Interactive minimax optimisation for integrated performance analysis and resource planning

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- Interactive multiple objective optimisation
- Performance analysis
- Resource planning
- Data envelopment analysis

**Abstract**

Analysing performances for future improvement and resource planning is a key management function. Data Envelopment Analysis (DEA) provides an analytical mean for performance modelling without assuming parametric functions. Multiple Objective Optimisation (MOO) is well-suited for resource planning. This paper reports an investigation in exploring relationships between DEA and MOO models for equivalent efficiency analysis in a MOO process. It is shown that under certain conditions minimax reference point models are identical to input-oriented dual DEA models for performance assessment. The former can thus be used for Hybrid Efficiency and Trade-off Analyses (HETA). In this paper, these conditions are first established and the equivalent models are explored both analytically and graphically to better understand HETA. Further investigation in the equivalence models leads to the modification of efficiency measures and the development of a minimax reference point approach for supporting integrated performance analysis and resource planning, with the Decision Maker’s (DM) preferences taken into account in an interactive fashion. Both numerical and case studies are conducted to demonstrate the proposed approach and its potential applications.

**1. Introduction**

DEA and MOO are tools in management control and planning and to an extent have been developed separately for several decades, with the former directed to analysing past performances as part of management control function and the latter to planning future performances [8]. As a performance measurement and analysis technique, DEA is a non-parametric frontier estimation methodology based on linear programming for evaluating the relative efficiency of a set of comparable Decision Making Units (DMUs) that share common functional goals. DEA has evolved tremendously and has been researched extensively since the original work of Charnes et al. [6].

DEA and MOO have much in common, though they retain their own distinctive traits [1,5,14,23,24]. For instance, Doyle and Green [10] suggested that DEA is a Multiple Criteria Decision Analysis (MCDA) method itself. Belton and Vickers [4] and Stewart [24] described similarities between the formulations of basic DEA models and classical multi-attribute value function approaches. Sarkis [21] termed DEA as a reactive approach to MCDA where different alternatives are evaluated objectively. In particular, Golany [11,12] developed an interactive model to allocate a set of input levels as resources and to select the most preferred output levels from a set of alternative points on the efficient frontier. Athanassopoulos [23] used goal programming and DEA to support resource allocation. Post and Spronk [19] combined the use of DEA and interactive goal programming to adjust the upper and lower feasible boundaries of the input and output levels. Joro et al. [14] showed the structural similarity between DEA and multiple objective linear programming for value efficiency analysis [13,15–17]. The above techniques require prior preference information and/or lead to variation from or restriction on the efficient frontier of an initial DEA model, although a hybrid approach was proposed for performance improvement without requiring prior preferences or changing an efficient frontier [28,29].

The attraction of DEA is that it provides an analytical mean for performance modelling without assuming parametric functions between inputs and outputs and an efficient frontier is formulated on which a DMU can plan its resources and set its future improvement targets proven achievable by its peers. It is therefore important that the DM of the DMU can explore the same efficient frontier in order to identify his most preferred performance target in comparison with a fixed benchmark embedded in an initial DEA model. Such exploration would be desirable if conducted in an interactive and consistent manner as the DM can have an opportunity to investigate and learn what efficient solutions are available and what resources need to be consumed to get to his most
preferred solution, so that a well-informed decision can be made without having to assume overall preferences a priori.

This paper reports an investigation into exploring relationships between DEA and MOO models for integrated efficiency analysis and resource planning. It is shown that minimax reference point models are identical to input-oriented dual DEA models under certain conditions. This equivalence relationship means that the former can be used for HETA on the same efficient frontier, so that performance targets can be set and required resources can be planned in an integrated and consistent manner, with the DM’s preferences taken into account in an interactive fashion. In this paper, computational studies are conducted to analyse efficiency analytically and graphically in an MOO process, leading to the construction of new efficiency measures for HETA. Based on the equivalent reference point models, a computational procedure is proposed to analyse data envelope and efficient frontier for interactive trade-off analysis and informed search for the most preferred performance targets. An interactive approach is then explored for HETA. Numerical examples and a case study are examined to demonstrate HETA and its potential application.

The remainder of the paper is organised as follows. Section 2 briefly presents typical input-oriented DEA models as an analytical means for performance modelling, and minimax reference point models as a basis for resource planning. In Section 3, an interactive minimax reference point approach is investigated for HETA. Section 4 reports a case study on supplier performance analysis to illustrate the interactive approach. The paper is concluded in Section 5.

2. Analytical models for performance analysis and resource planning

DEA was initially developed by Charnes, Cooper and Rhones in 1978 for measuring and analysing the relative efficiencies of comparable DMUs with incommensurable inputs and outputs. In DEA, an efficient frontier is formulated, separating efficient DMUs from inefficient ones. An efficient DMU means that no other DMU can either produce the same outputs by consuming fewer inputs, known as the input-oriented approach, or produce more outputs by consuming the same inputs, known as output-oriented approach. In this paper, we first briefly present DEA as a non-parametric means for performance modelling.

2.1. Input-oriented DEA models as a means for performance modelling

The original DEA model proposed by Charnes et al. [6] is a fractional non-linear programming model, known as the CCR model. The objective function in the model is to maximise the single ratio of weighted outputs over weighted inputs for a model. The objective function in the model is to maximise the modelling orientated approach. In this paper, we DMU comparable numerical examples and a case study are examined to demonstrate HETA and its potential application.

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Suppose there are n DMUs, each producing s outputs denoted by yrt (the rth output of DMU j) and consuming m inputs denoted by xtt (the ith input of DMU j). The input-oriented dual CCR model is then defined as follows [7]:

\[
\begin{align*}
\text{Min } & \quad \theta_j \\
\text{s.t. } & \quad \theta_j x_{ij} - \sum_{j'=1}^n \lambda_{j'} x_{ij'} \geq 0, \quad i = 1, \ldots, m \\
& \quad \sum_{j=1}^n \lambda_j y_{rj} \geq y_{rj}, \quad r = 1, \ldots, s; \quad \lambda_j \geq 0 \text{ for all } j
\end{align*}
\]

In the above model, \( \lambda_j \) is a reference parameter for DMU \( j = 1, \ldots, n \) and \( \lambda_j > 0 \) means that DMU \( j \) is used to construct a composite DMU as a benchmark for the observed DMU0. At the optimal solution of model (1), an optimal composite DMU is constructed that cannot underperform DMU0, which is referred to as the DEA efficient solution or the benchmark of DMU0 in this paper. This benchmark consumes at most a proportion \( \theta_j \) of the inputs of DMU0 and produces at least the same outputs as DMU0, \( \theta_j = 1 \) represents a full efficiency score and \( 0 < \theta_j < 1 \) reveals the presence of inefficiency. The parameter \( \theta_j \) indicates the degree to which DMU0 has to reduce the consumption of its resources (inputs) in order to become efficient. The reduction is employed concurrently to all inputs and results in a radial movement towards the envelopment surface [7].

Note that such a radial movement strategy is embedded in the DEA modelling mechanism a priori and does not necessarily take management preferences into consideration, so it is technical rather than preferential. In the following sections, we will explore how efficiency analysis can be conducted equivalently in an MOO process so that trade-off analysis can be consistently conducted in the same framework to plan resources, with management preferences incorporated in an interactive fashion.

2.2. Minimax reference point models as a basis for resource planning

In the input-oriented dual CCR model, an efficiency score is generated for an observed DMU0 by minimising inputs with outputs constrained as least at their required levels. This is in essence a MOO problem. In this section, we briefly describe basic MOO concepts and models, in particular the minimax reference point models as a basis for resource planning.

Suppose a MOO problem has \( m \) objectives to be minimised, in general represented by

\[
\text{Min } \quad f(\lambda) = [f_1(\lambda) \cdots f_m(\lambda)]
\]

s.t. \( \lambda \in \Omega = \{\lambda | g_j(\lambda) \leq 0, h_j(\lambda) = 0; j = 1, \ldots, k_1, l = 1, \ldots, k_2\} \) (2)

where \( \Omega \) is a feasible decision space, \( f_i(\lambda) \) \( (i = 1, \ldots, m) \) are continuously differentiable objective functions, and \( g_j(\lambda) \) \( (j = 1, \ldots, k_1) \) and \( h_j(\lambda) \) \( (l = 1, \ldots, k_2) \) are continuously differentiable inequality and equality constraint functions respectively. In this paper, \( f_i(\lambda), g_j(\lambda) \) and \( h_j(\lambda) \) are all assumed to be the linear functions of \( \lambda \).

In a MOO problem, we are interested in finding efficient solutions. A feasible solution \( \lambda^* \) is said to be efficient if there exists no other feasible solution which is better than \( \lambda^* \) at least on one objective and as good as \( \lambda^* \) on all other objectives. An efficient solution can be formally defined as follows.

**Definition 1.** In formulation (2), a feasible solution \( \lambda^* \in \Omega \) is an efficient solution if and only if there does not exist any other feasible solution \( \lambda \in \Omega \) such that \( f_i(\lambda) \leq f_i(\lambda^*) \) for all \( i = 1, \ldots, m \) and \( f_i(\lambda^*) < f_i(\lambda^*) \) for at least one \( \tau \in \{1, \ldots, m\} \).

Any efficient solutions of a MOO problem can be generated using a minimax formulation [22,27]. Suppose \( \lambda \) is an efficient solution of model (2) and \( f^* \) is the minimum feasible value of objective i. There exists a weighting vector \( w \) satisfying \( w_1 = 1 \) and \( w_i > 0 \) for \( i = 2, \ldots, m \) and a reference point \( f^* \) such that \( \lambda \) can be generated by solving the following weighted minimax reference point problem [27]:

\[
\text{Min } \quad \text{Max } \lambda \in \Omega \quad (w_i(f_i(\lambda) - f^*_i))
\]

s.t. \( \lambda \in \Omega \) (3)

The weighted minimax reference point formulation will be called the ideal point model if the ideal point \( f^* = [f^*_1 \cdots f^*_m]^\top \) is used as the reference point \( f^* = [f^*_1 \cdots f^*_m]^\top \). In the minimax reference point formulation, for a given weight vector, the DM is assumed to
be satisfied with an efficient solution $\lambda \in \Omega$ at which $f(\lambda)$ has the shortest weighted distance to $f^{\text{ref}}$ measured in $\infty$-norm in the objective space. If $f^{\text{ref}}$ is no worse than an ideal point, or $f^{\text{ref}} \leq f^{*}$, then the weighted maximin formulation given in formulation (3) can be equivalently transformed into the following form by introducing an auxiliary variable $\theta$ [18,27]:

$$\begin{align*}
\text{Min} & \quad \theta \\
\text{s.t.} & \quad w_i(f_i(\lambda) - f_i^{\text{ref}}) \leq \theta, \quad i = 1, \ldots, m; \quad \lambda \in \Omega; \quad \theta \geq 0
\end{align*} \tag{4}$$

It will be shown in the following section that under certain conditions the maximin formulations can be used to generate the same DEA efficiency scores, composite inputs and outputs as the input-oriented dual CCR models produce.

3. Interactive minimax reference point approach for HETA

In this section, we investigate an interactive approach based on minimax reference point models to support integrated performance analysis and resource planning. In the approach, DEA is primarily employed for non-parametric performance modelling, and equivalence between DEA models and minimax models are investigated to enable equivalent performance analysis in a MOO process. The characteristics of efficiency frontier are explored and consistent efficiency measures are investigated for HETA. A gradient projection method based on the ideal point model is then investigated to facilitate interactive trade-off analysis based on the results of efficiency analysis.

3.1. Equivalence between minimax and DEA models

The input-oriented dual CCR model given in formulation (1) can be equivalently rewritten by

$$\begin{align*}
\text{Min} & \quad \theta_i \\
\text{s.t.} & \quad \lambda_j x_{ik} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad \theta_i \geq 0 \\
\lambda & \in \Omega_i, \quad \theta \geq 0
\end{align*} \tag{5}$$

In this section, we prove that formulation (4) is equivalent to formulation (5) under certain conditions. In formulation (5) the ith composite input can be represented by $f_i(\lambda)$ as follows:

$$f_i(\lambda) = \frac{1}{\lambda_j} x_{ik} \quad (i = 1, \ldots, m) \quad \text{and} \quad \lambda = [\lambda_1, \ldots, \lambda_n]^T$$

In the following trade-off analysis, the ith composite input $f_i(\lambda)$ will be defined as an objective for minimisation, so that are $m$ objectives in total. The minimum feasible value of the ith composite input for the observed $DMU_0$ is denoted by $f_i^{\text{ref}} = f_i(\lambda^*)$ where $\lambda^*$ can be found by solving the following single objective optimisation problem:

$$\begin{align*}
\text{Min} & \quad f_i(\lambda) = \sum_{j=1}^{n} \lambda_j x_{ij} \\
\lambda & \in \Omega_i
\end{align*} \tag{6}$$

Note that $\lambda = [\lambda_1, \ldots, \lambda_m]^T$ is the ideal point in the objective space spanned on $\Omega_i$ by the $m$ objective functions defined in (6) for the observed $DMU_0$. Suppose the feasible decision space $\Omega$ in formulation (4) is set to be the same as defined in formulation (5), or $\Omega = \Omega_i$. The equivalence relationship between the input-oriented dual CCR model (1) or (5) and the minimax formulation (4) can be established by the following theorem.

**Theorem 1.** Suppose $x_{ij} \geq 0$ for any $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The input-oriented dual CCR model (5) can be equivalently transformed to the minimax formulation (4) using Eq. (6), formulation (7) and the following equations:

$$\begin{align*}
w_i &= \frac{1}{x_{ij}} \\
f_i^{\text{ref}} &= f_i^{\text{ref}} = \frac{\min_{\lambda \in \Omega_i}}{w_i} = x_{ij} f_i^{\text{ref}} \\
\theta &= \frac{\theta_i}{w_i} \\
m_{\lambda} &= \min_{1 \leq i \leq m} \left\{ w_i f_i^{\text{ref}} \right\}
\end{align*} \tag{7}$$

**Proof.** Using Eqs. (6) and (8), the input-oriented dual CCR model (5) can be equivalently rewritten as follows:

$$\begin{align*}
\text{Min} & \quad \theta_i \\
\text{s.t.} & \quad \frac{1}{w_i} f_i(\lambda) \geq 0, \quad i = 1, \ldots, m; \quad \lambda \in \Omega_i \\
\lambda & \in \Omega_i, \quad \theta_i \geq 0
\end{align*} \tag{12}$$

The first $m$ objective constraints in formulation (12) can be equivalently transformed as follows, where “$\Longleftrightarrow$” means “is equivalent to”. For any $i = 1, \ldots, m$, we have

$$\begin{align*}
\frac{1}{w_i} f_i(\lambda) \geq 0 & \Longleftrightarrow w_i f_i(\lambda) \leq \theta_i \\
\Longleftrightarrow w_i f_i(\lambda) - f_i^{\text{ref}} \leq \theta_i - f_i^{\text{ref}} \\
\Longleftrightarrow w_i (f_i(\lambda) - f_i^{\text{ref}}) \leq \theta_i
\end{align*} \tag{13}$$

Moreover, the objective function of formulation (12) becomes

$$\begin{align*}
\text{Min} & \quad \theta_i = \text{Min}(\theta_i - f_i^{\text{ref}}) = \text{Min} \theta_i \\
\text{s.t.} & \quad \text{for any } \lambda \in \Omega_i, \quad \text{from Eq. (11)} \quad \text{we have}
\end{align*} \tag{14}$$

$$\begin{align*}
f_i(\lambda) - f_i^{\text{ref}} \geq 0 & \quad \text{for any } \lambda \in \Omega_i \text{ and } i = 1, \ldots, m
\end{align*} \tag{15}$$

Also there is

$$\begin{align*}
\theta = \theta_i - f_i^{\text{ref}} \geq w_i f_i(\lambda) - f_i^{\text{ref}} \geq w_i f_i(\lambda) - f_i^{\text{ref}} \geq 0 & \quad \text{for any } i = 1, \ldots, m
\end{align*} \tag{16}$$

E.O.D.

**Theorem 1** shows that if the reference point in model (4) is set by $f^{\text{ref}} = [f_1^{\text{ref}}, \ldots, f_m^{\text{ref}}] = f^{\text{ref}} = [x_{i1}, \ldots, x_{im}]^T$ and $w_i$ by Eq. (8), then the input-oriented dual CCR model will be identical to the following minimax reference point formulation:

$$\begin{align*}
\text{Min} & \quad \theta \\
\text{s.t.} & \quad \left( \frac{1}{w_i} \lambda_j x_{ij} - f_i^{\text{ref}} \right) \leq \theta_i, \quad i = 1, \ldots, m; \quad \lambda \in \Omega_i
\end{align*} \tag{18}$$

By identical it is meant that they share the same decision and objective spaces and have the same optimal solution. Since $f^{\text{ref}} = F^{\text{ref}} = [x_{i1}, \ldots, x_{im}]^T = f^{\text{ref}} = [x_{i1}, \ldots, x_{im}]^T$, such a reference point $f^{\text{ref}}$ is called super-ideal point and the minimax reference point formulation established using the super-ideal point is therefore referred to as the super-ideal model in this paper. If $x_{ij} \geq 0$ for all $i = 1, \ldots, m, f_i^{\text{ref}} = F^{\text{ref}} = [x_{i1}, \ldots, x_{im}]^T$ such a point $F^{\text{ref}} = [x_{i1}, \ldots, x_{im}]^T$ lies in the same line emitting from the origin to the point $[x_{i1}, \ldots, x_{im}]^T$. So, the origin, or $f^{\text{ref}} = 0$, is a special super-ideal point. The super-ideal point model [18] can be used to generate the same efficiency score and efficient composite inputs and outputs of the observed $DMU_0$ as from the dual CCR model (1).
if in model (18) \( w_i \) is calculated by Eq. (8) and \( f_i^{cf} \) by Eqs. (9) and (11).

### 3.2. Equivalent efficiency analysis in an MOO process

In this section, we conduct analytical and graphical investigation into how the above equivalence relationship can be explored to support HETA. In models (5) and (18), \( \lambda = [\lambda_1, ..., \lambda_m] \) is a vector of decision variables and \( \Omega_e \) defines the feasible decision space for the observed DMUs. Any feasible solution \( \lambda \in \Omega_e \) represents a composite unit for the observed DMUs. The DEA efficient solution, denoted by \( \lambda^{DEA} \), is an efficient solutions in \( \Omega_e \) or \( \lambda^{DEA} \in \Omega_e \).

If \( \lambda_i \neq 0 \) and \( \lambda_j = 0 \) for all \( i = 1, ..., m \) with \( i \neq j \), then DMU \( j \) itself is a composite solution in \( \Omega_e \). In this case, the feasible value of \( \lambda_j \) can be calculated by requiring \( \lambda_j y_{ij} \geq y_{ij} \) for all \( r = 1, ..., s \), which leads to the following result:

\[
\tilde{\lambda}_j = \max_{1 \leq r \leq s} \left\{ \frac{y_{rj}}{y_{ij}} \right\}
\]

(19)

The value \( \tilde{\lambda}_j \), calculated using Eq. (19) is a scaling factor for DMU \( j \), which can be used to decide the location of DMU \( j \) in the objective space constructed for DMUs, as shown later.

\[
f_{i}(\lambda) = \sum_{j=1}^{n} \lambda_j x_{ij}
\]

is the \( i \)th composite input. In model (18), \( f_{i}(\lambda) \) \((i = 1, ..., m)\) together constitute an objective space. Therefore, the corresponding MOO problem, in which the super-ideal point model (18) is constructed, is the following \( m \) objective linear programming problem:

\[
\text{Min } \bar{f} = [f_1(\lambda), ..., f_m(\lambda)]^T
\]

s.t. \( \lambda = [\lambda_1, ..., \lambda_m]^T \in \Omega_e \)

(20)

The \( m \) dimensional objective space is expanded by the \( m \) objective functions \( f_{i}(\lambda) \) \((i = 1, ..., m)\). The feasible objective space, denoted by \( f(\Omega_e) \), is the projection of the feasible decision space \( \Omega_e \) using the objective functions. All the efficient solutions of model (20) are represented by \( E(f(\Omega_e)) \), which includes all possible efficient composite solutions for the observed DMUs to benchmark against. In this section, we show that the DEA efficient solution is one of the efficient solutions in \( E(f(\Omega_e)) \), or \( f(\lambda)^{DEA} \in E(f(\Omega_e)) \).

We will show how \( E(f(\Omega_e)) \) can be graphically and analytically generated and represented for an observed DMU \( j \) if there are up to three inputs with any number of outputs and any number of DMUs. In general, if there are more than three inputs, any solution in \( E(f(\Omega_e)) \) can be explored using the minimax reference point model (18) with the reference point set as the ideal point, or \( f^{cf} = f_B = [f_{B1}, ..., f_{Bn}]^T \), and \( w_r \) allowed to change in \( R^+ \).

Note that an original DMU \( j \) can be represented in \( f(\Omega_e) \) by

\[
f_{i}(\lambda) = \tilde{\lambda}_j x_{ij} \quad \text{for } i = 1, ..., m
\]

(21)

Since \( \tilde{\lambda}_j = 1 \), the observed DMUs \( j \) is represented in \( f(\Omega_e) \) using its true (un-scaled) input values. In general, all other original DMUs are represented in \( f(\Omega_e) \) using their scaled input values. The locations of the original DMUs using Eq. (21) provide a basis to characterise \( E(f(\Omega_e)) \) as shown later in this paper.

In the rest of this subsection, we generate \( E(f(\Omega_e)) \) for a simple DEA model with two inputs, two outputs and three DMUs in order to show analytically and graphically how efficiency analysis can be conducted in a MOO process on the basis of the reference point model (18). Consider a DEA model defined in Table 1, where \( x_1 \) and \( x_2 \) are the two inputs and \( y_1 \) and \( y_2 \) the two outputs.

Solving model (1) or model (18) for example 1, we can find that DMU \( B \) and DMU \( C \) are both efficient but DMU \( A \) is inefficient with an DEA efficiency score of 0.446. Furthermore, the values of the decision variables of the DEA efficient solution are \( \lambda = [\lambda_1, \lambda_2, \lambda_3]^T = [0.179, 0.179, 0.619]^T \) for DMU \( A \), with the optimal composite inputs given by \( x = [x_1, x_2]^T = [1.8, 0.9]^T \) and the optimal composite outputs by \( y = [y_1, y_2]^T = [3.94, 5.01]^T \). In the following, we first show how to generate these results graphically using an equivalent MOO model defined by formulation (20).

The equivalent MOO problem for DMU \( A \) in example 1 is given as follows:

\[
\text{Min } f = [f_1(\lambda) = 4\lambda_1 + 2\lambda_2 + 8\lambda_3, \ f_2(\lambda) = 2\lambda_1 + 4\lambda_2 + 3\lambda_3]
\]

s.t. \( \lambda = [\lambda_1, \lambda_2, \lambda_3]^T \in \Omega_4 \)

\[
\begin{align*}
2.5\lambda_1 + 16\lambda_2 + 6\lambda_3 & \geq 2.5 \\
5\lambda_1 + 8\lambda_2 + 20\lambda_3 & \geq 5 \\
\lambda_1 & \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0
\end{align*}
\]

(22)

The feasible decision space \( \Omega_4 \) of problem (22) is the first quadrant above the planes \( AC \) and \( BD \), as illustrated in Fig. 1(a), which is an unbounded region.

The feasible objective space of problem (22) is the projection of the feasible decision space to the objective space using the objective functions \( f_1(\lambda) \) and \( f_2(\lambda) \), denoted by \( f(\Omega_4) \). Since both \( f_1(\lambda) \) and \( f_2(\lambda) \) are linear functions, we can generate the feasible objective space \( f(\Omega_4) \) by locating the extreme points \( A, B, C \) and \( D \) in the objective space. The coordinates of these points in the decision space are shown in Fig. 1(a). Their coordinates in the objectives space can be calculated using the objective functions \( f_1(\lambda) \) and \( f_2(\lambda) \), as shown in the second and third columns in Table 2.
Table 2

<table>
<thead>
<tr>
<th>DMU</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$z_i x_1^r$, $x_2^r_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4</td>
<td>2</td>
<td>$1 + {4, 2}$</td>
</tr>
<tr>
<td>B</td>
<td>1.25</td>
<td>2.5</td>
<td>$0.625 + {2, 4}$</td>
</tr>
<tr>
<td>C</td>
<td>3.336</td>
<td>0.417</td>
<td>$0.417 + {8, 1}$</td>
</tr>
<tr>
<td>D</td>
<td>1.907</td>
<td>0.514</td>
<td>$0.074 + {2, 4} + 0.22 \times {8, 1}$</td>
</tr>
</tbody>
</table>

In general, the objective coordinates can be generated using the scaling factors of the DMUs, as shown in the last column in Table 2. For instance, the scaling factor for DMU B can be found using Eq. (19) by $\beta_B = \max\{2.5/16, 5/8\} = 0.625$. So, the coordinates of point B in $f(\Omega)$ can be generated by scaling the objective values of DMU B by $0.625 \times \{2, 4\}$ and point D in Fig. 1(a) is composed of DMU B scaled by a factor of 0.074 and DMU C scaled by a factor of 0.22. Its objective coordinates are thus given by $0.074 \times \{2, 4\} + 0.22 \times \{8, 1\} = \{1.907, 0.514\}$.

As shown by the shaded region in Fig. 1(b), the feasible objective space $f(\Omega)$ is an unbounded region. Note that point B is any point beyond point B on a line emitting from the origin through point B, referred to as the expansion point of point B, and point C is any point beyond point C on a line emitting from the origin through point C, referred to as the expansion point of point C. Therefore, the data envelope for DMU A in example 1 is composed of $\partial f(\Omega) \cup \partial f(\Omega)$. Its efficient frontier is given by $\partial f(\Omega) \cup \partial f(\Omega)$. In Fig. 1(b), the ideal point $f^\text{id}$ is given by taking the minimum feasible objective values, so $f^\text{id} = [1.25, 0.417]$. The super-ideal point $f^\text{sid}$ is given by $f^\text{sid} = f^\text{min}[4, 2]$ with $f^\text{min} = \min\{2.5/16, 0.417/2\} = 0.2085$. So $f^\text{sid} = [0.834, 0.417]$, which is on a line emitting from the origin to point A, represented by $2f_1 - 4f_2 = 0$, and is dominated by the origin point. The minimax contour at the origin is a rectangle with the origin as its centre and the line passing through its southwest and northeast corners.

Solving the input-oriented CCR dual model (5) is equivalent to solving the super-ideal point model (18). In Fig. 1(b), solving the super-ideal point model for DMU A is to expand the minimax contour until it just touches the feasible objective space at point A, which is the intersection of the above line and the efficient line segment $\partial f(\Omega)$ represented by $1.986f_1 + 0.657f_2 = 4.125$. So, the objective coordinates of point A are given by $[1.782, 0.891]$. The DEA efficient score $e_A$ for DMU A can then be calculated by $e_A = \frac{\overline{OA}}{\overline{OA}} = \frac{\sqrt{1.782^2 + 0.891^2}}{\sqrt{2^2 + 2^2}} = 0.446$ (23) which is the same efficiency score as the one generated by solving the input-oriented dual CCR model. The DEA composite inputs and outputs for DMU A can also be graphically generated as follows. Note that point A is a convex combination of point B and point D given by $A = \frac{0.265}{1 - 0.265} \times B + \frac{0.265}{1 - 0.265} \times D = 0.19 \times B + 0.81 \times D$ (24)

Since point B represents the scaled DMU B with $\beta_B = 0.625$ and point D represents a composite DMU composed of 0.074 of DMU B and 0.22 of DMU C, point A also represents a composite DMU composed of DMU B scaled by 0.179 (or 0.19 x 0.625 + 0.81 x 0.074) and DMU C scaled by 0.179 (or 0.81 x 0.22). So $x = [x_1, x_2, x_3]^T = [0.0179, 0.179]^T$, which leads to the same DEA composite inputs $x = [x_1, x_2]^T = [1.8, 0.9]^T$ and composite outputs $y = [y, y_2]^T = [3.94, 5.01]^T$.

Point A is one of the efficient solutions that DMU A can benchmark against. Other efficient solutions in $\partial f(\Omega) \cup \partial f(\Omega)$ could also be potential benchmarks for DMU A, depending upon the DMU’s preferences. In general, any efficient solutions for DMU A can be found by solving the following ideal point model with the weighting parameter $w_i$ changed systematically.

Min $\theta$

subject to $\sum_{i=1}^{n} \lambda_i x_i - f_i^\text{id} \leq \theta$, $i = 1, ..., m$; $\theta \geq 0$, $\lambda \in \Omega_\theta$ (25)

3.3. Computational procedure for exploring efficient frontier for HETA

In the previous subsection, we used a simple DEA example with three DMUs to show both the decision and objective spaces graphically. In this subsection, an analytical procedure is investigated to generate the data envelopes and efficient frontiers of complicated DEA problems for exploring the characteristics of HETA, so that an appropriate interactive method can be employed to support HETA, as discussed later in this section. The analytical procedure for generating a data envelope and its efficient frontier consists of the following steps.

Step 1: For a DEA problem, choose an observed DMU and define its objective functions as composite inputs, as shown in section 3.2.

Step 2: Calculate the scaling factors for all original DMUs using Eq. (19) and the coordinates of all scaled original DMUs in the objective space.

Step 3: Construct an initial data envelope by linking the identified points in the first quadrant, which includes all the scaled original DMUs, as shown in section 3.2.

Step 4: Expand the current data envelope by formulating a searching problem to check whether there is any feasible solution beyond a facet on the current envelope along the outwards normal direction of the facet.

Suppose there are two inputs and $I = [x_{11}, x_{12}]^T$ and $J = [x_{21}, x_{22}]^T$ are two adjacent extreme points on the current data envelope with $x_{12} \geq x_{22}$, so the line segment between I and J is a facet on the envelope. Then the searching function along the direction of the outwards normal vector of the facet is given by

$$g(f_1, f_2) = \begin{cases} -(x_{22} - x_{12})f_1(l) + (x_{21} - x_{11})f_2(l) & \text{if } N^\parallel \cdot f_1 \geq 0 \\ -(x_{12} - x_{22})f_1(l) + (x_{11} - x_{21})f_2(l) & \text{if } N^\parallel \cdot f_1 < 0 \end{cases}$$

(26)

where $N^\parallel$ is the outwards normal vector of the line $f_1$ and $f_1$ is the horizontal axis ($f_1$). $N^\parallel \cdot f_1 \geq 0$ in (26) means that the angle between $N^\parallel$ and $f_1$ is less than or equal to 90 degree whilst $N^\parallel \cdot f_1 < 0$ means that the angle is larger than 90°. The searching problem is defined by

Max $g(f_1^{(1)}, f_2^{(2)}(l))$

subject to $\lambda \in \Omega_\theta$(27)

where $\Omega_\theta$ is the decision space for the observed DMUs. If the optimal solution is $f_1^{(1)} = x_{11}$ and $f_2^{(2)} = x_{12}$ or $f_1^{(1)} = x_{21}$ and $f_2^{(2)} = x_{22}$ with $g^* = |x_{21}x_{22} - x_{11}x_{12}|$, there will be no feasible solution outside $f_1$, which is then recorded as the expanded facet of the data envelope. Otherwise, a new feasible solution is found and the data envelope is adjusted accordingly.

Step 5: If there is any unexpanded facet on the data envelope, go to Step 4.
3.4. Consistent measurement of technical and preferential efficiency

In Fig. 1(b), the line emitting from the origin to point A intersects with the efficient frontier at point A*, leading to the measurement of the DEA efficiency score, or $e_A = OA'/OA$, which is consistent with the suggestion of the composite DMU $A'$ as the benchmark for DMU A. However, such consistency may not always hold.

If the DEA model shown in Table 1 is slightly modified so that the input levels for DMU A are changed to $x_1 = 1.5$ and $x_2 = 3.5$ respectively, solving the dual CCR model for each DMU of this modified example shows that DMUs B and C are still efficient whilst DMU A is inefficient with a new DEA efficiency score of 0.833. The DEA composite unit for DMU A is DMU B scaled by a factor of 0.625 with the composite input levels given by 1.25 and 2.5 respectively. The objective space for DMU A in this modified example is drawn in Fig. 2. Its efficient frontier is still $E_1(f(x_A)) = BD$ and $CD$, but its data envelope becomes $A'X'$ and $A'B'D$ and $A'C'D$. Point A becomes an extreme point on the data envelope but is inefficient.

As shown in Fig. 2, solving the super-ideal point model (18) with the origin as the reference point is to expand the minimax contour until its extreme point on the data envelope but is inefficient with a new technical efficiency score, or $e_A = OA'/OA$, which is consistent with the benchmark for DMU A as the reference point and its targeted input vector by $x_A'' = [1.25, 2.917]^T$. From Fig. 2, two issues can be observed in the conventional DEA. The first issue is that the DEA efficiency score is calculated with point A' assumed as the benchmark for DMU A, or $e_A = OA'/OA = \frac{\sqrt{1.25^2 + 2.917^2}}{\sqrt{1.5^2 + 3.5^2}} = 0.833$.

However, point A' is not feasible. This leads to the second issue. In the conventional DEA it is Point B that is actually suggested as the benchmark for DMU A, which is inconsistent with the efficiency measurement. That is, it is not a radial movement from A to A' that makes DMU A become efficient.

To address the above mentioned two issues, it seems appropriate to define a new technical efficiency score, or TES for short, for DMU A as follows:

$$TE_{SA} = \frac{OT}{OT + AT} = \sqrt{\sum_{i=1}^{m} x_{Pi}^2} / \sqrt{\sum_{i=1}^{m} x_{Ai}^2 + \sum_{i=1}^{m} (x_{Pi} - x_{Ai})^2} = 0.731$$

In example 1 the DEA efficiency score $e_A$ and technical efficiency score $TES_A$ for DMU A are the same. In the modified example, we have $TES_A < e_A$. In general, the technical efficiency score can be defined as follows.

**Definition 2.** Suppose a DEA problem has $m$ inputs. In the objective space, suppose the observed DMU $A$ is represented by $A = [x_{A1}, \ldots, x_{Am}]^T$ and its targeted input vector by $T = [x_{T1}, \ldots, x_{Tm}]^T$. Then the technical efficiency score for DMU A is defined by

$$TES_A = \frac{OT}{OT + AT} = \sqrt{\sum_{i=1}^{m} x_{Pi}^2} / \sqrt{\sum_{i=1}^{m} x_{Ai}^2 + \sum_{i=1}^{m} (x_{Pi} - x_{Ai})^2}$$

The technical efficiency score of a DMU can be intuitively interpreted as a degree of minimum efforts that the DMU needs to make to reduce its inputs for achieving feasible efficiency. The closer the technical efficiency score is to one, the smaller the efforts needed to achieve the feasible efficiency. Analytically, a technical efficiency score measures the relative distance from a DMU to its nearest feasible efficient benchmark identified in the objective space through minimax optimisation. Regarding the relationship between the DEA efficiency score and the technical efficiency score, it can be shown that the following conclusion holds. Suppose $T = [x_{T1}, \ldots, x_{Tm}]^T$ is the suggested target for DMU A with $A = [x_{A1}, \ldots, x_{Am}]^T$. Then, the DEA efficiency score of DMU A will be equal to its technical efficiency score if the following conditions are met:

$$\frac{x_{Ai}}{x_{Ti}} = \frac{x_{Pi}}{x_{Pi}} \forall i = 1, \ldots, m$$

If the suggested target is the most preferred solution of the decision maker maximising his implicit utility function, the technical efficiency score defined in Eq. (30) will be referred to as the preferential efficiency score, measuring the degree to which the observed DMU needs to reduce or balance its inputs to achieve the most preferred target, defined as follows.

**Definition 3.** Suppose a DEA problem has $m$ inputs. In the objective space, suppose the observed DMU $A$ is represented by $A = [x_{A1}, \ldots, x_{Am}]^T$ and its preferred input vector by $P = [x_{P1}, \ldots, x_{Pm}]^T$. Then a preferential efficiency score for DMU A is defined by

$$PES_A = \frac{OP}{OP + AB} = \sqrt{\sum_{i=1}^{m} x_{Pi}^2} / \sqrt{\sum_{i=1}^{m} x_{Ai}^2 + \sum_{i=1}^{m} (x_{Pi} - x_{Ai})^2}$$

The preferential efficiency score of a DMU can be intuitively interpreted as a degree of minimum efforts that the DMU needs to make to balance its inputs for achieving the most preferred solution. The closer the preferential efficiency score is to one, the smaller the efforts needed to achieve the most preferred DMU. Analytically, a preferential efficiency score measures the relative distance from a DMU to its most preferred DMU in the context of minimax optimisation.

3.5. Interactive trade-off analysis and minimax optimisation

As a result of efficiency analysis, a benchmark is generated for a DMU, such as Point A' in Fig. 1(b) and Point B in Fig. 2 for DMU A. A benchmark can be justified for the consistent and objective comparison of efficiency if it is created on a common basis across all DMUs. This is because consistency and objectivity are paramount in efficiency assessment. However, a question arises as to whether the benchmark should also be used to plan resources for the DMU. If not, a further question is how to find an alternative efficient solution as a benchmark for resource planning, given that in resource planning it is the DMU's preferences that become paramount. To help address the questions raised above, an interactive approach is proposed, in which the DMU's indifference trade-offs are acquired to identify a new reference point and then a minimax optimisation problem is
solved to find a more preferred benchmark. This method is summarised as the following main steps.

Step 1: Check whether a current benchmark (e.g. generated from efficiency analysis) is what the DM prefers the most. While this question may be answered in different ways, in MCDA a popular criterion is to check whether the DM’s utility (or value) is maximised at this solution. Suppose the DM’s utility function is denoted by \( u(\lambda) = u(f_1(\lambda), \ldots, f_m(\lambda)) \). If \( u(\lambda) \) is given \textit{a priori}, a benchmark will be the DM’s most preferred solution if \( u(\lambda) \) is maximised at the benchmark within \( \Omega_0 \). The problem is that \( u(\lambda) \) is unknown in general, so this straightforward test is not feasible. However, local information about DM’s partial preferences may be acquired. For example, the DM may be capable of judging whether a given amount of the increased consumption of one resource (input) can be offset by a certain amount of the decreased consumption of another resource without changing his overall utility. In this paper, such an indifference trade-off or marginal rate of substitution is suggested as a means to acquire partial preference information from the DM as it can be used to estimate the local utility gradient of \( u(\lambda) \), such as \( \vec{u} (B) \) at Point B in Fig. 3 where the corresponding utility contour and its tangent line at Point B are also illustrated by the dotted curve and line. Without loss of generality, suppose the first input \( f_1(\lambda) \) is chosen as reference for increase by a given amount of \( \Delta f_1 \) from the current solution \( \lambda \). Suppose \( \Delta f_1 \) can be exactly offset by decreasing other inputs \( f_i(\lambda) \) by \( \Delta f_i \) for \( i = 2, \ldots, m \) without changing the DM’s overall utility. Then, the gradient of the utility function \( u(\lambda) \) at \( \lambda \) can be estimated as follows [27]:

\[
\nabla u(\lambda) = \left[ \frac{\partial u}{\partial f_1}, \ldots, \frac{\partial u}{\partial f_m} \right]^T = \frac{du}{d\lambda} \vec{u}(\lambda)
\]

with \( \vec{u}(\lambda) \approx \left[ \frac{\Delta f_1}{\Delta f_2}, \ldots, \frac{\Delta f_1}{\Delta f_m} \right]^T \)

(33)

Suppose the normal vector of the facet of the efficient frontier at \( \lambda \) is given by \( \vec{N}(\lambda) \). It is proven [27] that \( \vec{u}(\lambda) \) will overlap \( \vec{N}(\lambda) \) if \( \lambda \) is the most preferred solution maximising \( u(\lambda) \), with \( \vec{N}(\lambda) \) calculated by

\[
\vec{N}(\lambda) = [w_1\beta_1, \ldots, w_m\beta_m]^T = w_1\beta_1 \left[ \begin{array}{c} w_2\beta_2 \\ \frac{w_2\beta_2}{w_1\beta_1} \\ \vdots \\ \frac{w_m\beta_m}{w_1\beta_1} \end{array} \right]^T
\]

(34)

where \( \beta_i \) is the dual variable of the \( i \)-th objective constraint of the following ideal point formulation:

\[
\text{Min } \theta \quad \text{s.t. } w_i \left( \sum_{j=1}^n x_{ij} - f^0_i \right) \leq \theta \quad \text{for } i = 1, \ldots, m; \quad \lambda \in \Omega_0
\]

(35)

whose optimal solution will be \( \lambda \) if \( w_i \) is given by

\[
w_i = 1/(f_1(\lambda) + \delta - \vec{f}^0_i) \quad \text{for } i = 1, \ldots, m
\]

(36)

\( \delta \) is a perturbation parameter which is sufficiently small, with \( \delta = 0 \) if \( f_i(\lambda) > \vec{f}^0_i \). Checking whether the current solution \( \lambda \) is the DM’s most preferred solution is therefore equivalent to checking whether \( u(\lambda) \) is proportional to \( \vec{N}(\lambda) \) at \( \lambda \), or

\[
\left[ \frac{\Delta f_1}{\Delta f_2}, \ldots, \frac{\Delta f_1}{\Delta f_m} \right]^T \times \left[ \begin{array}{c} w_2\beta_2 \\ \frac{w_2\beta_2}{w_1\beta_1} \\ \vdots \\ \frac{w_m\beta_m}{w_1\beta_1} \end{array} \right] = w_1\beta_1 \left[ \begin{array}{c} w_2\beta_2 \\ \frac{w_2\beta_2}{w_1\beta_1} \\ \vdots \\ \frac{w_m\beta_m}{w_1\beta_1} \end{array} \right]
\]

or

\[
\Delta f_i = w_i\beta_i/\Gamma \vec{f}^0_1 \quad \text{for } i = 2, \ldots, m
\]

(37)

\( \Delta f_i = (w_i\beta_i/\Gamma \vec{f}^0_1)\Delta f_1 \) is the optimal indifference trade-off between inputs \( f_i(\lambda) \) and \( f_1(\lambda) \), which can be used to guide the acquisition of the DM’s preferences, as discussed later. Condition (37) means that if \( \lambda \) is already efficient it will be infeasible to move in the direction of a normal vector at \( \lambda \) on the efficient frontier. So, it will be infeasible to improve \( u(\lambda) \) if \( \vec{u}(\lambda) \) is in the same direction as \( \vec{N}(\lambda) \). In Fig. 3, the condition given by Eq. (37) is met at point M but not at point B.

Step 2: Find a new reference point with higher utility by utility gradient projection. If \( \vec{u}(\lambda) \) is not proportional to \( \vec{N}(\lambda) \), then a better efficient solution with higher utility than the current value of \( u(\lambda) \) can always be found around \( \lambda \) on the efficient frontier. The projection of \( \vec{u}(\lambda) \) onto the efficient frontier, as shown by \( \Delta \vec{u} \) in Fig. 3, provides a direction along which the utility \( u(\lambda) \) can be improved. The projection of \( \vec{u}(\lambda) \) to a facet with \( \vec{N} \) as its normal vector is given by

\[
\Delta \vec{u} = \begin{bmatrix} \Delta f_1 \\ \vdots \\ \Delta f_m \end{bmatrix} = -\vec{u} + \frac{\vec{u}}{\vec{N}} \cdot \frac{\vec{N}}{\vec{N} \cdot \vec{N}}
\]

(38)

Suppose \( \alpha \) is a step size. A new reference point can then be given by

\[
f^{\alpha} = [f^{\alpha}_1, \ldots, f^{\alpha}_m]^T = \text{with } f^{\alpha}_i = f_i(\lambda) + \alpha \Delta f_i
\]

(39)

If \( \alpha \) is sufficiently small, there will be \( u(f^{\alpha}) > u(\lambda) \) [26].

Step 3: Update the weighting parameters of the ideal point problem (35) and resolve it as follows:

\[
w_i = 1/(f_i - \vec{f}^0_i) \quad \text{for } i = 1, \ldots, m
\]

(40)

Solving the ideal point problem (35) again with \( w_i \) updated by Eq. (40) results in a new efficient solution \( \lambda^{\text{new}} \), which lies on the intersection of the efficient frontier and a line passing the ideal point and the reference point \( f^{\alpha} \). For a sufficiently small step size \( \alpha \), there will be \( u(\lambda^{\text{new}}) > u(\lambda) \). At the new solution, we can go back to step 1 to check whether it is the DM’s most preferred solution. This interactive process can be repeated until condition (37) is met such as at point M in Fig. 3.

4. A case study for supplier assessment of fashion retailer

4.1. Problem description and conventional DEA results

This case study concerns a fashion accessory retailer that owns a network of chain stores specialised in high street fashion jewellery such as necklace, bracelets and earrings and other
accessories including belts and bags [25]. In the case study, two inputs, two outputs and twelve DMUs (the suppliers of the retailer) are taken into account. The two quantitative inputs chosen are average price per product and percentage of late deliveries. The two outputs are the quantitative attribute of total purchase and the qualitative attribute of product quality in terms of expected average utility. More measures were used by the retailer for performance analysis such as customer service. However, one widely adopted rule of thumb suggests that the number of DMUs should be at least two or three times larger than the total number of inputs and outputs; otherwise the discrimination power of DEA would be affected [9,20,30]. For this case study, we therefore only choose the above two input and two output measures. This is also because there is little correlation between the two inputs or between the two outputs as discussed later.

The quantitative data was directly obtained from the company, with the qualitative data being the average utility scores for the DMUs generated using the multiple criteria assessment, which reflect the value judgments of the management of the company [25]. The input and output values used for performance modelling are shown in Table 3. It should be noted that the correlation coefficient between the two outputs “Total purchase” and “Product quality” is 0.076, showing that the two outputs are almost uncorrelated. The correlation coefficient between the two inputs “Average price” and “Late deliveries” is –0.1, indicating a quite low degree of negative correlation between the two inputs. Also note that the number of DMUs is three times larger than the total number of inputs and outputs. As such, this is a practical case that can be studied using DEA for performance analysis.

The DEA efficiency scores and composite inputs and outputs can be found by running the input-oriented dual CCR model (1) or the identical super-ideal point model (18) with the origin as the reference point. The DEA efficiency scores of all the suppliers are shown in Table 4. It can be seen that suppliers 3, 4, 5, 6, 10, 11 and 12 are all inefficient and the other suppliers are efficient. Each inefficient supplier should benchmark against the efficient suppliers. For instance, supplier 12 has a DEA efficiency score of 0.498 and its DEA composite unit or benchmark is composed of supplier 1 by a degree of 0.374 and supplier 2 by a degree of 0.324.

The DEA composite inputs and outputs of the benchmark DMU for each supplier are shown in Table 5. For each efficient supplier, they are the same as its original inputs and outputs. For each inefficient supplier, its DEA composite inputs are at most as large as its original inputs whilst its DEA composite outputs are no smaller than its original outputs. For the inefficient supplier 12, condition (31) is not met. In fact, we have 9/194 ≠ 274/97. So, the DEA efficiency score for supplier 12 is not measured consistently and from Tables 4 and 5, its technical efficiency score is given by

\[
T_{E12} = \frac{0.127}{0.12 + 1.12} = \sqrt{\frac{0.97^2 + 2.74^2}{(1.94 - 0.97)^2 + (9 - 2.74)^2}} = 0.315
\]

(41)

Table 3

<table>
<thead>
<tr>
<th>Supplier</th>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.34</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>1.44</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>2.13</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>2.01</td>
<td>9</td>
</tr>
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<td>5</td>
<td>3.44</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>2.98</td>
<td>4</td>
</tr>
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<td>3</td>
</tr>
<tr>
<td>12</td>
<td>1.94</td>
<td>9</td>
</tr>
</tbody>
</table>

where point 12\(T\) represents the target for supplier 12, as shown for supplier 12 (DMU 12) in Table 5. It can be shown that condition (31) is not met for suppliers 3, 4 and 10 either, but it is met for the other inefficient suppliers. For example, there is 4/3.44=1.21/1.04 for supplier 5. So, only for inefficient suppliers 5, 6 and 11 and for the efficient suppliers the technical efficiency score is the same as the DEA efficiency score. In the following, we use the concepts and techniques investigated in the previous sections to analyse the above results.

4.2. Equivalent efficiency analysis in a MOO process

In Table 5, the suppliers are divided into two types: efficient and inefficient. In this subsection, we generate and analyse the data envelopes and efficient frontiers for the efficient supplier 1 and the inefficient supplier 5 to illustrate the various features of efficiency analysis and provide a basis for resource allocation through interactive trade-off analysis to be discussed in the next section.

The MOO formulation (20) for the efficient supplier 1 is constructed as follows:

Since there are twelve decision variables, it is not possible to draw the decision space of the above MOO problem. However, the procedure investigated in Section 3.3 can be used to generate the data envelope and the efficient frontier in the objective space for supplier 1. The scaling factors and coordinates for the twelve suppliers in the objective space constructed for supplier 1 are
shown in Table 6. The twelve suppliers are drawn in the objective space as shown in Fig. 4, leading to the initial data envelope of $12', 12, 12$ and $12$, the vertex points of which are composed of the original DMUs only, with point $12'$ being the expansion point of point $12$ and point $8'$ the expansion point of point $8$.

Following the procedure investigated in Section 3.3, we find that there is no more feasible solution outside this initial data envelope. The efficient frontier for supplier 1 is given by $13, 13$ and $13$, any of which could be a potential target for supplier 1, depending upon the DM's preferences. Since point 1 is on the efficient frontier, supplier 1 is efficient. Solving the super-ideal point model (18) for supplier 1 with the origin as the reference point is to expand the minimax contour (M.C. in Fig. 4) along the line emitting from the origin to point 1. The northeast corner of the minimax contour touches the efficient frontier at point 1 with the DEA efficiency score of $e_1 = 0.71$. Note that points 2 and 7 are not on the efficient frontier constructed for supplier 1, though suppliers 2 and 7 are both efficient.

The MOO formulation (20) for the inefficient supplier 5 is constructed as follows:

$$\text{Min } f = [f_1(\lambda), f_2(\lambda)]$$

S.T.  $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}] \in \Omega_5$  \hspace{1cm} (43)

where $f_1(\lambda)$ and $f_2(\lambda)$ are the same as defined in problem (42) and $\Omega_5$ is different from $\Omega_1$ only in that the right hand side constants 5640.8 and 0.69 given in $\Omega_1$ for supplier 1 are replaced by 3249 and 0.36 respectively in $\Omega_5$ for supplier 5.

The feasible objective space for supplier 5 is drawn in Fig. 5, and the vertex points of the data envelopes are composed of the original DMUs and a new composite DMU (point 13 in Fig. 5), given by $12, 12, 12$, and $12, 12$, the vertex points of which are composed of the original DMUs only, with point $12$ being the expansion point of point 12 and point $8$ the expansion point of point $8$.

Note that points 1, 2 and 7 are not on the efficient frontier for supplier 5, although suppliers 1, 2 and 7 are all efficient. It is important to observe that in Table 4 the DEA benchmark for supplier 5 is composed of supplier 1 by a degree of 0.229 and supplier 9 by 0.525, whilst suppliers 1 is not on the efficient frontier in Fig. 5. How can this happen? The answer is that point 13 is on the frontier, which is expanded from and composed of suppliers 1 and 9. This observation is important to help design an appropriate interactive process to search for a benchmark most preferred by the decision maker, as shown for supplier 12 in next section.
It is important to observe that the efficiency analysis for supplier 1, whose function is given by $0.47$ respectively in Fig. 7, to solve the super-ideal point model (18) for supplier 12, the objective space constructed for supplier 12 is shown in Fig. 6. The initial data envelope is composed of $\Omega_{12}$, whose function is given by $0.69$ in Fig. 5. Efficiency analysis for supplier 12 is constructed as follows:

$$\begin{align*}
\text{Min} & \quad f = [f_{1}(\lambda), f_{2}(\lambda)] \\
\text{s.t.} & \quad \lambda = [\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}, \lambda_{5}, \lambda_{6}, \lambda_{7}, \lambda_{8}, \lambda_{9}, \lambda_{10}, \lambda_{11}, \lambda_{12}]^{T} \in \Omega_{12}
\end{align*}$$

where $f_{1}(\lambda)$ and $f_{2}(\lambda)$ are the same as defined in problem (42) and $\Omega_{12}$ is different from $\Omega_{1}$ only in that the right hand side constants 5640.8 and 0.69 in $\Omega_{1}$ for supplier 1 are replaced by 14723 and 0.47 respectively in $\Omega_{12}$ for supplier 12. The objective space constructed for supplier 12 is shown in Fig. 7 in detail where the local feasible region round the efficient frontier for supplier 12 is amplified. The coordinates for point 12 is given by (1.94, 9), point 13 by (1.02, 2.07), point 14 by (0.97, 2.74) and point 15 by (1.29, 1.65). It is important to observe that the efficient DMUs 1, 2 and 7 are not on the efficient frontier constructed for supplier 12, whilst in Table 4 the DEA benchmark for supplier 12 is composed of supplier 1 by a degree of 0.374 and supplier 2 by 0.324. The explanation for such a phenomenon is that the benchmark for supplier 12 is actually the composite DMU 14 (point 14), as shown below, which is composed of suppliers 1 and 2.

As shown in Fig. 7, to solve the super-ideal point model (18) for supplier 12 with the origin as the reference point is to expand the minimax contour along the line emitting from the origin to point 12.

\[
e_{12} = \frac{0.12}{0.12} = \frac{0.97^2 + 4.5^2}{1.94^2 + 9^2} = 0.5
\]  

### Table 6

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The right side of the minimax contour touches the efficient frontier at point 14, with $f_{1}(\lambda_{14}) = 0.97$ and $f_{2}(\lambda_{14}) = 2.74$, and the DEA efficiency score is calculated by $\hat{e}_{12} = 0.12\big/\Omega_{12}$ where point 12 is the northeast corner of the minimax contour with the coordinates given by $[0.97, 4.5]^{T}$, which is the intersection of the line emitting from the origin to point 12, whose function is given by $9f_{1} - 1.94f_{2} = 0$, and the line of the right side of the minimax contour, whose function is given by $f_{1} = 0.97$. So, the DEA efficiency score is given by

\[
\hat{e}_{12} = \frac{0.12}{0.12} = \frac{0.97^2 + 4.5^2}{\sqrt{1.94^2 + 9^2}} = 0.5
\]
This is the same DEA efficiency score as shown in Table 4 for supplier 12. Since point 12 is not feasible, however, point 14 rather than point 12 is actually suggested as the benchmark for supplier 12 in the conventional DEA as shown in Table 5, which is inconsistent with the measurement of the DEA efficiency score. The following technical efficiency score should be used to measure the degree to which supplier 12 needs to make the better use of its resources to achieve full efficiency:

\[
\text{TES}_{12} = \frac{\theta_{14}}{0.97 + 2.74^2} = \frac{\theta_{14}}{0.97 + 2.74^2} = 0.315
\]

(47)

Note that the benchmark for supplier 12 generated from the conventional DEA process, as given in Table 5 (or point 12 in Eq. (41)), is the same as point 14 in Fig. 7 and Eq. (47). This explains why the result of Eq. (47) is the same as that of Eq. (41).

In the enlarged efficient frontier shown in Fig. 7, the ideal point \( f^{12} \) takes the minimum feasible values of 0.97 for the first composite input \( f_1(\lambda) \) and 1.19 for the second composite input \( f_2(\lambda) \). The DEA benchmark for supplier 12 is given by point 14, which is an efficient solution at the upper end of the efficient frontier, where the lowest feasible Average Price \( f_2(\lambda) \) is achieved but the percentage of Late Deliveries \( f_2(\lambda) \) is the highest among all solutions on the efficient frontier.

Point 14 was suggested as the target for supplier 12 to benchmark against in the conventional DEA simply because of the fixed radial movement strategy along the line emitting from the origin to point 12, technically embedded in the input-oriented dual CCR model (1) without necessarily taking into account the DM's preferences. In general, this strategy may not be what the DM prefers the most and trade-off analysis between \( f_1(\lambda) \) and \( f_2(\lambda) \) may need to be conducted to take into account the DM's preferences for planning resources.

As shown in Fig. 7, improving \( f_2(\lambda) \) at the expense of \( f_1(\lambda) \) along the efficient line \( \lambda T_1 \) seems attractive, as indicated by its normal vector \( \overrightarrow{N}_{14} = [-0.0661, -0.0504]^T = -0.0661[1, 1/13.2]^T \), which shows that a small sacrifice (increase) of \( f_1(\lambda) \) can lead to a large improvement (decrease) of \( f_2(\lambda) \). More precisely, the sacrifice of one unit of \( f_1(\lambda) \) along the efficient line \( \lambda T_1 \) can lead to the improvement of 13.2 units of \( f_2(\lambda) \). On the other hand, trade-offs along the efficient line \( \lambda T_3 \) lead to 1.55 units of improvement for \( f_2(\lambda) \) at the expense of each unit of \( f_1(\lambda) \), because its normal vector

\[
\overrightarrow{N}_{13} = [-0.4217, -0.2714]^T = -0.4217[1, 1/1.55]^T.
\]

Along the efficient line \( \lambda T_3 \), however, the sacrifice of \( f_1(\lambda) \) becomes even less attractive to improve \( f_2(\lambda) \), as \( \overrightarrow{N}_{13} = [-0.426, -0.4283]^T = -0.4261[1, 0.995]^T \). In other words, one unit of sacrifice from \( f_1(\lambda) \) can only lead to 0.995 unit of improvement for \( f_2(\lambda) \) along \( \lambda T_3 \). Furthermore, improving \( f_2(\lambda) \) along the efficient line \( \lambda T_3 \) does not make sense because one unit of improvement for \( f_2(\lambda) \) would need 31.6 units of sacrifice from \( f_1(\lambda) \), because \( \overrightarrow{N}_{13} = [-0.0413, -1.3052]^T = -1.3052[1, 31.6]^T \). The above trade-off analysis is discussed in association with the normal vector of an efficient facet (line). This association is useful because the normal vectors of an efficient frontier for a MOO problem can be generated using the minimax formulation without extra costs as shown in Section 3.5.

The interactive trade-off analysis process investigated in Section 3.5 is applied to analyse supplier 12 and demonstrated as follows. For example, suppose the DM of supplier 12 decides to investigate whether or not his overall utility could be improved from point 14 that was initially suggested as the benchmark for DMU 12 in the conventional DEA. The normal vector of the efficient frontier at Point 14 can be found by constructing the ideal point problem (35) and using Eq. (36) to set the weighting parameter, which is the same as \( \overrightarrow{N}_{14} \), or \( \overrightarrow{N}(12) = \overrightarrow{N}_{14} = [0.6661, 1.132]^T \). An optimal indifference trade-off question can then be presented to the DM by asking ‘whether his overall utility would remain unchanged if from the current level of the benchmark for ‘Average Price’ and ‘Late Deliveries’, or \( f_1(\lambda) = 0.97 \) and \( f_2(\lambda) = 2.74 \), ‘Average Price’ is increased by one unit and the ‘Late Deliveries’ is decreased by 13.2 units”. If the answer is “yes”, Point 14 will be the DM’s most preferred solution and the process terminated. Otherwise, the process continues.

Suppose the DM’s answer to the above optimal indifference trade-off question is “no”. Note that since Point 14 is an extreme point the DM can only improve his overall utility by improving “late deliveries” \( f_2(\lambda) \) at the expense of “average price” \( f_1(\lambda) \). Indeed, he should do so along a line overlapping the efficient facet \( T_1 T_3 \). Since \( T_1 T_3 \) has been drawn already and the improvement direction is known at Point 14, there is no need to estimate the DM’s local utility gradient. To search for a reference point along this line, suppose the DM is prepared to sacrifice 0.09 units of \( f_1(\lambda) \) to improve \( f_2(\lambda) \) by an estimated amount of 1.888 (or 0.09 \times 13.2) units along the line \( T_1 T_3 \). A new reference point is then given by \( T = [0.97 + 0.09, 2.74 – 1.888] = [1.06, 1.552]^T \), as shown in Fig. 7, which is not feasible.

Note that the ideal point for DMU 12 is given by \( \lambda T^{12} = [0.97, 1.19]^T \). Use Eq. (36) to update the weights by \( w_1 = 1/(0.06 – 0.97) = 11.1111 \) and \( w_2 = 1/(1.552 – 1.19) = 2.7624 \). To find the efficient solution closest to the new target \( T \), formulate the following minimax ideal point problem:

\[
\text{Min } \theta \quad \text{s.t. } w_1 (f_1(\lambda) – 0.97) \leq \theta, \quad w_2 (f_2(\lambda) – 1.19) \leq \theta, \quad \lambda \in \Omega_{12}
\]

(48)

The optimal solution of the above problem can be generated by using Solver in Excel and given by \( MPS = [1.146, 1.898]^T \), which is the intersection point of the efficient line \( T_3 T_5 \) and the line emitting from the ideal point \( \lambda T^{12} \) to the efficient frontier through the reference point \( T \) as shown in Fig. 7.

The Lagrange (simplex) multipliers or the dual variables of the first two objective constraints in formulation (48) are given by

\[
\beta = [\beta_1, \beta_2]^T = [-0.278, -0.722]^T,
\]

which can be obtained from the Sensitivity Report generated by Solver in Excel. The normal vector of the efficient frontier at point MPS is then given from Eq. (34) by

\[
\overrightarrow{N}_{MPS} = [w_1 \beta_1, w_2 \beta_2]^T = [-11.1111 \times 0.278, -2.7624 \times 0.722]^T = [-3.0889, -1.9945]^T = -3.0889[1, 1/1.55]^T,
\]

which is in parallel...
with $N = -0.4217(1/1.55)^2$, the normal vector of the efficient facet $T_3.T_5$.

The $DM$ is then presented with a new optimal indifference trade-off question by asking “whether he agrees that one unit of $f_1(\lambda)$ can be exactly offset by 1.55 units of $f_2(\lambda)$ from this current target of $f_1(MPS^{12}) = 1.146$ and $f_2(MPS^{12}) = 1.898$”. If the answer is “no”, the trade-off process will continue. Otherwise, $MPS$ will be the most preferred solution that maximises the DMs implicit utility function around this current target. The preferential efficiency score for supplier 12 is then given by

$$PE_{12} = \frac{0}{MPS^{12}}$$

$$= \frac{\sqrt{1.146^2 + 1.898^2}}{\sqrt{1.146^2 + 1.898^2 + \sqrt{(1.94 - 1.146)^2 + (9 - 1.898)^2}} - 0.2368}$$

(49)

The very low $PE_{12}$ value indicates that supplier 12 has to make a lot of efforts for improving its performance to achieve this most preferred efficiency target. In Fig. 7, this means graphically that point 12 is far away from point $MPS$ compared with the distance between point $MPS$ and the origin.

5. Conclusion

This paper reported an investigation in exploring equivalence relationships between DEA and MOO models. It was shown that minimax reference point models are equivalent to input-oriented dual CCR models under certain conditions. This leads to the development of an interactive minimax reference point approach for Hybrid Efficiency and Trade-off Analyses (HETA), with the decision maker’s preferences taken into account interactively. The proposed HETA approach can be used to support past performance assessment and future resource planning in a consistent and interactive manner. The graphical and analytical procedures explored in this paper for generating data envelopes and efficient frontiers helped to deepen the understanding of HETA as a MOO process and led to the construction of the technical and preferential efficiency measures for HETA. The case study showed the implementation process of the interactive HETA approach and its potential applications. Both the graphical and analytical procedures can be applied to analyse performance analysis problems having many DMUs. The graphical procedure can be used to analyse higher dimensional DEA problems with three or more inputs by exploring two inputs at a time. If all inputs need to be explored simultaneously, the analytical procedure can be applied.

It is worth emphasising that both input-oriented dual CCR models and minimax reference point models share the same Pareto frontier, although the former is formulated to find a particular solution on the Pareto frontier, defined as the DEA efficient solution in this paper. This property of sharing the same Pareto frontier ensures that the performance of a DMU is assessed objectively in conjunction with other DMUs, on the same basis of which the performance target is set for future improvement with the DM preferences taken into account interactively. This feature makes the proposed HETA approach practical and distinctive from other methods by preserving the consistency and objectivity in performance assessment as well as the flexibility of taking into account the DMs preferences in resource planning for performance improvement.

Performance analysis and resource planning is a complex process. While the proposed HETA approach provides a useful analytical means to support this process, more factors need to be considered which could be both quantitative and qualitative and involve uncertainty such as incomplete data and imprecise preference information. Also, this investigation is focused on using input-oriented dual DEA models for non-parametric performance modelling. Other modelling techniques may provide alternative means for similar purposes, which needs to be further investigated. Finally, it should be noted that the computational procedure proposed to explore efficient frontier is rather generic and is mainly used to illustrate the concepts and computational process of HETA. To deal with large scale DEA problems with hundreds of DMUs and more inputs/outputs, more powerful algorithms and user interfaces need to be developed to explore efficient frontiers to support informative trade-off analysis.

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