$. If $\frac{1}{u(H_N)-u(H_1)}$ is not multiplied, $D(e_{ij}(a_l)) = u(H_N) - u(H_1)$. Since the utility of $H_n (n = 1, 2, 3, \cdots, N)$ reflects the subjective judgment of an individual or a group of DMs, the utility of H_1 may be more than 0, whereas the utility of H_N may be less than one considering the personal difference on background, knowledge and risk preference. When $u(H_1) > 0$ or $u(H_N) < 1$, $D(e_{ij}(a_l)) < 1$. So the largest dissimilarity between two BDs will not certainly lead to the result that $D(e_{ij}(a_l)) = 1$. Comparatively, the proposed dissimilarity in Eq. (12) equals to one when $\beta_{1,|i-j|}(a_l) = 1$ and $\beta_{N,|i-j|}(a_l) = 1$ no matter how the utility of H_1 or H_N is set. So the improved dissimilarity measure defined in Eq. (12) has the following properties:

1 > $0 \le D(e_{ij}(a_l)) \le 1$;
2 > $D(e_{ij}(a_l)) = D(e_{ji}(a_l))$;
3 > $D(e_{ij}(a_l)) = 1$ iff $\beta_{1,|i-j|}(a_l) = 1$ and $\beta_{N,|i-j|}(a_l) = 1$;
4 > $D(e_{ij}(a_l)) = 0$ iff $\forall\, n \in \{1, 2, \cdots, N\}, \beta_{n,|i-j|}(a_l) = 0$;
5 > $D(e_{ii}(a_l)) = 0$.

Here, Property 2 > is the symmetry of the measure. Property 3 > and 4 > indicate that $D(e_{ij}(a_l))$ defined in Eq. (12) is bounded between 0 and 1. Property 5 > is the reflexivity of the dissimilarity measure.

Definition 4. Suppose the distributed dissimilarity vector between two BDs $S(e_i(a_l))$ and $S(e_j(a_l))$ is denoted by Eq. (10). Then based on the improved dissimilarity measure given in Def.3, the *conflict measure on the alternative level* between two attributes e_i and e_j $(i, j = 1, 2, \cdots, L)$ respect to all the S alternatives is defined as follows:

$$Con^1(e_i, e_j) = \frac{1}{S}\sum_{l=1}^{S} D(e_{ij}(a_l)) \tag{13}$$

Remark 1. If $\forall\, l \in \{1, 2, \cdots, S\}$, $D(e_{ij}(a_l)) = 1$, then from Def.4, we will have $Con^1(e_i, e_j) = 1$; On the contrary, $\forall\, l \in \{1, 2, \cdots, S\}$, $D(e_{ij}(a_l)) = 0$, we will have $Con^1(e_i, e_j) = 0$. So the following properties are satisfied:

1 > $0 \le Con^1(e_i, e_j) \le 1$;
2 > $Con^1(e_i, e_j) = Con^1(e_j, e_i)$;
3 > $Con^1(e_i, e_i) = 0$;
4 > $Con^1(e_i, e_j) = 1$ iff $\forall\, l \in \{1, 2, \cdots, S\}, D(e_{ij}(a_l)) = 1$;
5 > $Con^1(e_i, e_j) = 0$ iff $\forall\, l \in \{1, 2, \cdots, S\}, D(e_{ij}(a_l)) = 0$.

When $Con^1(e_i, e_j) = 1$, it is clear that the BDs between e_i and e_j on each alternative are completely opposite, that is, $\forall\, l \in \{1, 2, \cdots, S\}$, $GD(e_{ij}(a_l)) = (1, 0, 0\cdots, 0, 1)$. The more discordant the BDs of the S alternatives on e_i and e_j, the higher the value of $Con^1(e_i, e_j)$.

Example 2. Given the frame of discernment $H = \{H_1, H_2, \cdots, H_5\}$. Table 1 shows the BDs of two attributes e_1 and e_2 on two alternatives a_1 and a_2. It is clear that there is no uncertainty and ignorance contained in any of the four BDs. From Eq. (12), we have $D(e_{12}(a_1)) = D(e_{12}(a_2)) = 1$. And the conflict measure on the alternative level between e_1 and e_2 denoted by $Con^1(e_1, e_2)$ equals to 1.

In this case, a_1 and a_2 cannot be compared only from e_1 and e_2 because they are both assessed to be bad on one attribute and excellent on the other one. According to (Fu et al., 2018), the dissimilarity based discriminating power on e_i represented by $\bar{D}(e_i)$ can be generated as follows:

$$\bar{D}(e_i) = \frac{1}{S}\sum_{l=1}^{S} \bar{D}(e_i(a_l)) \tag{14}$$

Here, $\bar{D}(e_i(a_l))$ denotes the average dissimilarity measure between $S(e_i(a_l))$ and the BDs of e_i on other $S-1$ alternatives represented by $S(e_i(a_m)) (m = 1, 2, \cdots, S, m \ne l)$, and it is computed by

$$\bar{D}(e_i(a_l)) = \frac{1}{S-1}\sum_{m=1}^{S} D'(e_i(a_{lm})) \tag{15}$$

In Eq. (15), $D'(e_i(a_{lm}))$ is generated from $D(e_i(a_{lm}))$ in Eq. (7) which is multiplied by $\frac{1}{u(H_N)-u(H_1)}$. Here, $\bar{D}(e_1) = \bar{D}(e_2) = 1$.

Example 3. Similar with Example 2, the BDs of e_1 and e_2 on a_1 and a_2 shown in Table 2 are also certain and complete. Differently, the two alternatives are assessed to be the same on either e_1 or e_2. Here, $D(e_{12}(a_1)) = D(e_{12}(a_2)) = 1$, and $Con^1(e_1, e_2) = 1$ which are identical with Example 2.

Because there is no difference between the BDs of the two alternatives on $e_i (i = 1, 2)$, $\bar{D}(e_1) = \bar{D}(e_2) = 0$. So a_1 and a_2 cannot be simply compared only from e_1 and e_2. According to the objective weight assignment method such as entropy method, SD method or discriminating power method, the weight of e_1 and e_2 is 0 in both Example 2 and 3 if there are other attributes included in the assessment and no more alternatives involved. It is not reasonable because the conflict measure

Table 1
Belief distributions of two attributes on two alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0	a_1	0	0	0	0	1
a_2	0	0	0	0	1	a_2	1	0	0	0	0

Table 2
Belief distributions of two attributes on two alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0	a_1	0	0	0	0	1
a_2	1	0	0	0	0	a_2	0	0	0	0	1

between e_1 and e_2 is 1 in the two cases. The difference lies in that two opposite assessments are possessed by each attribute respect to the two alternatives in Example 2, while the BDs of different alternatives on a specific attribute are the same in Example 3. In sum, the second dimension which measures the conflict between the BDs of each pair of attributes should be considered in the generation of weights, especially when the first dimension which quantifies the discrepancy among different alternatives on a specific attribute cannot discriminate the importance of different attributes properly.

Example 4. Given the frame of discernment $H = \{H_1, H_2, \cdots, H_5\}$. The BDs of three attributes e_1, e_2 and e_3 on five alternatives $a_i (i = 1, 2, \cdots, 5)$ are shown in Table 3. It is clear that a_1 is the worst one because its belief degree on H_1 for all the three attributes are 1, while a_4 is the best option since it is assessed to H_5 with the belief degree of 1 on each of the three attributes.

In this example, the dissimilarity based discriminating power on the three attributes are identical such that $\bar{D}(e_1) = \bar{D}(e_2) = \bar{D}(e_3) = 0.6$. The SD or Gini's mean difference (GMD) based discriminating power (Fu et al., 2018) in this case are also identical on the three attributes such that $\bar{S}(e_1)=\bar{S}(e_2) = \bar{S}(e_3) = 0.2$; $\bar{G}(e_1)=\bar{G}(e_2) = \bar{G}(e_3) = 0.36$. So the objective weights generated from any one of the three discriminating powers will be the same such that $w_1 = w_2 = w_3 = \frac{1}{3}$. But the conflict measure on the alternative level between each pair of the three attributes is different, i.e., $Con^1(e_1, e_2) = 0.4$, $Con^1(e_1, e_3) = 0.6$, $Con^1(e_2, e_3) = 0.2$. According to the viewpoint of CRITIC and CCSD, the correlation between each pair of the three attributes should be considered in the process of generating weights. So the discrepancy on the correlation among the three attributes will lead to the difference on attribute weights.

Based on the conflict measure defined in Def.4, a *conflict measure matrix* can be constructed as follows:

$$\begin{pmatrix} Con^1(e_1, e_1) & Con^1(e_1, e_2) & \cdots & Con^1(e_1, e_L) \\ Con^1(e_2, e_1) & Con^1(e_2, e_2) & \cdots & Con^1(e_2, e_L) \\ \cdots & \cdots & \cdots & \cdots \\ Con^1(e_L, e_1) & Con^1(e_L, e_2) & \cdots & Con^1(e_L, e_L) \end{pmatrix} \tag{16}$$

It's a symmetric matrix that reflects the conflict measure of each pair of attributes. From remark 1, the elements on the diagonal line of the matrix in Eq. (16) are 0. If the average conflict measure between e_i and other $L-1$ attributes is large, e_i should probably be given a higher weight. Thus, the average conflict measure is defined as follows.

Definition 5. Given the dissimilarity measure and conflict measure defined in Eqs. (12) and (13) respectively, the *average conflict measure on the alternative level* created from e_i with respect to other $L-1$ attributes is defined as follows:

$$Con^1(e_i) = \frac{1}{L-1} \sum\nolimits_{j=1}^{L} Con^1(e_i, e_j) \tag{17}$$

Properties:

1 > $0 \le Con^1(e_i) \le 1$;
2 > $Con^1(e_i) = 1$ iff $\forall j \in \{1, 2, \cdots, L\} j \ne i$, $Con^1(e_i, e_j) = 1$;
3 > $Con^1(e_i) = 0$ iff $\forall j \in \{1, 2, \cdots, L\}$, $Con^1(e_i, e_j) = 0$.

Property 2 > is a special case that all attributes except e_i have the same BDs on each alternative, while the BDs between e_i and any one of the other $L-1$ attributes on each alternative are absolutely incompatible, i.e., $\forall\ j \in \{1, 2, \cdots, i-1, i+1, \cdots, L\}$; $l \in \{1, 2, \cdots, S\}$

$$GD(e_{ij}(a_l)) = ((H_1, 1), (H_2, 0), \cdots, (H_n, 0), \cdots, (H_{N-1}, 0), (H_N, 1))$$

and $\forall\ j, k \in \{1, 2, \cdots, i-1, i+1, \cdots, L\}$; $l \in \{1, 2, \cdots, S\}$

$$GD(e_{jk}(a_l)) = ((H_1, 0), (H_2, 0), \cdots, (H_n, 0), \cdots, (H_{N-1}, 0), (H_N, 0))$$

Otherwise, $Con^1(e_i) < 1$. Property 3 > indicates that all the L attributes have the same BDs on each alternative although it rarely happens in real MADM problems.

3.2. Conflict measure on the evaluation grade level

Given the belief degree of dissimilarity defined in Eq. (9), then the *belief degree vector of dissimilarity* on H_n between e_i and e_j considering all the S alternatives is denoted by

$$\beta_{n,|i-j|}(\mathbf{A}) = (\beta_{n,|i-j|}(a_1), \beta_{n,|i-j|}(a_2), \cdots, \beta_{n,|i-j|}(a_S))^T \tag{18}$$

So the distributed dissimilarity matrix shown in Eq. (11) can be represented by $\mathbf{GD} = (\beta_{1,|i-j|}(\mathbf{A}), \beta_{2,|i-j|}(\mathbf{A}), \cdots, \beta_{N,|i-j|}(\mathbf{A}))$. As such, the *average belief degree of dissimilarity* on H_n between e_i and e_j is defined as follows:

$$\bar{\beta}_{n,|i-j|}(a_l) = \frac{1}{S} \sum\nolimits_{l=1}^{S} \beta_{n,|i-j|}(a_l) \tag{19}$$

Definition 6. Suppose the average belief degree of dissimilarity is

Table 3
Belief distributions of three attributes on five alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0	a_1	1	0	0	0	0
a_2	0	0	0	0	1	a_2	1	0	0	0	0
a_3	1	0	0	0	0	a_3	1	0	0	0	0
a_4	0	0	0	0	1	a_4	0	0	0	0	1
a_5	1	0	0	0	0	a_5	0	0	0	0	1

e_3	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0
a_2	1	0	0	0	0
a_3	0	0	0	0	1
a_4	0	0	0	0	1
a_5	0	0	0	0	1

computed by Eq. (19) based on the belief degree vector of dissimilarity presented in Eq. (18), then the *conflict measure on the evaluation grade level* between e_i and e_j respect to all the S alternatives is defined by

$$Con^2(e_i, e_j) = \frac{1}{u(H_N) - u(H_1)} \sum_{n=1}^{N-1} \sum_{s=n+1}^{N} \bar{\beta}_{n,|i-j|}(a_l)\bar{\beta}_{s,|i-j|}(a_l)u(H_{|s-n|}) \tag{20}$$

Properties:

1 > $0 \le Con^2(e_i, e_j) \le 1$;
2 > $Con^2(e_i, e_j) = Con^2(e_j, e_i)$;
3 > $Con^2(e_i, e_i) = 0$;

4 > $Con^2(e_i, e_j) = 1$ iff $\bar{\beta}_{n,|i-j|}(a_l) = 1$ for $n = 1, N$ and $\bar{\beta}_{n,|i-j|}(a_l) =$; 0 for $n = 2, \cdots, N-1$
5 > $Con^2(e_i, e_j) = 0$ iff $\forall\, l \in \{1, 2, \cdots, S\}$, $S(e_i(a_l)) = S(e_j(a_l))$.

Property 4 > and 5 > represent two special cases, i.e., the evaluation values of two attributes are either completely conflicting or completely consistent. In Property 5 > , $S(e_i(a_l)) = S(e_j(a_l))$ will lead to the result that $\bar{\beta}_{n,|i-j|}(a_l) = 0$ for $n = 1, 2, \cdots, N$, while the condition in Property 4 > is caused by the fact that $GD(e_{ij}(a_l)) = (1, 0, 0\cdots, 0, 1)$ ($\forall\, l \in \{1, 2, \cdots, S\}$).

Definition 7. Suppose the conflict measure is calculated by Eq. (20), then the *average conflict measure on the evaluation grade level* created from e_i is computed by

$$Con^2(e_i) = \frac{1}{L-1} \sum_{j=1}^{L} Con^2(e_i, e_j) \tag{21}$$

Example 5. Given the frame of discernment $H = \{H_1, H_2, \cdots, H_5\}$. Table 4 shows the BDs of four attributes $e_i (i = 1, 2, 3, 4)$ on five alternatives $a_l (l = 1, 2, \cdots, 5)$. It is clear that the BDs of e_2, e_3 and e_4 are identical with each other on all alternatives, while the BD of e_1 on any one of the five alternatives is completely opposite to e_2, e_3 and e_4. From Eq. (12), we have $D(e_{1j}(a_l)) = 1$ for $j = 2, 3, 4$; $l = 1, 2, \cdots, 5$, and $D(e_{ij}(a_l)) = 0$ for $i, j = 2, 3, 4$. The conflict measure on the alternative level between e_1 and $e_j (j = 2, 3, 4)$ denoted by $Con^1(e_1, e_j)$ equals to 1, while $Con^1(e_i, e_j) = 0$ for $i, j = 2, 3, 4$. So the average conflict measure defined in Eq. (17) can be generated as: $Con^1(e_1) = 1$, $Con^1(e_i) = \frac{1}{3}(i = 2, 3, 4)$. By Eq. (20), the conflict measure on the evaluation grade level between e_i and e_j can also be generated as follows: $Con^2(e_1, e_j) = 1 (j = 2, 3, 4)$, $Con^2(e_i, e_j) = 0$ for $i, j = 2, 3, 4$. Then the average conflict measure defined in Eq. (21) is $Con^2(e_1) = 1$, $Con^2(e_i) = \frac{1}{3}(i = 2, 3, 4)$.

3.3. Comparisons with other conflict measures

The conflict measures between two pieces of evidence in the mathematical framework of evidence theory have been widely studied in recent years. It is also termed as distance or dissimilarity. The conflict belief K in the Dempster's combination rule (Ronald & Yager, 2018) has previously been regarded as the sole quantification of conflict measure between BDs. But this factor can be inappropriate to be a dissimilarity measure because some counterintuitive results may be generated, especially when two pieces of evidence are identical. Afterwards, more than 15 popular dissimilarity measures have been proposed. Jousselme classified them into five categories (Jousselme & Maupin, 2012), including the composite distances (Tessem, 1993), the Minkowski family (Jousselme et al., 2001), the inner product family (Ristic & Smets, 2006), the fidelity family (Florea & Bossé, 2009), the information-based distances (Denoeux, 2000) and the two-dimensional distances (Liu, 2006).

Here, we divided them into three categories: (1) the probability-based distance such as Tessem's distance (Tessem, 1993) and combined dissimilarity measure (Liu et al., 2011); (2) the mass-based distance such as Jousselme's distance (Jousselme et al., 2001), Cosine similarity (Wen et al., 2008) and correlation coefficient (Jiang, 2018); (3) distance considering two factors such as Liu's distance (Liu, 2006). Up to now, there is no one measure that is perfect to quantify the dissimilarity between two BDs in all cases. That is to say, each one of them has its advantages together with some weakness in specific situation. For instance, Tessem's distance and Tanimoto's similarity is invalid when the Pignistic probability of two pieces of evidence is identical even though the BDs may be different. Jousselme's distance may generate counter-intuitive result when the intersection of the focal elements associated with two BDs is empty. Cosine similarity may generate irrational result when there is no identical focal element for two BDs although the cores of the two BDs have common elements.

It is not within the scope of this paper to conduct a comprehensive survey of the existing conflict measures. The big advantage of the conflict measure proposed in Defs.4 and 6 lies in the consideration of the utilities of evaluation grades against the non-definition on the utility of hypotheses included in the frame of discernment in any of the above mentioned measures. So the above listed conflict/dissimilarity measures cannot be applied in a MADM or GDM problem unless the utilities of different evaluation grades are distinguished. Instead, they are more applicable in the situation such as sensor fusion and target identification where each element in the frame of discernment represents a probable target rather than an evaluation grade.

Example 6 Given the frame of discernment = {Worst(H_1), Poor(H_2), Average(H_3), Good(H_4), Excellent(H_5)}. The utilities of the five evaluation grades are set to be $u(H_1) = 0$, $u(H_2) = 0.25$, $u(H_3) = 0.5$, $u(H_4) = 0.75$, $u(H_5) = 1$. Table 5 shows the BDs of three attributes $e_i (i = 1, 2, 3)$ on five alternatives

Table 4
Belief distributions of four attributes on five alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0	a_1	0	0	0	0	1
a_2	0	0	0	0	1	a_2	1	0	0	0	0
a_3	1	0	0	0	0	a_3	0	0	0	0	1
a_4	0	0	0	0	1	a_4	1	0	0	0	0
a_5	1	0	0	0	0	a_5	0	0	0	0	1
e_3	H_1	H_2	H_3	H_4	H_5	e_4	H_1	H_2	H_3	H_4	H_5
a_1	0	0	0	0	1	a_1	0	0	0	0	1
a_2	1	0	0	0	0	a_2	1	0	0	0	0
a_3	0	0	0	0	1	a_3	0	0	0	0	1
a_4	1	0	0	0	0	a_4	1	0	0	0	0
a_5	0	0	0	0	1	a_5	0	0	0	0	1

Table 5
Belief distributions of three attributes on five alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5	e_3	H_1	H_2	H_3	H_4	H_5
a_1	1	0	0	0	0	a_1	0	0	0	0	1	a_1	0	0	0	0	1
a_2	0	0	0	0	1	a_2	1	0	0	0	0	a_2	0	0	1	0	0
a_3	1	0	0	0	0	a_3	0	0	0	0	1	a_3	0	0	0	0	1
a_4	0	0	0	0	1	a_4	1	0	0	0	0	a_4	0	0	1	0	0
a_5	1	0	0	0	0	a_5	0	0	0	0	1	a_5	0	0	0	0	1

$a_l (l = 1, 2, \cdots, 5)$. It is clear that the BD of e_1 on any one of the five alternatives is completely opposite to e_2.

According to Def.2, the distributed dissimilarity matrix is generated and shown in Table 6. From Eq. (12), we have $D(e_{12}(a_l)) = 1$ for $l = 1, 2, \cdots, 5$, $D(e_{13}(a_l)) = 1 (l = 1, 3, 5)$ and $D(e_{13}(a_l)) = 0.5 (l = 2, 4)$. So the conflict measure on the alternative level between e_1 and e_2 defined in Eq. (13) denoted by $Con^1(e_1, e_2)$ equals to 1, while $Con^1(e_1, e_3) = 0.8$, $Con^1(e_2, e_3) = 0.2$. As such, the average conflict measure defined in Eq. (17) is generated as: $Con^1(e_1) = 0.9$, $Con^1(e_2) = 0.6$, $Con^1(e_3) = 0.5$. By Eq. (20), the conflict measure on the evaluation grade level between e_i and e_j can also be generated as follows: $Con^2(e_1, e_2) = 1$, $Con^2(e_1, e_3) = 0.92$, $Con^2(e_2, e_3) = 0.08$. Then the average conflict measure defined in Def.7 is $Con^2(e_1) = 0.96$, $Con^2(e_2) = 0.54$, $Con^2(e_3) = 0.5$. Obviously, $D(e_{12}(a_2)) \neq D(e_{13}(a_2))$, $D(e_{12}(a_4)) \neq D(e_{13}(a_4))$. It is reasonable because the dissimilarity between e_1 and e_2 respect to a_2 or a_4 is larger than that between e_1 and e_3. If we use any one of the above mentioned dissimilarity measures to replace Eq. (12), the induced value of conflict measure between e_1 and e_2 will be 1, which is equal to that between e_1 and e_3.

3.4. Comprehensive weight assignment method

Just as discussed in (Fu et al., 2018) and (Chin et al., 2015), the larger the discriminating power or deviation incompatibility of an attribute, the larger weight should be assigned. It is in accord with the perspective of EWAM (Song et al., 2017; Zhou et al., 2019) and SD method (Chin et al., 2015; Diakoulaki et al., 1995) where the EWAM or SD is the measurement of discrepancy among different alternatives on a specific attribute. The CRITIC or CCSD method adds a second aspect of intrinsic information contained in attributes, namely the conflict among attributes or from an opposite point of view, the correlation between each pair of attributes. It is known to us all that the more conflict between the values of an attribute and any other attributes respect to all alternatives, the attribute is to be assigned with a higher weight. But it should be mentioned that the correlation in either the CRITIC or CCSD method is derived from crisp numerical values. When the assessments of attributes are represented by BDs, the Spearman correlation coefficient cannot be directly used. For instance, given the BDs of two attributes be assessed on five alternatives which are shown in Table 7. The utilities of the five evaluation grades are set to be $u(H_1) = 0$, $u(H_2) = 0.25$, $u(H_3) = 0.5$, $u(H_4) = 0.75$, $u(H_5) = 1$. So the utilities of $a_l (l = 1, 2, \cdots, 5)$ on e_1 and e_2 are identical such that $u(e_1(a_1)) = u(e_2(a_1)) = 0.25$, $u(e_1(a_2)) = u(e_2(a_2)) = 0.5$, $u(e_1(a_3)) = u(e_2(a_3)) = 0.75$, $u(e_1(a_4)) = u(e_2(a_4)) = 0.5$, $u(e_1(a_5)) = u(e_2(a_5)) = 1$. The intrinsic information contained in the BDs is not preserved when the utility based correlation coefficient is applied. The conflict measure calculated by Eq. (13) is 0.375. It is reasonable because the dissimilarity between e_1 and e_2 on all alternatives in Eq. (13) is extracted from the BDs although the utilities of each alternative on e_1 and e_2 are the same. So the rationality of the method to measure the correlation or conflict between attributes represented by BDs is significant for objectively obtaining the weights of attributes.

It should be mentioned that results created by the average conflict measure on the alternative and evaluation grade level are not necessarily the same in any case. So a weighted averaging operation for the two conflict measures is conducted as follows:

$$Con(e_i) = \sum_{k=1}^{2} \delta_k Con^k(e_i) \tag{22}$$

where $0 \leq \delta_k \leq 1 (k = 1, 2)$, and $\sum_{k=1}^{2} \delta_k = 1$. Here, we call $Con(e_i)$ the *weighted average conflict measure*. When $\delta_1 = 1$, $Con(e_i)$ becomes the average conflict measure on the alternative level. Conversely, Eq. (22) becomes the average conflict measure on the evaluation grade level provided that $\delta_2 = 1$. So the selection of δ_k will determine the value of $Con(e_i)$ to some extent.

In (Fu et al., 2018), a combination of the dissimilarity, SD and GMD based discriminating power is conducted. Inspired by the entropy-based weight assignment method (Zhou et al., 2019), the entropy based discriminating power is added as the fourth factor. Then the *comprehensive discriminating power* on e_i can be computed as follows:

$$D(e_i) = \theta^D \bar{D}(e_i) + \theta^S \bar{S}(e_i) + \theta^G \bar{G}(e_i) + \theta^E \bar{E}(e_i) \tag{23}$$

In Eq. (23), $\bar{S}(e_i)$, $\bar{G}(e_i)$ and $\bar{E}(e_i)$ signify the SD, GMD and entropy

Table 6
Distributed dissimilarity matrix between attributes.

$GD(e_{12}(a_l))$	H_1	H_2	H_3	H_4	H_5	$D(e_{12}(a_l))$	$GD(e_{13}(a_l))$	H_1	H_2	H_3	H_4	H_5	$D(e_{13}(a_l))$
a_1	1	0	0	0	1	1	a_1	1	0	0	0	1	1
a_2	1	0	0	0	1	1	a_2	0	0	1	0	1	0.5
a_3	1	0	0	0	1	1	a_3	1	0	0	0	1	1
a_4	1	0	0	0	1	1	a_4	0	0	1	0	1	0.5
a_5	1	0	0	0	1	1	a_5	1	0	0	0	1	1
$\bar{\beta}_{n,\|1-2\|}(a_l)$	1	0	0	0	1		$\bar{\beta}_{n,\|1-3\|}(a_l)$	0.6	0	0.4	0	1	

$GD(e_{23}(a_l))$	H_1	H_2	H_3	H_4	H_5	$D(e_{23}(a_l))$
a_1	0	0	0	0	0	0
a_2	1	0	1	0	0	0.5
a_3	0	0	0	0	0	0
a_4	1	0	1	0	0	0.5
a_5	0	0	0	0	0	0
$\bar{\beta}_{n,\|2-3\|}(a_l)$	0.4	0	0.4	0	0	

Table 7
Belief distributions of two attributes on five alternatives.

e_1	H_1	H_2	H_3	H_4	H_5	e_2	H_1	H_2	H_3	H_4	H_5
a_1	0.5	0	0.5	0	0	a_1	0	1	0	0	0
a_2	0	0.5	0	0.5	0	a_2	0	0	1	0	0
a_3	0	0	0.5	0	0.5	a_3	0	0	0	1	0
a_4	0.5	0	0	0	0.5	a_4	0	0	1	0	0
a_5	0	0	0	0	1	a_5	0	0	0	0	1

based discriminating power respectively. $0 \le \theta^h \le 1 (h = D, S, G, E)$, and $\theta^D + \theta^S + \theta^G + \theta^E = 1$. It is obvious that different assignment of θ^h will lead to different value of $D(e_i)$. Based on the weighted average conflict measure and the comprehensive discriminating power on e_i, the weight of e_i can be generated by the following equation:

$$w_i = \frac{D(e_i) * \sqrt{Con(e_i)}}{\sum_{j=1}^{L} D(e_i) * \sqrt{Con(e_i)}} \quad (24)$$

It need to be mentioned that if the BDs of e_i on all S alternatives are identical, $\bar{D}(e_i)$, $\bar{S}(e_i)$, $\bar{G}(e_i)$ or $\bar{E}(e_i)$ equals to 0 which means e_i can be deleted before generating the attribute weights. A special case is that $\forall\ i \in \{1, 2, \cdots, L\}$, $D(e_i) = 0$, which means all the S alternatives are assessed to be the same on any attribute. In this situation, the attribute weights cannot be generated by Eqs. (22)–(24), but it rarely happens in real decision-making problems because any two alternatives do have differentia on some attributes.

4. Dissimilarity measure on alternative

In a decision-making problem, the support degree of assessment for the final decision is pivotal because it determines the extent that we can rely on the aggregated result. However, if the assessment value of each alternative differs significantly over different attributes, it may be difficult to make a ranking order firmly. On the contrary, it is easier to make a choice provided that the BDs of attributes achieve a high degree of consistency no matter the assessment is positive or negative. Specifically, if an alternative is assessed on some attributes with high grades, whereas it is assumed to be relatively poor on some other attributes, the divergence of different attributes should be considered before the final decision be made only depending on the aggregated assessment. So how to measure the total distinction among different attributes associated with an alternative is significant for the final selection or ranking. In this section, the definition of dissimilarity measure on alternative is given to calculate the incompatibility of the BDs on different attributes assessed to an alternative. We call it the external divergence of different attributes which is the first factor to influence the support degree of assessment. Thus, the smaller the value of the dissimilarity measure on an alternative, the easier to give a final judgment on the alternative, and vice versa.

4.1. Dissimilarity measure on alternative

The *dissimilarity measure matrix* on a_l can be constructed based on Eq. (12) as follows:

$$\boldsymbol{D} = [D(e_{ij}(a_l))]_{L\times L} = \begin{pmatrix} D(e_{11}(a_l)) & D(e_{12}(a_l)) & \cdots & D(e_{1L}(a_l)) \\ D(e_{21}(a_l)) & D(e_{22}(a_l)) & \cdots & D(e_{2L}(a_l)) \\ \cdots & \cdots & \cdots & \cdots \\ D(e_{L1}(a_l)) & D(e_{L2}(a_l)) & \cdots & D(e_{LL}(a_l)) \end{pmatrix} \quad (25)$$

It is a non-negative symmetric matrix where the diagonal line value is zero.

Definition 8. Given the dissimilarity measure matrix on a_l shown in Eq. (25), the *average dissimilarity measure* between $S(e_i(a_l))$ and the BDs of a_l on other $L-1$ attributes can be determined by

$$D(e_i(a_l)) = \frac{1}{L-1}\sum_{j=1}^{L} D(e_{ij}(a_l)) \quad (26)$$

The average dissimilarity measure defined in Def.8 reflects the discrepancy that a_l be assessed on e_i and other $L-1$ attributes. It is a measure of incompatibility between the subjective judgment of a specific attribute and the BDs of other attributes on an alternative. The larger the value of $D(e_i(a_l))$, the more inconsistency between the BDs of e_i and other attributes on a_l.

Properties:

1 > $0 \le D(e_i(a_l)) \le 1$;
2 > $D(e_i(a_l)) = 1$ iff $\forall\ j \in \{1, 2, \cdots i-1, i+1, \cdots, L\}$, $D(e_{ij}(a_l)) = 1$;
3 > $D(e_i(a_l)) = 0$ iff $\forall\ j \in \{1, 2, \cdots, L\}$, $D(e_i(a_l)) = D(e_j(a_l))$.

Property 2 > indicates that $D(e_i(a_l))$ attains to the maximum value when e_i is assessed to be completely opposite with other $L-1$ attributes on a_l such that $\beta_{N,i}(a_l) = 1$ and $\beta_{1,j}(a_l) = 1 (j = 1, 2, \cdots, i-1, i+1, \cdots, L)$, or $\beta_{1,i}(a_l) = 1$ and $\beta_{N,j}(a_l) = 1 (j = 1, 2, \cdots, i-1, i+1, \cdots, L)$. Otherwise, $D(e_i(a_l)) < 1$, especially when all the attributes are assessed to be the same, we will have $D(e_i(a_l)) = 0$ no matter the assessment is certain or contain ignorance.

Definition 9. Based on the average dissimilarity measure proposed in Def.8, the *dissimilarity measure on alternative* a_l is defined as follows:

$$D(a_l) = \frac{1}{L}\sum_{i=1}^{L} D(e_i(a_l)) \quad (27)$$

Properties:

1 > $0 \le D(a_l) \le 1$;
2 > $D(a_l) = 0$ iff $\forall\ i, j \in \{1, 2, \cdots, L\}$, $D(e_{ij}(a_l)) = 0$;
3 > $D(a_l) = 1$ iff $L = 2$, and $D(e_{12}(a_l)) = 1$.

The dissimilarity measure on an alternative defined in Eq. (27) measures the divergence of the BDs that an alternative be assessed to different attributes. The smaller the value of $D(a_l)$, the higher consistency among different attributes assessed on an alternative, and vice versa. So it is easier for a DM to conclude a comprehensive judgment on an alternative when $D(a_l)$ is small. From another point of view, when the aggregating assessments of two alternatives are similar, $D(a_l)$ could be considered for the comparison of them. It has the similar meaning with standard deviation where the assessments of e_i on a_l are numerical values. Property 2 > is a special case that the BDs of a_l on all the L attributes are the same which leads to $D(a_l) = 0$. It should be mentioned that even if the BDs of all the attributes are the same, the aggregated BD may be probably different with the BDs of original attributes when the ER approach is applied. Moreover, if $D(a_l) = D(a_k)(l, k \in \{1, 2, \cdots, L\}; l \neq k)$, we cannot conclude that $D(e_i(a_l)) = D(e_i(a_k))$ ($\forall\ i \in \{1, 2, \cdots, L\}$). The reason lies in that $D(a_l)$ is the general dissimilarity measure between every two attributes respect to a_l. So $D(e_i(a_l))$ may be worse than $D(e_i(a_k))$ on some attributes, while better than $D(e_i(a_k))$ on other attributes which may lead to the result that $D(a_l)$ equals to $D(a_k)$.

Example 7. Suppose the BDs that a_l be assessed on e_1, e_2 and e_3 are given in Table 8. Let the BDs on e_2 and e_3 be fixed, while the BD on e_1 be changing steadily, i.e., for $l = 1, 2, \cdots, 10$, $\beta_{5,1}(a_l) = 1 - 0.1\bullet l$ and $\beta_{4,1}(a_l) = 0.1\bullet l$, $\beta_{4,1}(a_l^{'}) = 1 - 0.1\bullet l$ and $\beta_{3,1}(a_l^{'}) = 0.1\bullet l$, $\beta_{3,1}(a_l^{''}) = 1 - 0.1\bullet l$ and $\beta_{2,1}(a_l^{''}) = 0.1\bullet l$, $\beta_{2,1}(a_l^{'''}) = 1 - 0.1\bullet l$ and $\beta_{1,1}(a_l^{'''}) = 0.1\bullet l$. The dissimilarity measures on alternatives a_l, $a_l^{'}$, $a_l^{''}$ and $a_l^{'''}$ $(l = 1, 2, \cdots, 10)$ can then be calculated by Eq. (27) and shown in Fig. 1. Obviously, with the difference between the BDs of e_1 and

$e_i(i=2,3)$ increases gradually from a_1 to $a_{10}^{'''}$, the dissimilarity measure also increases. When the BD of e_1 is assessed on $a_{10}^{'''}$ such that $\beta_{1,1}(a_{10}^{'''})=1$ and $\beta_{n,1}(a_{10}^{'''})=0(n=2,3,4,5)$, it attains the maximum dissimilarity measure of 0.667.

Here, the dissimilarity measure can also be calculated by the standard deviation as follows:

$$D'(a_l)=\frac{1}{u(H_N)-u(H_1)}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\sigma_n(a_l)\sigma_s(a_l)u(H_{|s-n|}) \tag{28}$$

where $\sigma_n(a_l)$ denotes the standard deviation of the belief degree that a_l be assigned to H_n on all L attributes.

$$\sigma_n(a_l)=\sqrt{\frac{1}{L}\sum_{i=1}^{L}\left(\beta_{n,i}(a_l)-\frac{\sum_{i=1}^{L}\beta_{n,i}(a_l)}{L}\right)^2}\ (n=1,2,\cdots,N) \tag{29}$$

It can be proved easily that Eq. (28) also satisfies the three properties listed for Eq. (27).

4.2. Considering the incompleteness of BDs

The above conflict measure or dissimilarity measure assumes that the BDs are all complete assessments. In other words, there is no ignorance included in any one of the BDs. However, incompleteness and ignorance are common in real decision-making problems, such as GDM situation (Hwang & Lin, 1987; Zhou et al., 2018), lack of professional knowledge from experts, unreliability of equipment where data are acquired from, and so on. In these circumstances, the ignorance presented in Eq. (2) is more than 0 such that $\beta_{H,i}(a_l)>0$. Here, the conflict measure on the alternative level and dissimilarity measure on alternative will be discussed in the situation that BDs are incomplete assessments.

Based on Eq. (12), the minimum and maximum value of the improved dissimilarity measure between $S(e_i(a_l))$ and $S(e_j(a_l))$ can be generated by the following pair of optimization models:

$$\text{Min/Max}D(e_{ij}(a_l))$$

s.t. Eqs. (2), (8)

$$D(e_{ij}(a_l))=\frac{1}{u(H_N)-u(H_1)}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta^*_{n,|i-j|}(a_l)\beta^*_{s,|i-j|}(a_l)u(H_{|s-n|})$$

$$\beta^*_{n,|i-j|}(a_l)=|\beta^*_{n,i}(a_l)-\beta^*_{n,j}(a_l)|\ n=1,2,\cdots,N$$

$$\beta_{n,i}(a_l)\le\beta^*_{n,i}(a_l)\le\beta_{n,i}(a_l)+\beta_{H,i}(a_l)n=1,2,\cdots,N$$

Thus, the improved dissimilarity measure will be included in an interval such that $D(e_{ij}(a_l))\in[D^-(e_{ij}(a_l)),D^+(e_{ij}(a_l))]$ where $D^-(e_{ij}(a_l))$ and $D^+(e_{ij}(a_l))$ are computed by the above pair of models.

The conflict measure on the alternative level in Eq. (13) will also be an interval, which can be generated by solving the following pair of programming models:

$$\text{Min/Max}Con^1(e_i,e_j)$$

s.t. Eqs. (2), (8)

$$Con^1(e_i,e_j)=\frac{1}{S(u(H_N)-u(H_1))}\sum_{l=1}^{S}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta^*_{n,|i-j|}(a_l)\beta^*_{s,|i-j|}(a_l)u(H_{|s-n|})$$

$$\beta^*_{n,|i-j|}(a_l)=|\beta^*_{n,i}(a_l)-\beta^*_{n,j}(a_l)|\ n=1,2,\cdots,N;\ l=1,2,\cdots,S$$

$$\beta_{n,i}(a_l)\le\beta^*_{n,i}(a_l)\le\beta_{n,i}(a_l)+\beta_{H,i}(a_l)n=1,2,\cdots,N;\ l=1,2,\cdots,S$$

So we have $Con^1(e_i,e_j)\in[Con^1(e_i,e_j)^-,Con^1(e_i,e_j)^+]$. Then the left and right extensions of average conflict measure defined in Eq. (17) can be computed by the following pair of models:

$$\text{Min/Max}Con^1(e_i)$$

s.t. Eqs. (2), (8)

$$Con^1(e_i)=\frac{1}{S(L-1)(u(H_N)-u(H_1))}\sum_{j=1}^{L}\sum_{l=1}^{S}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta^*_{n,|i-j|}(a_l)\beta^*_{s,|i-j|}(a_l)u(H_{|s-n|})$$

$$\beta^*_{n,|i-j|}(a_l)=|\beta^*_{n,i}(a_l)-\beta^*_{n,j}(a_l)|\ n=1,2,\cdots,N;\ l=1,2,\cdots,S;\ j=1,2,\cdots,L$$

$$\beta_{n,i}(a_l)\le\beta^*_{n,i}(a_l)\le\beta_{n,i}(a_l)+\beta_{H,i}(a_l)n=1,2,\cdots,N;\ l=1,2,\cdots,S;\ j=1,2,\cdots,L$$

By solving the above model, we have $Con^1(e_i)\in[Con^1(e_i)^-,Con^1(e_i)^+]$.

Similarly, the average dissimilarity measure proposed in Def.8 will be in an interval such that $D(e_i(a_l))\in[D^-(e_i(a_l)),D^+(e_i(a_l))]$, which can be calculated by the following pair of models:

$$\text{Min/Max}D(e_i(a_l))$$

s.t. Eqs. (2), (8)

$$D(e_i(a_l))=\frac{1}{(L-1)(u(H_N)-u(H_1))}\sum_{j=1}^{L}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta^*_{n,|i-j|}(a_l)\beta^*_{s,|i-j|}(a_l)u(H_{|s-n|})$$

$$\beta^*_{n,|i-j|}(a_l)=|\beta^*_{n,i}(a_l)-\beta^*_{n,j}(a_l)|\ n=1,2,\cdots,N;\ j=1,2,\cdots,L$$

$$\beta_{n,i}(a_l)\le\beta^*_{n,i}(a_l)\le\beta_{n,i}(a_l)+\beta_{H,i}(a_l)n=1,2,\cdots,N;\ j=1,2,\cdots,L$$

As such, the minimum and maximum value of dissimilarity measure on a_l proposed in Def.9 can be generated by the following pair of models:

$$\text{Min/Max}D(a_l)$$

s.t. Eqs. (2), (8)

$$D(a_l)=\frac{1}{L(L-1)(u(H_N)-u(H_1))}\sum_{i=1}^{L}\sum_{j=1}^{L}\sum_{n=1}^{N-1}\sum_{s=n+1}^{N}\beta^*_{n,|i-j|}(a_l)\beta^*_{s,|i-j|}(a_l)u(H_{|s-n|})$$

$$\beta^*_{n,|i-j|}(a_l)=|\beta^*_{n,i}(a_l)-\beta^*_{n,j}(a_l)|\ n=1,2,\cdots,N;\ i,j=1,2,\cdots,L$$

$$\beta_{n,i}(a_l)\le\beta^*_{n,i}(a_l)\le\beta_{n,i}(a_l)+\beta_{H,i}(a_l)n=1,2,\cdots,N;\ i,j=1,2,\cdots,L$$

So we have $D(a_l)\in[D^-(a_l),D^+(a_l)]$.

5. The uncertainty measure of BDs

The second factor to determine the support degree of assessment is the internal indeterminacy of a BD. It can be measured by the uncertainty of the original assessment provided that subjective judgments are extracted from an individual or a group of DMs. If the BDs assessed on most attributes are associated with a high degree of uncertainty, the support degree of the aggregated result will be low no matter which combination rule is used. An extreme case is that all attributes are assessed to be *Good* and *Bad* with the belief degree of 0.5 and 0.5 on an alternative, which represents a confusing situation to any DM.

Table 8
Belief distributions of alternative a_l on three attributes.

a_l	H_1	H_2	H_3	H_4	H_5
e_1	0	0	0	0	1
e_2	0	0	0	0	1
e_3	0	0	0	0	1

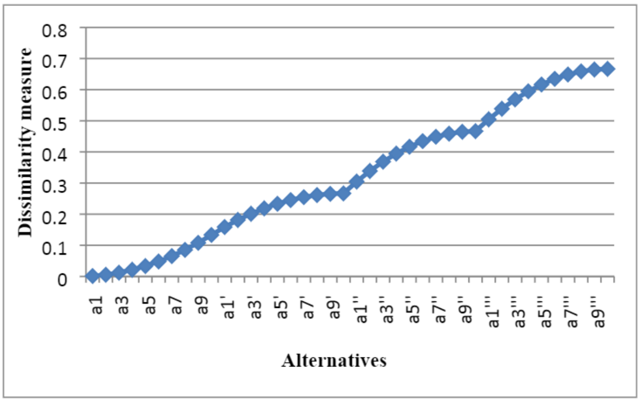

Fig. 1. Dissimilarity measure on alternative.

Moreover, the aggregated BD by the ER algorithm may be an uncertain assessment even though all the basic attributes are certain BDs as long as the BDs of at least two attributes are different. So the uncertainty of the final combined BD not only results from the uncertainty of the BD on each basic attribute, but also the incompatibility among the BDs of all attributes. In recent years, a lot of researchers have proposed the uncertainty measure of mass functions (Jiroušek & Shenoy, 2018; Jousselme et al., 2006; Klir & Ramer, 1990; Ronald & Yager, 2018; Yang & Han, 2016). As discussed in the introduction, all of them are not considered in a decision-making situation where the utility of each evaluation grade is different. Here, to capture the utility difference among focal elements, two types of uncertainty measure are defined as the uncertainty of the original BD and the uncertainty of the final aggregated BD. The relationship between these two types of uncertainty is also studied, which will deduce the total incompatibility measure among BDs.

5.1. Uncertainty measure

Definition 10. Suppose the belief degree that a_l be assessed on e_i to H_n is denoted by Eq. (2), the *uncertainty measure* to the BD of $S(e_i(a_l))$ can be calculated as follows:

$$Un(e_i(a_l)) = \frac{4}{u(H_N) - u(H_1)} \sum_{n=1}^{N-1} \sum_{s=n+1}^{N} \beta_{n,i}(a_l)\beta_{s,i}(a_l)u(H_{|s-n|}) \tag{30}$$

Properties:

1 > $0 \le Un(e_i(a_l)) \le 1$;
2 > $Un(e_i(a_l)) = 0$, iff$\beta_{h,i}(a_l) = 1 (h \in \{1, 2, \cdots, N\}), \beta_{n,i}(a_l) = 0 (n = 1, 2, \cdots, N, n \ne h)$;
3 > $Un(e_i(a_l)) = 1$, iff$\beta_{1,i}(a_l) = 0.5, \beta_{N,i}(a_l) = 0.5, \beta_{n,i}(a_l) = 0 (n = 2, \cdots, N-1)$.

The uncertainty measure proposed in Def.10 is a multivariable quadratic function with N variables, i.e. $\beta_{n,i}(a_l)(n = 1, 2, \cdots, N)$ such that $0 \le \beta_{n,i}(a_l) \le 1$, $\sum_{n=1}^{N} \beta_{n,i}(a_l) \le 1$. Properties 2 > and 3 > can be proved by calculating the following model:

Min/Max$Un(e_i(a_l))$

s.t. Eqs. (8), (30)

$0 \le \beta_{n,i}(a_l) \le 1 n = 1, 2, \cdots, N$

$\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1$

Property 2 > indicates that if the BD is absolutely assigned to an evaluation grade, the uncertainty measure is 0. In individual decision-making problems, the uncertainty of a BD is subjective that can be interpreted as the likelihood of a proposition, so the uncertainty measure to a BD on an attribute in Def.10 may be seen as the hesitation degree from an individual. In a multiple attribute group decision making (MAGDM) problem, the belief degree reflects objective uncertainty to some extent, so $Un(e_i(a_l))$ can be interpreted as the inconsistency degree among the judgments of DMs. The condition in Property 3 just reflects the largest conflict among different DMs. Thus, the proposed uncertainty measure can be seen as the unreliability of the assessment or the conflict involved in a BD.

The above model assumes that there is no ignorance contained in a BD. When the assessment is incomplete, the following pair of models can be conducted to calculate the minimum and maximum value of the uncertainty measure defined in Eq. (30).

Min/Max$Un(e_i(a_l))$

s.t. Eqs. (2), (8)

$$Un(e_i(a_l)) = \frac{4}{u(H_N) - u(H_1)} \sum_{n=1}^{N-1} \sum_{s=n+1}^{N} \beta^*_{n,i}(a_l)\beta^*_{s,i}(a_l)u(H_{|s-n|})$$

$\beta_{n,i}(a_l) \le \beta^*_{n,i}(a_l) \le \beta_{n,i}(a_l) + \beta_H(a_l) n = 1, 2, \cdots, N$

$\sum_{n=1}^{N} \beta^*_{n,i}(a_l) = 1$

Let $Un^-(e_i(a_l))$ and $Un^+(e_i(a_l))$ be the minimum and maximum value of the above pair of programming models respectively, then we have $Un(e_i(a_l)) \in [Un^-(e_i(a_l)), Un^+(e_i(a_l))]$.

Example 8. Given the frame of discernment $H = \{H_1, H_2, H_3, H_4\}$. The utilities of the four evaluation grades are set to be $u(H_1) = 0$, $u(H_2) = 0.35$, $u(H_3) = 0.7$, $u(H_4) = 1$. Suppose the belief degree of e_i be evaluated to a_l on grade H_1 represented by $\beta_{1,i}(a_l)$ changes from 0 to 1 with a step of 0.1. Three cases are to be conducted. 1) The belief degree on H_2 represented by $\beta_{2,i}(a_l)$ is set to be $1 - \beta_{1,i}(a_l)$. 2) The belief degree on H_3 represented by $\beta_{3,i}(a_l)$ is assumed to be $1 - \beta_{1,i}(a_l)$. 3) $\beta_{4,i}(a_l)$ is assumed to be $1 - \beta_{1,i}(a_l)$. Fig. 2 shows the uncertainty measure generated by Eq. (30) with the change of $\beta_{n,i}(a_l)(n = 2, 3, 4)$. '$H_2$', '$H_3$' and '$H_4$' represent the results from the above three mentioned cases respectively.

From Fig. 2, we can see that the three curves are all symmetric. For a specific value of $\beta_{n,i}(a_l)$, the uncertainty measure to the BD increases when n changes from 2 to 4. The reason lies in that the larger the value of n, the greater the utility difference represented by $u(H_{|1-n|})$ that makes the BD more indeterminate. When $\beta_{n,i}(a_l) = 0.5(n = 2, 3, 4)$, the uncertainty measure attains the maximum value for each of the three cases. A special situation is that $Un(e_i(a_l)) = 1$ when the BD is given by $S(e_i(a_l)) = \{(H_1, 0.5), (H_2, 0), (H_3, 0), (H_4, 0.5)\}$. In this case, the alternative has two extreme opposite characters that we cannot easily make a choice. In a MAGDM problem, it reflects that half of the DMs assume a_l to be excellent on e_i, while the other half object to it.

Theorem 1. *If the uncertainty measure of $S(e_i(a_l))$ is defined by Eq. (30), it is continuous with respect to $\beta_{n,i}(a_l)$.*

The proof of Theorem 1 is shown in the appendix.

Definition 11. Given the BD represented by Eq. (2) and the uncertainty measure proposed in Def.10, the *average uncertainty measure* on a_l considering the uncertainty measure of all the L attributes can be calculated as follows:

$$Un(a_l) = \frac{1}{L} \sum_{i=1}^{L} Un(e_i(a_l)) \tag{31}$$

Def.11 gives us a view about the overall uncertainty measure on an alternative respect to all the L attributes. Obviously, the larger the value of $Un(a_l)$, the more subjective uncertainty contained in some of the L attributes, which will lead to the decrease of support degree of assessment. Here, we also define an uncertainty measure just from the aggregated belief degrees as follows.

Definition 12. Given the aggregated belief degrees represented by Eq.

(4), the *global uncertainty measure* to the aggregated BD $S(a_l)$ is calculated by

$$Un(S(a_l)) = \frac{4}{u(H_N) - u(H_1)} \sum_{n=1}^{N-1} \sum_{s=n+1}^{N} \beta_n(a_l)\beta_s(a_l)u(H_{|s-n|}) \tag{32}$$

Properties:

1> $0 \le Un(a_l) \le 1$, $0 \le Un(S(a_l)) \le 1$;
2> $Un(a_l) = 0$, *iff* $\forall\, i \in \{1, 2, \cdots, L\}$, $\exists\, h^i \in \{1, 2, \cdots, N\}$, $\beta_{h^i,i}(a_l) = 1$, ; and $\beta_{n,i}(a_l) = 0 (n = 1, 2, \cdots, N,\, n \ne h^i)$
3> $Un(a_l) = 1$, *iff* $\forall\, i \in \{1, 2, \cdots, L\}$, $\beta_{1,i}(a_l) = 0.5$, $\beta_{N,i}(a_l) =$; 0.5, and $\beta_{n,i}(a_l) = 0 (n = 1, 2, \cdots, N,\, n \ne 1, N)$
4> $Un(a_l) = Un(S(a_l))$, *iff* $S(e_1(a_l)) = S(e_2(a_l)) = \cdots = S(e_L(a_l)) = S(a_l)$;
5> $Un(S(a_l)) = 0$, *iff* $\beta_{n,1}(a_l) = \beta_{n,2}(a_l) = \cdots = \beta_{n,L}(a_l) = 1 (n \in \{1, 2, \cdots, N\})$and $\beta_{m,i}(a_l) = 0 (m \ne n,\, i = 1, 2, \cdots, L)$;
6> $Un(S(a_l)) = 1$ *iff* $S(e_1(a_l)) = S(e_2(a_l)) = \cdots = S(e_L(a_l))$ and $\forall\, i \in \{1, 2, \cdots, L\}$, $\beta_{1,i}(a_l) = 0.5$, $\beta_{N,i}(a_l) = 0.5$.

$Un(S(a_l))$ in Def.12 reflects the uncertainty of the aggregated BD, while $Un(a_l)$ in Def.11 shows the uncertainty extracted from each attribute assessed on a_l. Property 2 > indicates that when the assessments that a_l be assessed to all the L attributes are certain, then $Un(a_l) = 0$. So if $\exists\, i, j \in \{1, 2, \cdots, L\}$, $S(e_i(a_l)) \ne S(e_j(a_l))$, the aggregated BD is uncertain by the ER approach. Then from Def.12, we will have$Un(S(a_l)) > 0$ because the aggregated belief degree is positive on at least two evaluation grades. But the average uncertainty measure $Un(a_l)$ proposed in Def.11 can still attain to 0 if the condition in Property 2 > is satisfied.

Property 4 > indicates that the average uncertainty measure equals to the global uncertainty measure when $S(a_l) = S(e_i(a_l)) (i = 1, 2, \cdots, L)$. There are two situations satisfying the condition in Property 4 > . Firstly, the BDs that a_l be assessed on all L attributes are the same and certain such that $\beta_{n,1}(a_l) = \beta_{n,2}(a_l) = \cdots = \beta_{n,L}(a_l) = 1 (n \in \{1, 2, \cdots, N\})$ and $\beta_{m,i}(a_l) = 0 (m \ne n,\, i = 1, 2, \cdots, L)$. This will lead to the result that $\beta_n(a_l) = 1$ and $\beta_m(a_l) = 0 (m = 1, 2, \cdots, n-1, n+1, \cdots, N)$. It is a special case of the condition in Property 2 > where h^i is not necessary to be identical with $h^j (i, j \in \{1, 2, \cdots, L\})$. Secondly, the total belief degree of one is evenly assigned to a set of evaluation grades for all L attributes no matter how the weights are assigned to these attributes. Specifically, suppose $H_i^+ = \{H_n | \beta_{n,i}(a_l) > 0\}$, and $\beta_{n,i}(a_l) = \frac{1}{|H_i^+|} (H_n \in H_i^+)$, $\beta_{n,i}(a_l) = 0 (H_n \notin H_i^+)$ where $|H_i^+|$ is the cardinality of H_i^+. Here, $H_i^+ = H_j^+ (i, j \in \{1, 2, \cdots, L\})$ and $|H_i^+| \le N$. In the first situation of Property 4 > , $Un(a_l) = Un(S(a_l)) = 0$, while $Un(a_l) = Un(S(a_l)) > 0$ in the second situation. The first situation can also be seen as a special case of situation 2 where H_i^+ consists of only one element.

Example 9. Three BDs are given as follows:

$$S(e_1(a_l)) = \{(H_1, 0), (H_2, 1/3), (H_3, 1/3), (H_4, 1/3), (H_5, 0)\}$$

$$S(e_2(a_l)) = \{(H_1, 0), (H_2, 1/3), (H_3, 1/3), (H_4, 1/3), (H_5, 0)\}$$

$$S(e_3(a_l)) = \{(H_1, 0), (H_2, 1/3), (H_3, 1/3), (H_4, 1/3), (H_5, 0)\}$$

The aggregated distribution is $S(a_l) = \{(H_1, 0), (H_2, 1/3), (H_3, 1/3), (H_4, 1/3), (H_5, 0)\}$ no matter how the weights are allocated to the attributes. So in this case, $Un(a_l) = Un(S(a_l)) > 0$ which accords with the second situation in Property 4 > .

Property 5 > is just the first situation in Property 4 > that all L attributes are definitely assessed to a specific evaluation grade, which leads to the combined BD being a certain assessment such that $\beta_n(a_l) = 1 (n \in \{1, 2, \cdots, N\})$ and $\beta_m(a_l) = 0 (m \ne n)$. Property 6 > is a special case of the second situation in Property 4 > that all the L attributes are assessed with the largest uncertainty which leads to the global uncertainty measure attains to 1.

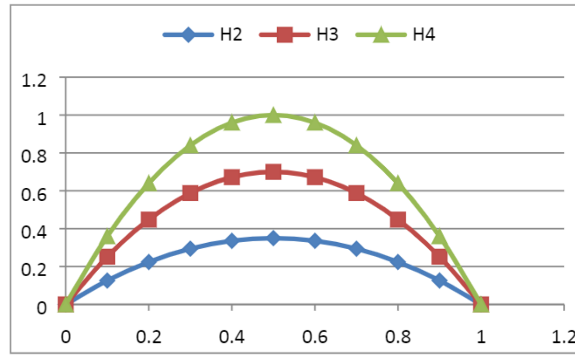

Fig. 2. Uncertainty measure subject to the changes of $\beta_{n,i}(a_l)$ $(n = 2, 3, 4)$.

Example 10. Two BDs are given as follows:

$$S(e_1(a_l)) = \{(H_1, 1), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 0)\}$$

$$S(e_2(a_l)) = \{(H_1, 0), (H_2, 1), (H_3, 0), (H_4, 0), (H_5, 0)\}$$

The aggregated BD is $S(a_l) = \{(H_1, 0.5), (H_2, 0.5), (H_3, 0), (H_4, 0), (H_5, 0)\}$ provided that the weights of e_1 and e_2 are assumed to be equal. Suppose $u(H_1) = 0$, $u(H_2) = 0.3$, $u(H_3) = 0.6$, $u(H_4) = 0.8$ and $u(H_5) = 1$, then from Eqs. (30)–(32), we have $Un(a_l) = 0$, $Un(S(a_l)) = 0.3$.

It can be seen that although the uncertainty measures of $S(e_1(a_l))$ and $S(e_2(a_l))$ defined in Eq. (30) are zero, the global uncertainty measure $Un(S(a_l))$ is greater than 0 and purely determined by the discrepancy of the two BDs in this case. So we conclude that the inequation $Un(a_l) \ne Un(S(a_l))$ satisfies in almost all of the cases. Even if $S(e_1(a_l)) = S(e_2(a_l)) = \cdots = S(e_L(a_l))$, we will have $Un(a_l) \ne Un(S(a_l))$ as long as $\beta_{n,i}(a_l) \ne \frac{1}{|H_i^+|} (H_n \in H_i^+)$. The reason lies in that $S(a_l)$ does not equal to $S(e_i(a_l)) (i = 1, 2, \cdots, L)$ no matter how the weights are assigned to the attributes.

Example 11. Three BDs are given as follows:

$$S(e_1(a_l)) = \{(H_1, 0), (H_2, 0.5), (H_3, 0.3), (H_4, 0.2), (H_5, 0)\}$$

$$S(e_2(a_l)) = \{(H_1, 0), (H_2, 0.5), (H_3, 0.3), (H_4, 0.2), (H_5, 0)\}$$

$$S(e_3(a_l)) = \{(H_1, 0), (H_2, 0.5), (H_3, 0.3), (H_4, 0.2), (H_5, 0)\}$$

The aggregated BD is $S(a_l) = \{(H_1, 0), (H_2, 0.5280), (H_3, 0.2886), (H_4, 0.1834), (H_5, 0)\}$ provided that $w_1 = w_2 = w_3 = 1/3$. It is clear that $Un(a_l) \ne Un(S(a_l))$. $Un(a_l) = 0.428$, $Un(S(a_l)) = 0.419$. Here, $Un(a_l) > Un(S(a_l))$ because the aggregated BD becomes more certain after the combination of the three same BDs by the ER approach. Specifically, the aggregated belief degree on H_2 becomes larger while the fused belief degree on H_3 and H_4 become smaller. This can be easily interpreted because the more confident a focal element is judged by different evidences compared with other focal elements, the higher combined belief degree to the focal element. So when $Un(a_l) > Un(S(a_l))$ just as presented in Example 11, the BDs are consistent to a certain degree that the aggregated BD becomes more certain than any one of the original BD. On the contrary, if $Un(a_l) < Un(S(a_l))$, the BDs are incompatible to an extent that the combined BD is more uncertain than the original BDs.

So the global uncertainty measure to the aggregated BD reflects the differences between each pair of the BDs to some extent. From this point of view, we conclude that $Un(S(a_l))$ reflects the uncertainty from the final aggregated distribution. It reveals not only the uncertainties from the original BD of each attribute, but also the incompatibility among the BDs of L attributes. It can be easily explained from Example 10 that $Un(S(a_l)) = 0.3$ derives from the differences between $S(e_1(a_l))$ and $S(e_2(a_l))$.

Comparatively, $Un(a_l)$ reflects the uncertainty only caused by the original BD of each attribute. With the rise of $Un(a_l)$, the uncertainty or unreliability of assessment increases since the judgments to the basic attributes by the DM may be probably more hesitate. Property 3 > reveals the situation where the maximum average uncertainty is contained in the original BDs no matter how the weights are assigned to the L attributes. It is just a special case of Property 4 >, so $Un(a_l) = Un(S(a_l)) = 1$ in this situation.

From the above analysis, the total incompatibility measure among the BDs of attributes on a specific alternative can be defined.

Definition 13. Given the average uncertainty measure in Def.11 and the global uncertainty measure in Def.12, the *total incompatibility measure* among the BDs of all the L attributes on a_l is computed by

$$Incom\left(\prod_i^L S(e_i(a_l))\right) = Un(S(a_l)) - Un(a_l) \tag{33}$$

The symbol $\prod$ in Eq. (33) does not signify the continuous multiplication, it only represents the total effect of the BDs of L attributes.

Properties:

1 > $Incom\left(\prod_i^L S(e_i(a_l))\right) \le 1$;
2 > $Incom\left(\prod_i^L S(e_i(a_l))\right) = 0$ iff $Un(S(a_l)) = Un(a_l)$;
3 > $Incom\left(\prod_i^L S(e_i(a_l))\right) = 1$ iff $Un(a_l) = 0$ and $Un(S(a_l)) = 1$.

The situation in Property 3 > happens when half of the attributes are assigned with $\beta_{1,i}(a_l) = 1$ and $\beta_{n,i}(a_l) = 0 (n = 2, \cdots, N)$, while the other half are given the BDs of $\beta_{N,i}(a_l) = 1$ and $\beta_{n,i}(a_l) = 0 (n = 1, 2, \cdots, N - 1)$.

Example 12. Four BDs are given as follows:

$$S(e_1(a_l)) = \{(H_1, 1), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 0)\}$$

$$S(e_2(a_l)) = \{(H_1, 1), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 0)\}$$

$$S(e_3(a_l)) = \{(H_1, 0), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 1)\}$$

$$S(e_4(a_l)) = \{(H_1, 0), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 1)\}$$

The aggregated BD is $S(a_l) = \{(H_1, 0.5), (H_2, 0), (H_3, 0), (H_4, 0), (H_5, 0.5)\}$ provided that the weights of e_1 to e_4 satisfy the condition of either $w_1 = w_3$, $w_2 = w_4$ or $w_1 = w_4$, $w_2 = w_3$. Suppose $u(H_1) = 0$, $u(H_2) = 0.3$, $u(H_3) = 0.6$, $u(H_4) = 0.8$ and $u(H_5) = 1$, we have $Un(a_l) = 0$, $Un(S(a_l)) = 1$ from Eqs. (31) and (32). This means that the original assessment of each attribute is certain, while the aggregated BD of the four attributes is a high uncertain outcome. The total incompatibility measure attains the maximum value, i.e., $Incom\left(\prod_i^L S(e_i(a_l))\right) = 1$. If the condition of either $w_1 = w_3$, $w_2 = w_4$ or $w_1 = w_4$, $w_2 = w_3$ is not satisfied, the aggregated BD will not be $\beta_1(a_l) = 0.5$ and $\beta_5(a_l) = 0.5$, even though we set $w_1 + w_2 = w_3 + w_4$. As a result, $Un(S(a_l)) < 1$. From this point of view, the global uncertainty measure is not only determined by the uncertainty measure to the original BD of each attribute and the incompatibility among the BDs of attributes, but also influenced by the attribute weights. It can be observed from the above result that the complete high conflict between subset $\{e_1, e_2\}$ and $\{e_3, e_4\}$ leads to the maximum value of total incompatibility measure. Thereby, in a MADM problem, the higher value of the total incompatibility measure, the larger discrepancy among the assessments of different attributes, and vice versa. So it will be more difficult for a DM to make a decision when the total incompatibility measure is large, especially in the above case where the assessment of each attribute is certain while the aggregated assessment is highly uncertain. In a decision-making problem where we intend to choose alternatives with distinguished features, an alternative with larger total incompatibility measure may probably be selected if the aggregated assessments of several alternatives are similar. The selection of football players can sometimes represent this situation. On the contrary, inadecision-making problem where alternatives should not be deficient on any attribute, an alternative with smaller total incompatibility measure has priority over others with similar aggregated assessments. The college entrance examination in China is a typical case of the second situation. A student cannot do poorly in any of the courses if he/she wants to be admitted to a good college. The selection of a leader is also such a case where many aspects of a candidate ought to be considered. A candidate cannot be selected if his/her morality is low even though he/she has excellent professional capability. In consistence with the Cannikin law, the following conclusion can be drawn: if some alternatives have similar performance in general, an alternative with smaller total incompatibility will be preferred over others in an incomplete compensatory MADM problem; otherwise, we should choose the alternative with larger total incompatibility in a complete compensatory MADM problem. The total incompatibility measure designed in Eq. (33) and the dissimilarity measure defined in Eq. (27) or Eq. (28) have similar meanings. They reflect the discrepancy of different evidences (e.g., attributes in MADM problem or DMs in GDM problem) represented by uncertain BDs. The former is an indirect concept generated from Eqs. (31) and (32), while the latter is a direct definition generated from the dissimilarity measure matrix.

5.2. The procedure of decision-making

In sum, the whole process of the proposed method is shown in Fig. 3, where the weight assignment method is further extended in Fig. 4. There are three steps included in the whole decision-making process.

(1) Before applying the proposed method, the first step is to construct the hierarchical structure of attributes for a MADM problem, followed by the data collection from either an individual or a group of DMs. For a quantitative attribute, the acquired numerical value should be transformed to a BD associated with the general frame of discernment according to the rules provided in (Zhou et al., 2019). Each qualitative attribute may be assigned with a unique frame of discernment which also ought to be transformed to the general frame of discernment according to specific rules established in real problems. As such, a decision-making matrix in the form of BDs transformed from the values of both quantitative and qualitative attributes presented by Eq. (3) is constructed.
(2) The second step is to generate attribute weights from BDs which is the main topic discussed in Section 3. Two dimensions are to be involved in the calculation. The first dimension is the discriminating power derived from the BD matrix computed by Eq. (23) according to (Fu et al., 2018). It is assumed that the larger the value of discriminating power, the more weight will be allocated to the attribute. Meanwhile, the improved dissimilarity measure between two attributes on an alternative proposed in Eq. (12) can be computed from the distributed dissimilarity vector defined in Eq. (10). Then it is used to calculate the conflict measure on the alternative or evaluation grade level respectively, which induce the second dimension called the weighted average conflict measure defined in Eq. (22). The larger the value of the weighted average conflict measure, the less interdependency between the attribute and all other attributes, thus the attribute will be assigned with higher importance. As such, the weight derived from both the above two dimensions is generated by Eq. (24).
(3) The third step of the decision making process is to aggregate the BDs of all attributes on each alternative by the ER algorithm. The ER algorithm provides a conjunctive probabilistic reasoning process, and is both commutative and associative. The final generated assessment on an alternative is presented in the form shown in Eq. (4). To facilitate the ranking or selection of alternatives, average dissimilarity measure is calculated by Eq. (26), followed by the computation of dissimilarity measure on alternative defined by Eq. (27) in Section 4. It can be then used to assist the DM to make a ranking order or choice. The smaller the value of dissimilarity measure on an alternative, the more support degree that the

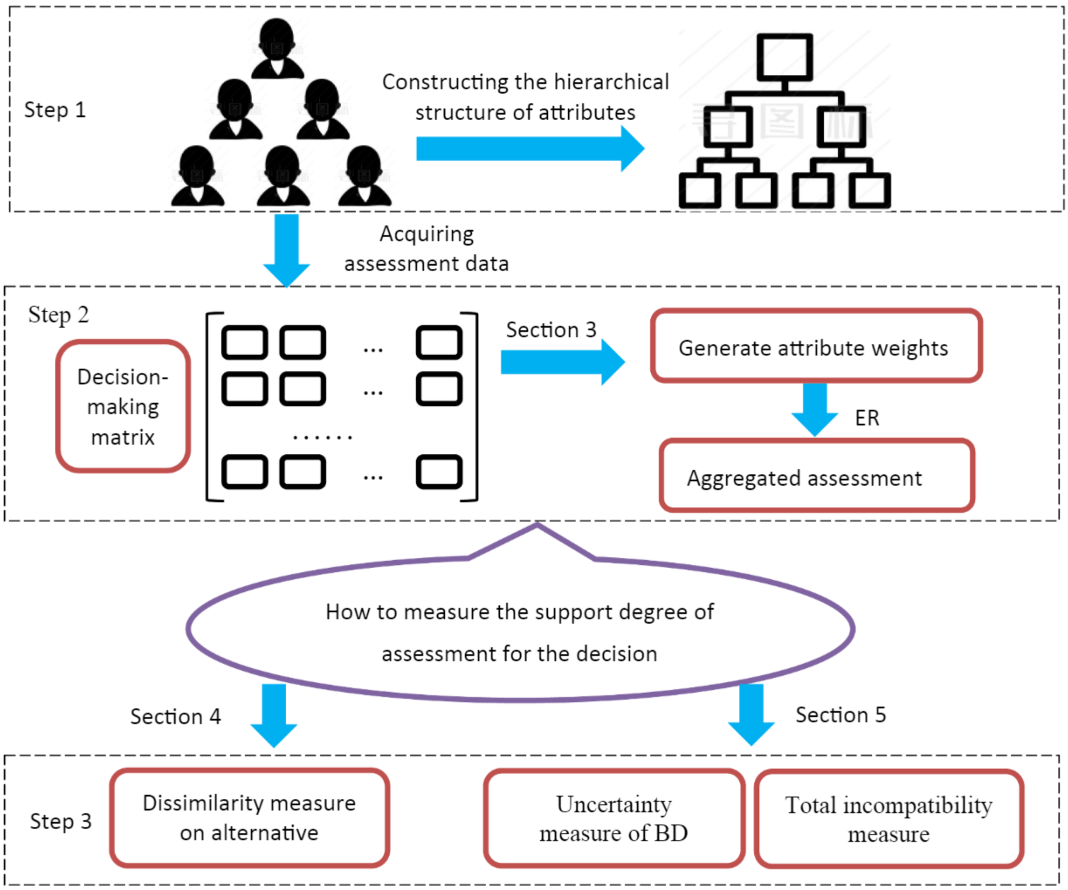

Fig. 3. The general decision process under subjective judgment.

aggregated result can be considered as the basis of the final decision, and vice versa. Furthermore, the uncertainty measure proposed in Section 5 can also be utilized to quantify the support degree of the final aggregated result. The dissimilarity measure and the average uncertainty measure on an alternative are two sides of an assessment. The former reflects the external divergence among the assessments of different attributes, while the latter reveals the internal indeterminacy of each attribute represented by linguistic judgment.

6. Case study

It was discussed in the introduction section that a variety of methodological methods to determine attribute weights have been designed in order to handle different situations. Each of them has its unique feature which differentiates from other techniques. In this subsection, a car selection problem adapted from (Yang, 2001) is chosen to illustrate the effectiveness and applicability of the proposed method. Several different weight assignment methods are compared with the proposed method to demonstrate their respective properties.

6.1. Generating weights using different methods

In a car selection problem, seven attributes are included in the assessment and ranking order of six cars. Six linguistic evaluation grades are set to assess the alternative cars as follows:

$$H = \{\mathrm{Worst}(W), \mathrm{Poor}(P), \mathrm{Average}(A), \mathrm{Good}(G), \mathrm{Excellent}(E) \\ \mathrm{andTop}(T)\}$$

Table 9 shows the BDs assessed on qualitative attributes and the transformed BDs from original numerical values on quantitative attributes. For the generation of the overall performance on each car, the ER approach is to be used to combine the BDs in Table 9. In (Yang, 2001), the weights are supposed to be equal such that $w_i = 1/7 (i = 1, 2, \cdots, L)$.

The specific meaning of the BD in Table 9 has been illustrated in Example 1. It can be seen that some of the assessments are incomplete. Intuitively, from the viewpoint of the first dimension, the weight of e_4 ought to be large because the divergence of assessments on the six cars respect to e_4 "Horsepower" is very large. Additionally, on the second dimension, the state of e_4 for Car 2 to grade *Poor* is 0.533, while the belief degree of e_4 for Car 5 to grade *Worst* and *Poor* is 0.467 and 0.533. For other six attributes, the six alternative cars are rarely assessed to *Worst* and *Poor*. So the conflict between e_4 and other attributes is large, which will lead to more importance allocated to e_4. Comparatively, the assessments of e_5 or e_7 on all the six cars are quite similar. That is to say, these two attributes should not be paid much attention in the decision process. So e_5 and e_7 will be allocated with less weight in the final result no matter which methods are applied because each one of the following listed methods employs this dimension.

Without loss of generality, the utilities of the six evaluation grades are set to be equidistantly distributed such that $u(W) = 0$, $u(P) = 0.2$, $u(A) = 0.4$, $u(G) = 0.6$, $u(E) = 0.8$, $u(T) = 1$. It is worth mentioning that the utility functions of different DMs may not be the same due to their discrepancy on knowledge, background and expertise. So the utility of the worst evaluation grade may be larger than 0, while the utility of the best evaluation grade may be less than 1. As such, our proposed dissimilarity measure in Eq. (12) is comparatively rational in this circumstance. From this point of view, different utility functions may affect the final result. Fig. 5 shows the generated weights by different methods, including the GMD (Fu et al., 2018) method, SD (Chin et al., 2015; Diakoulaki et al., 1995) method, CCSD (Wang & Luo,

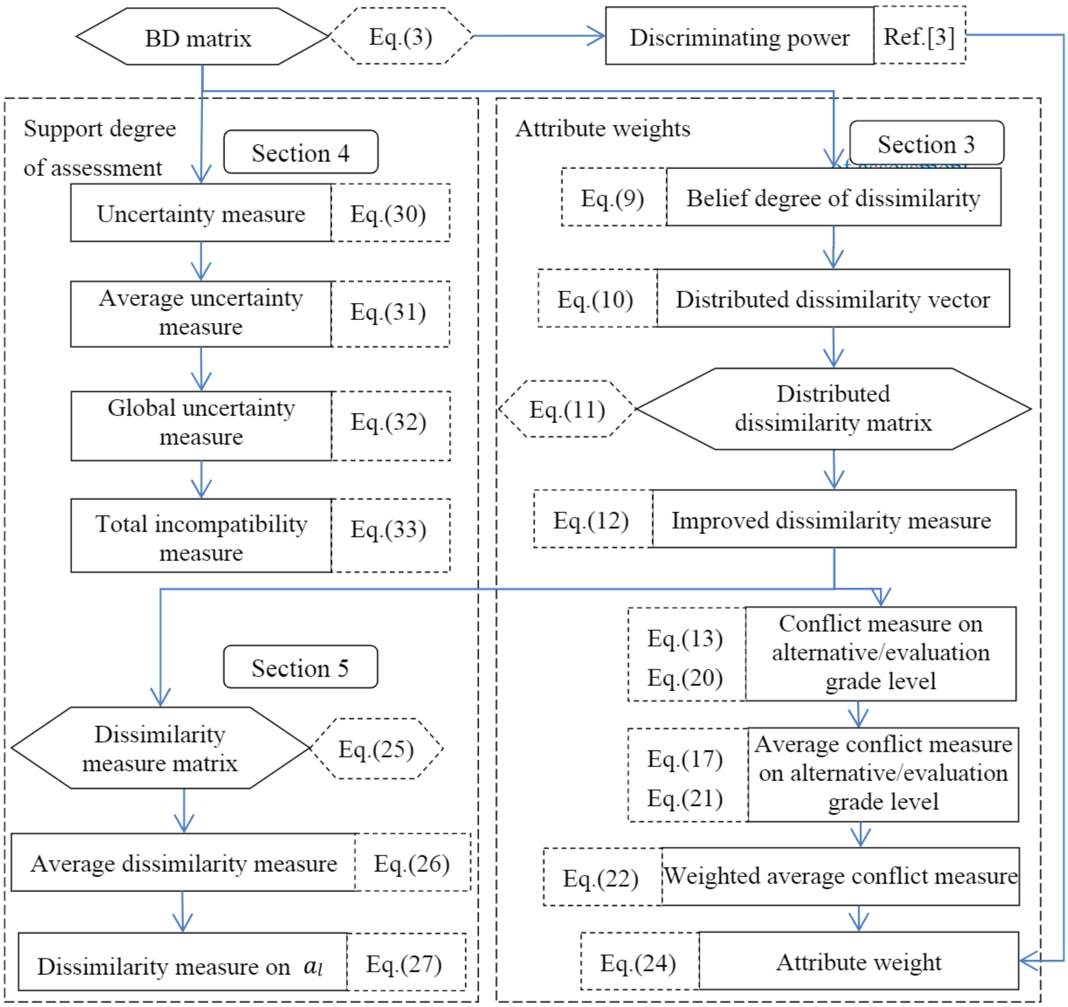

Fig. 4. The detailed process of generating attribute weights from BDs.

2010), CRITIC (Diakoulaki et al., 1995), methods in (Fu et al., 2018) and (Zhou et al., 2019). The horizontal axis represents the serial number of attribute, while the vertical axis denotes the generated weights. Since the first four methods are all designed to tackling with numerical assessment values, we transformed the BDs in Table 9 to utilities such that $u(e_i(a_l)) = \sum_{n=1}^{N} \beta_{n,i}(a_l)u(H_n) + \beta_{H,i}(a_l)\frac{u(H_N)+u(H_1)}{2}$ to apply these methods. In the transformation process, information loss will inevitably occur because different BDs may be transformed to the same utility value. The method in (Fu et al., 2018) considers the first dimension of information, while the method in (Zhou et al., 2019) extended the entropy method to cope with the situation where BDs are the representation of assessments. Although the divergence of the assessment of e_4 is large, the weight on e_4 generated by the method in (Zhou et al., 2019) is a little larger. Comparatively, the weight of e_4 computed by the proposed method is 0.2959 which is much less than 0.3768. And it is also a little larger than the other five methods. As such, we could distinguish more clearly the attribute that should be focused on. CCSD also creates a relatively high weight on e_4, but the weight on e_2 generated by it is small. From Table 9, we can see that both Car 4 and Car 5 are assessed on e_2 to *Poor* with the belief degree of 0.75 and 1, while other attributes are rarely assessed to these two grades except e_4. So from the second dimension, the average conflict measure of e_2 is large. Meanwhile, from the first dimension, the assessment of Car 2 on e_2 to grade *TOP* is 0.667, which leads to a large divergence of the six alternative cars on e_2. Thus, e_2 ought to be allocated with a large weight considering the two dimensions. The method in (Fu et al., 2018) also creates a set of relatively reasonable weights, but the weights of e_2 and e_4 are similar. Intuitively, both the divergence and average conflict measure of e_4 are larger than e_2, so the difference between the weight of them should be more obvious just as the proposed method does.

The standard deviations of the generated weights by the seven methods are shown in Table 10. Obviously, the weights created by the method in (Zhou et al., 2019) have the largest variation, whereas the CRITIC is the smallest. The proposed method provides a relatively compromised result that the important attributes can be reflected, while the distinction between the weights of attributes will not be overstated.

The dissimilarity measures and average uncertainty measures on the six cars are generated as shown in Fig. 6. The horizontal and vertical axes denote the alternative cars and dissimilarity/average uncertainty measure respectively. They reflect the extent that the final aggregated result could support the DM for a decision. It can be seen that the dissimilarity measures of Car 2 and Car 5 are relatively high. The reason lies in that Car 2 is assessed to be *Poor* on e_4 “Horsepower” with the belief degree of 0.533, while it is assumed to be *Top* on e_2 “Breaking” and e_3 “Handling” with the belief degree of 0.667 and 0.4. So both the advantages and disadvantages of Car 2 are obvious that would make a DM confused. It is the same case for Car 5 because it is assessed to be *Poor* on e_4 “Horsepower” and e_2 “Braking” with the belief degree of 0.533 and 1.0 respectively, while it is considered to be not bad on e_1 “Acceleration” and e_6 “Powertrain”. Car 3 owns the smallest dissimilarity measure because the assessments on it respect to different attributes don’t differentiate much. So the aggregated BD of Car 3 can be used more easily for the ranking order. With respect to the average uncertainty measure, the values are all relatively small because no assessment is allocated to more than two evaluation grades. Generally speaking, the average uncertainty has little impact on the reliability of the final result for any car, while the dissimilarity plays a primary role in determining the support degree of the final aggregated results.

6.2. Generating the final aggregated results

If we use utility to represent the assessment of cars on each attribute, the BDs in Table 9 can be transformed to utilities in Table 11. Although the transformation from Tables 9 to 11 loses some original information, it gives us a panoramic view on the performance of six cars. For instance, it is clear that Car 3 is assessed to be the best on e_1, e_6 and e_7, while Car 5 is assessed to be the worst on e_2, e_4 and e_7.

The overall assessment of each car under the proposed weight assignment method can be generated by the ER algorithm. For example, the aggregated distributed result of Car 1 using the proposed weights is shown in Fig. 7. It is obvious that Car 1 is assessed to be *Good* with a belief degree higher than 0.5. It is caused by the fact that e_2, e_3, e_5, e_6 and e_7 are all assessed to be *Good* with certain belief degrees. Similarly, the general performance on Car 1 is not assessed to be *Worst* because no basic attribute is assessed to this grade on Car 1.

Table 12 shows the final aggregated results and the global uncertainty measures of the six cars provided that the weights are calculated by the proposed methods. They are distributed assessment results which enable us to make a specific insight into the performance of each car. The aggregated result of Car 2 is the most uncertain assessment because the global uncertainty measure of Car 2 is the largest among the six cars. Comparatively, the performance of Car 3 is more determinate since its global uncertainty measure is the smallest.

The overall distributions of Car 3, Car 5 and Car 6 contain ignorance because the original assessments on these three cars include global ignorance. The overall utilities and ranking orders of the six cars under the above mentioned weight assignment methods are summarized in Table 13, in which "Averaging" means the seven attributes are of equal importance as stated in (Yang, 2001). From Table 13, we can see that the averaging weights, GMD, SD, and CRITIC have generated the same ranking order. It may be caused by the fact that these methods transform the BDs of each attribute to utilities before generating weights except the averaging weights. Since the original assessments of each car on all the attributes in Table 9 only involve a maximum of two adjacent evaluation grades, the information loss in the utility transformation is limited. When the assessment to an alternative is quite uncertain, the BDs may involve more than two adjacent evaluation grades. So the average uncertainty measure defined in Eq. (31) is large to some extent. The utility transformation will incur more information loss in this circumstance.

The results in Table 13 show that Car 3 is assessed to be the best by most of the methods. It can be interpreted that Car 3 is assessed to be *Top* on e_1 and e_6 to a certain degree, while it is assessed to be neither *Worst* nor *Poor* on any attribute. Considering the low dissimilarity and uncertainty depicted in Fig. 6, the support degree of assessment on Car 3 is high. So Car 3 is surely the most preferred alternative. Car 1 is ranked as the first by the method in (Zhou et al., 2019). It can be seen

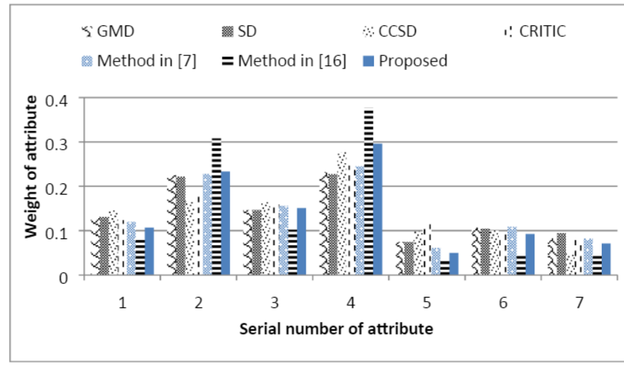

Fig. 5. Weights of attributes computed by different methods.

from Fig. 5 that e_4 is allocated with an abnormal large weight by EWAM, and Car 1 is assessed to be *Top* and *Excellent* on e_4, so the advantage of Car 1 is enlarged.

The results show that Car 2 is superior to Car 1 by GMD, SD, and Method in (Fu et al., 2018), while Car 1 is considered to be better than Car 2 by CCSD and the proposed method. It is caused by the fact that the former three methods only employ one dimension of information to generate attribute weights. So the correlation between each pair of attributes is not utilized in the weight assignment procedure. From Tables 12 and 13, we can also see that although the final utilities of Car 1 and Car 2 by most of the methods are similar, while their overall distributed assessments are quite different. Take the proposed method for example, the aggregated BD of Car 2 is a relatively averaging assessment on grade *P*, *A*, *G*, *E* and *T* (i.e., the fourth column in Table 12). Comparatively, the distributed overall assessment on Car 1 is mostly assigned to *G* and *T* (i.e., the third column in Table 12). This can also be seen from the dissimilarity calculated in Fig. 6 and global uncertainty generated in Table 12. The dissimilarity or global uncertainty of Car 1 is smaller than that of Car 2. The reason lies in that the original assessments of Car 1 on the seven attributes are quite focused on a set of evaluation grades, whereas Car 2 is assessed to all the six evaluation grades on different attributes. For example, Car 2 is assessed to be *Top* on e_2 and e_3, while assessed to be *Poor* on e_4. So the performance of Car 2 on different attributes is obviously contradictory which will probably make the DM a little confused. From this point of view, the support degree of the final assessment for Car 2 is less than Car 1. Car 5 is assessed to be the worst by all the weight assignment methods, and its dissimilarity shown in Fig. 6 is also too large. From Table 9, it is obvious that Car 5 is evaluated to be *Poor* and *Worst* on e_2 and e_4, whereas it is assessed to be *Good* on e_1, e_3, e_5 and e_6. But it is not considered to be *Top* on any attribute. If these six cars are the alternative products on market for a car manufacturer, it is significant for the product manager to find out the way of improving the poor performance on e_2 and e_4 for Car 5. The total incompatibility measure can also be generated by the average uncertainty measure and global uncertainty measure as shown in Fig. 8. As it has the similar trend with the dissimilarity measure depicted in Fig. 6, the two measures play the same role in this problem.

It should be mentioned that the utility estimation also determines

Table 9
BDs that six cars be assessed on seven attributes.

	Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
e_1	Acceleration	{(P,0.2), (A,0.8)}	{(G,0.5), (E,0.5)}	{(E,0.75), (T,0.25)}	{(A,0.4), (G,0.6)}	{(G,0.4), (E,0.4), (H,0.2)}	{(G,0.25), (E,0.75)}
e_2	Braking	{(G,1.0)}	{(E,0.333), (T,0.667)}	{(G,0.5), (E,0.5)}	{(P,0.75), (A,0.25)}	{(P,1.0)}	{(E,1.0)}
e_3	Handling	{(A,0.4), (G,0.6)}	{(E,0.6), (T,0.4)}	{(A,0.4), (G,0.4), (H,0.2)}	{(A,1.0)}	{(G,1.0)}	{(E,0.5), (T,0.4), (H,0.1)}
e_4	Horsepower	{(E,0.333), (T,0.667)}	{(P,0.533), (A,0.467)}	{(G,0.462), (E,0.538)}	{(G,0.385), (E,0.615)}	{(W,0.467), (P,0.533)}	{(A,0.267), (G,0.733)}
e_5	Ride quality	{(G,0.6), (E,0.4)}	{(A,1.0)}	{(H,1.0)}	{(G,1.0)}	{(G,1.0)}	{(H,1.0)}
e_6	Powertrain	{(A,0.4), (G,0.6)}	{(G,1.0)}	{(E,0.5), (T,0.4), (H,0.1)}	{(A,0.4), (G,0.6)}	{(G,0.6), (E,0.4)}	{(E,0.5), (T,0.3), (H,0.2)}
e_7	Fuel economy	{(G,1.0)}	{(G,1.0)}	{(E,1.0)}	{(G,1.0)}	{(A,1.0)}	{(G,1.0)}

Table 10
The standard deviations of the attribute weights by different methods.

	GMD	SD	CCSD	CRITIC	Method in (Fu et al., 2018)	Method in (Zhou et al., 2019)	Proposed
Standard deviation	0.0589	0.0561	0.0673	0.0548	0.0654	0.1293	0.0839

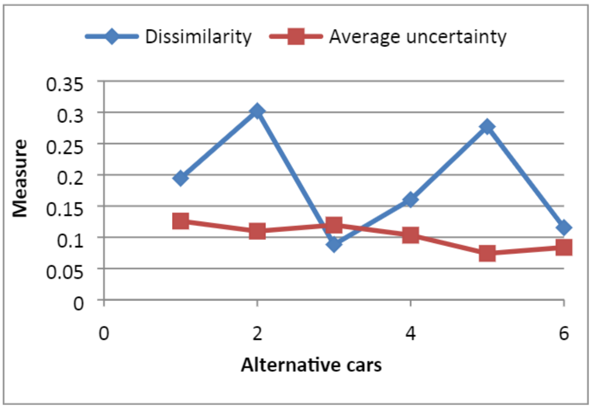

Fig. 6. Dissimilarity and average uncertainty measures on cars.

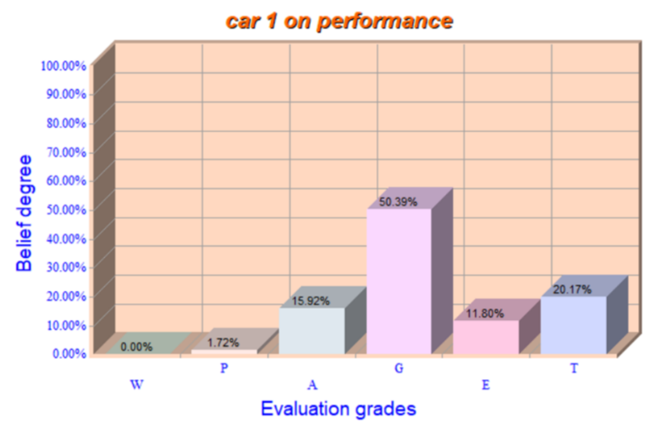

Fig. 7. The distributed overall assessment on Car 1 by the proposed weights.

the final results of all the cars. If the DM is risk-averse, the utility of six evaluation grades may be $u(W) = 0$, $u(P) = 0.35$, $u(A) = 0.6$, $u(G) = 0.8$, $u(E) = 0.95$, $u(T) = 1$. Then the aggregated utilities of six cars by the proposed weights are $u(\text{Car1}) = 0.8185$, $u(\text{Car2}) = 0.7612$, $u(\text{Car3}) = 0.8584$, $u(\text{Car4}) = 0.6973$, $u(\text{Car5}) = 0.4802$, $u(\text{Car6}) = 0.8479$. Although the ranking order is still Car3≻Car6≻Car1≻Car2≻Car4≻Car5, the final utilities of all the six cars increase. Furthermore, the degree of risk aversion will also influence the final assessment of each car.

6.3. Comparative analysis

Table 14 shows the comparison of the proposed method with several objective weight assignment methods.

Here, 'Num' means numerical value in the above table. The well-known GMD method, SD method, CRITIC, and CCSD to derive objective attribute weights are all based on the premise that assessments are presented in the form of numerical values. When qualitative attributes are assessed by subjective judgements such as BDs, these methods are not applicable to create the weights just from the assessment itself, unless the subjective method (such as AHP) is used. The proposed method is just designed to generate attribute weights by analyzing the intrinsic information contained in subjective assessments provided by DMs. In this sense, DMs don't need to provide their judgments on the importance of attribute passively. Otherwise, two aspects of subjective information should be extracted from DMs, which will probably make the decision-making process more complex and the DMs impatient. Thus the proposed method has the property of (i) and (ii).

(iii) and (iv) are associated with the proposed method due to the fact that the assessments are extracted in the form of BDs, which are easy to get from an individual or a group of experts when attributes cannot be quantified by numerical values. So the generated weights contain subjectivity to some extent. It is the same circumstance with the techniques in Refs. (Fu et al., 2018) and (Zhou et al., 2019). Since the GMD, SD, CRITIC and CCSD methods derive the relative importance from extracting quantitative values of attributes, the generated weights are unbiased and contain no subjectivity. The risk preference of evaluation grade is considered here because different types of utility function may be constructed due to the divergent attitude of DMs towards risk. This is reflected from the definitions of dissimilarity and conflict measure in Eqs. (12) and (13).

(v) and (vi) reflect the information utilized in the determination of attribute weights. The main difference between the proposed method and the techniques in Refs. (Fu et al., 2018) or (Zhou et al., 2019) lies in that whether the correlation between attributes is considered. The proposed method considers this dimension together with the divergence of different alternatives on each attribute, whereas Refs. (Fu et al., 2018) or (Zhou et al., 2019) only considers the dimension in (v). Although CCSD and CRITIC both take into account the dimensions in (v) and (vi), as they termed by 'the contrast intensity' and 'the conflicting character of the evaluation criteria', they are designed to cope with the situation where attributes are all assessed by numeric values. It can be easily reflected from the cases illustrated in these two methods (Diakoulaki et al., 1995; Wang & Luo, 2010). So how to measure the conflict or interdependency between BDs is not defined in the two techniques. Although the specific calculation process may be different, the above listed 7 methods all regard the dimension in property (v) as a factor to determine the attribute weights. For example, both SD, CRITIC and CCSD use the standard deviation as the divergence of different

Table 11
The utilities of 6 cars be assessed on 7 attributes.

	Acceleration		Braking		Handing		Horsepower		Ride quality		Powertrain		Fuel economy	
	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank
Car 1	0.36	6	0.60	4	0.52	4	0.9334	1	0.68	1	0.52	5	0.60	2
Car 2	0.70	3	0.9334	1	0.88	1	0.2934	5	0.40	6	0.60	4	0.60	2
Car 3	0.85	1	0.70	3	0.50	5	0.7076	3	0.50	4	0.85	1	0.80	1
Car 4	0.52	5	0.25	5	0.40	6	0.7230	2	0.60	2	0.52	5	0.60	2
Car 5	0.66	4	0.20	6	0.60	3	0.1066	6	0.60	2	0.68	3	0.40	6
Car 6	0.75	2	0.80	2	0.85	2	0.5466	4	0.50	4	0.80	2	0.60	2

Table 12
The final aggregated BDs of the six cars from the weights generated by the proposed method.

	Evaluation grade	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Performance	W	0	0	0	0	0.1445	0
	P	0.0172	0.1697	0	0.1671	0.4391	0
	A	0.1592	0.1962	0.0475	0.2796	0.0564	0.0818
	G	0.5039	0.1948	0.3259	0.3640	0.2771	0.3284
	E	0.1180	0.2209	0.5135	0.1893	0.0658	0.4536
	T	0.2017	0.2184	0.0495	0	0	0.0742
	H	0	0	0.0635	0	0.0172	0.0620
$Un(S(a_l))$		0.4326	0.6319	0.2371	0.4318	0.5072	0.2805

Table 13
The overall utilities and ranking orders of the six cars by some weight assignment methods.

	Averaging		GMD		SD		CCSD	
	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank
Car 1	0.5988	4	0.6324	4	0.6310	4	0.6502	3
Car 2	0.6306	3	0.6464	3	0.6468	3	0.6040	4
Car 3	0.7159	1	0.7110	1	0.7119	1	0.7012	1
Car 4	0.5222	5	0.5034	5	0.5043	5	0.5312	5
Car 5	0.4713	6	0.3834	6	0.3851	6	0.3926	6
Car 6	0.7017	2	0.7088	2	0.7077	2	0.6943	2
	CRITIC		Method in (Fu et al., 2018)		Method in (Zhou et al., 2019)		Proposed	
	Utility	Rank	Utility	Rank	Utility	Rank	Utility	Rank
Car 1	0.6445	4	0.6387	4	0.7135	1	0.6655	3
Car 2	0.6180	3	0.6467	3	0.6027	4	0.6244	4
Car 3	0.6980	1	0.7093	2	0.7062	2	0.7066	1
Car 4	0.5216	5	0.5031	5	0.5180	5	0.5151	5
Car 5	0.3937	6	0.3734	6	0.2550	6	0.3378	6
Car 6	0.6944	2	0.7095	1	0.6804	3	0.6979	2

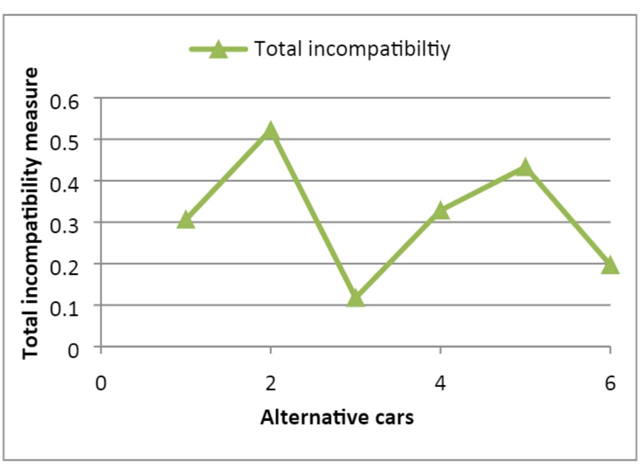

Fig. 8. Total incompatibility measure on cars.

alternatives on a certain attribute, comparatively, the method in Ref. (Zhou et al., 2019) utilize the concept of Shannon entropy as the measurement of dimension in (v).

With regard to (vii) and (viii), since the GMD method, SD, CRITIC and CCSD are all designed on the premise that attributes are represented by crisp numerical values, ignorance is not involved. Comparatively, Refs. (Fu et al., 2018); (Zhou et al., 2019) and the proposed method consider the ignorance embedded in BDs, which lead to the construction of optimization models for generating the range of the final results. As for (ix), the proposed method uses the original BDs to quantify the dimensions in (v) and (vi) for the elicitation of weights, so no information is lost or distorted. Ref. (Zhou et al., 2019) extended the entropy weight assignment method to tackling with the situations where attributes are assessed by interval numerical values, BDs with accurate belief degrees and interval BDs. But the factor of (v) is measured by the utilities converted from BDs instead of the direct discrepancy of BDs. So information contained in the BD may lose to a certain extent in the conversion. It is worth mentioning that all the above listed techniques require more than two alternatives included in a decision-making problem. That is to say, these methods are invalid if there is no alternative.

Last but not least, the proposed method can generate the support degree of assessment for decision-making which helps the DM to know how dependency the assessment is. Other methods don't incorporate this step, so only a final aggregated result can be generated. The support degree reflects the intrinsic characteristic of the assessment information provided by DMs. So the credibility or reliability of the assessment can be manifested to a certain degree.

7. Conclusions

In this paper, a novel weight assignment method for MADM problems is proposed on condition that the assessments are presented by BDs. The main principle behind the method is that the weight is correlated with the divergence of subjective judgment among different alternatives and the conflict between each pair of attributes. To do that, the conflict measures on both the alternative and evaluation grade levels between two attributes are designed. A comprehensive weight assignment method is then presented based on the above two aspects. Secondly, an alternative approach is proposed to quantify the support degree of assessment for decision-making in a MADM problem. It is

Table 14
Comparison of several weight assignment methods.

	Property	Method						
		Proposed	GMD (Fu et al., 2018)	SD (Diakoulaki et al., 1995)	CCSD (Wang & Luo, 2010)	CRITIC (Diakoulaki et al., 1995)	Method in Fu et al. (2018)	Method in Zhou et al. (2019))
i	Assessment form	BD	Num	Num	Num	Num	BD	BD
ii	Ability to tackle with subjective judgements	√	×	×	×	×	√	√
iii	The extent of subjectivity	certain	×	×	×	×	certain	certain
iv	Considering the risk preference	√	×	×	×	×	×	√
v	Considering the divergence of different alternatives	√	√	√	√	√	√	√
vi	Considering the correlation between attributes	√	×	×	√	√	×	×
vii	Consideration of ignorance	√	×	×	×	×	√	√
viii	Optimization models	√	×	×	√	×	√	√
ix	No information loss	√	√	√	√	√	√	×
x	Practicability related to alternative number	≥2	≥2	≥2	≥2	≥2	≥2	≥2
xi	Support degree of assessment for decision	√	×	×	×	×	×	×

based on the logic that the more discrepancy among different attributes on an alternative, the less supportive of the aggregated value to the final decision. Meanwhile, the more uncertainty contained in the assessment of each attribute on an alternative, the less support degree of the combined result. As such, the dissimilarity on alternative and uncertainty measure of BD are defined and analyzed. Some properties and comparisons are given to illustrate the effectiveness and applicability of the proposed methods. It is hoped that the method proposed in this paper can contribute to the development of the ER approach for MADM problem. In the future, the dissimilarity measure between interval belief distributions (IBDs) is to be studied, and how to reflect the ignorance contained in the original IBDs in the generated weights will be further analyzed.

CRediT authorship contribution statement

Mi Zhou: Conceptualization, Methodology, Investigation, Writing - original draft. **Yu-Wang Chen:** Review, Grammar checking. **Xin-Bao Liu:** Formal analysis, Supervision. **Ba-Yi Cheng:** Review. **Jian-Bo Yang:** Supervision.

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