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Interactive minimax optimisation for integrated performance analysis and resource planning

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ABSTRACT

Analysing performances for future improvement and resource planning is a key management function. *Data Envelopment Analysis* (*DEA*) provides an analytical mean for performance modelling without assuming parametric functions. *Multiple Objective Optimisation* (*MOO*) is well-suited for resource planning. This paper reports an investigation in exploring relationships between *DEA* and *MOO* models for equivalent efficiency analysis in a *MOO* process. It is shown that under certain conditions minimax reference point models are identical to input-oriented dual *DEA* models for performance assessment. The former can thus be used for *Hybrid Efficiency and Trade-off Analyses* (*HETA*). In this paper, these conditions are first established and the equivalent models are explored both analytically and graphically to better understand *HETA*. Further investigation in the equivalence models leads to the modification of efficiency measures and the development of a minimax reference point approach for supporting integrated performance analysis and resource planning, with the *Decision Maker*'s (*DM*) preferences taken into account in an interactive fashion. Both numerical and case studies are conducted to demonstrate the proposed approach and its potential applications.

1. Introduction

DEA and *MOO* are tools in management control and planning and to an extent have been developed separately for several decades, with the former directed to analysing past performances as part of management control function and the latter to planning future performances [8]. As a performance measurement and analysis technique, *DEA* is a non-parametric frontier estimation methodology based on linear programming for evaluating the relative efficiency of a set of comparable *Decision Making Units* (*DMUs*) that share common functional goals. *DEA* has evolved tremendously and has been researched extensively since the original work of Charnes et al. [6].

DEA and *MOO* have much in common, though they retain their own distinctive traits [1,5,14,23,24,]. For instance, Doyle and Green [10] suggested that *DEA* is a *Multiple Criteria Decision Analysis* (*MCDA*) method itself. Belton and Vickers [4] and Stewart [24] described similarities between the formulations of basic *DEA* models and classical multi-attribute value function approaches. Sarkis [21] termed *DEA* as a reactive approach to *MCDA* where different alternatives are evaluated objectively. In particular, Golany [11,12] developed an interactive model to allocate a set of input levels as resources and to select the most preferred output levels from a set of alternative points on the efficient frontier. Athanassopoulis [2,3] used goal programming and *DEA* to support resource allocation. Post and Spronk [19] combined the use of *DEA* and interactive goal programming to adjust the upper and lower feasible boundaries of the input and output levels. Joro et al. [14] showed the structural similarity between *DEA* and multiple objective linear programming for value efficiency analysis [13,15–17]. The above techniques require *prior* preference information and/or lead to variation from or restriction on the efficient frontier of an initial *DEA* model, although a hybrid approach was proposed for performance improvement without requiring *prior* preferences or changing an efficient frontier [28,29].

The attraction of *DEA* is that it provides an analytical means for performance modelling without assuming parametric functions between inputs and outputs and an efficient frontier is formulated on which a *DMU* can plan its resources and set its future improvement targets proven achievable by its peers. It is therefore important that the *DM* of the *DMU* can explore the same efficient frontier in order to identify his most preferred performance target in comparison with a fixed benchmark embedded in an initial *DEA* model. Such exploration would be desirable if conducted in an interactive and consistent manner as the *DM* can have an opportunity to investigate and learn what efficient solutions are available and what resources need to be consumed to get to his most

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preferred solution, so that a well-informed decision can be made without having to assume overall preferences *a priori*.

This paper reports an investigation into exploring relationships between *DEA* and *MOO* models for integrated efficiency analysis and resource planning. It is shown that minimax reference point models are identical to input-oriented dual *DEA* models under certain conditions. This equivalence relationship means that the former can be used for *HETA* on the same efficient frontier, so that performance targets can be set and required resources can be planned in an integrated and consistent manner, with the *DM*'s preferences taken into account in an interactive fashion. In this paper, computational studies are conducted to analyse efficiency analytically and graphically in an *MOO* process, leading to the construction of new efficiency measures for *HETA*. Based on the equivalent reference point models, a computational procedure is proposed to analyse data envelope and efficient frontier for interactive trade-off analysis and informed search for the most preferred performance targets. An interactive approach is then explored for *HETA*. Numerical examples and a case study are examined to demonstrate *HETA* and its potential application.

The remainder of the paper is organised as follows. Section 2 briefly presents typical input-oriented *DEA* models as an analytical means for performance modelling, and minimax reference point models as a basis for resource planning. In Section 3, an interactive minimax reference point approach is investigated for *HETA*. Section 4 reports a case study on supplier performance analysis to illustrate the interactive approach. The paper is concluded in Section 5.

2. Analytical models for performance analysis and resource planning

DEA was initially developed by Charnes, Cooper and Rhones in 1978 for measuring and analysing the relative efficiencies of comparable *DMU*s with incommensurate inputs and outputs. In *DEA*, an efficient frontier is formulated, separating efficient *DMU*s from inefficient ones. An efficient *DMU* means that no other *DMU* can either produce the same outputs by consuming fewer inputs, known as the input-orientated approach, or produce more outputs by consuming the same inputs, known as output-orientated approach. In this paper, we first briefly present *DEA* as a non-parametric means for performance modelling.

2.1. Input-orientated DEA models as a means for performance modelling

The original *DEA* model proposed by Charnes et al. [6] is a fractional non-linear programming model, known as the *CCR* model. The objective function in the model is to maximise the single ratio of weighted outputs over weighted inputs for a particular *DMU*, referred to as an observed *DMU* and denoted by DMU_0. Since the fractional model is non-linear and difficult to solve, it is transformed into equivalent linear programming models. In this section, the input-oriented dual *CCR* model is presented as an analytical means for non-parametric performance modelling.

Suppose there are n *DMU*s, each producing s outputs denoted by y_{rj} (the rth output of *DMU* j) and consuming m inputs denoted by x_{ij} (the ith input of *DMU* j). The input-oriented dual *CCR* model is then defined as follows [7]:

$$\begin{aligned} \text{Min} \quad & \theta_{j_0} \\ \text{s.t.} \quad & \theta_{j_0} x_{ij_0} - \sum_{j=1}^{n} \lambda_j x_{ij} \geq 0, \quad i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rj_0}, \quad r = 1, \ldots, s; \ \lambda_j \geq 0 \text{ for all } j \end{aligned} \tag{1}$$

In the above model, λ_j is a reference parameter for *DMU* j ($j=1, \ldots, n$) and $\lambda_j > 0$ means that *DMU* j is used to construct a composite *DMU* as a benchmark for the observed DMU_0. At the optimal solution of model (1), an optimal composite *DMU* is constructed that cannot underperform DMU_0, which is referred to as the *DEA* efficient solution or the benchmark of DMU_0 in this paper. The benchmark consumes at most a proportion θ_{j_0} of the inputs of DMU_0 and produces at least the same outputs as DMU_0. $\theta_{j_0} = 1$ represents a full efficiency score and $0 < \theta_{j_0} < 1$ reveals the presence of inefficiency. The parameter θ_{j_0} indicates the degree to which DMU_0 has to reduce the consumption of its resources (inputs) in order to become efficient. The reduction is employed concurrently to all inputs and results in a radial movement towards the envelopment surface [7]. Note that such a radial movement strategy is embedded in the *DEA* modelling mechanism *a priori* and does not necessarily take management preferences into consideration, so it is technical rather than preferential. In the following sections, we will explore how efficiency analysis can be conducted equivalently in an *MOO* process so that trade-off analysis can be consistently conducted in the same framework to plan resources, with management preferences incorporated in an interactive fashion.

2.2. Minimax reference point models as a basis for resource planning

In the input-oriented dual *CCR* model, an efficiency score is generated for an observed DMU_0 by minimising inputs with outputs constrained as least at their required levels. This is in essence a *MOO* problem. In this section, we briefly describe basic *MOO* concepts and models, in particular the minimax reference point models as a basis for resource planning.

Suppose a *MOO* problem has m objectives to be minimised, in general represented by

$$\begin{aligned} \text{Min} \quad & f(\lambda) = [f_1(\lambda) \cdots f_i(\lambda) \cdots f_m(\lambda)] \\ \text{s.t.} \quad & \lambda \in \Omega = \{\lambda | g_j(\lambda) \leq 0, \ h_l(\lambda) = 0; \ j = 1, \ldots, k_1, \ l = 1, \ldots, k_2\} \end{aligned} \tag{2}$$

where Ω is a feasible decision space, $f_i(\lambda)$ ($i=1,\ldots, m$) are continuously differentiable objective functions, and $g_j(\lambda)$ ($j=1,\ldots, k_1$) and $h_l(\lambda)$ ($l=1,\ldots, k_2$) are continuously differentiable inequality and equality constraint functions respectively. In this paper, $f_i(\lambda)$, $g_j(\lambda)$ and $h_l(\lambda)$ are all assumed to be the linear functions of λ.

In a *MOO* problem, we are interested in finding efficient solutions. A feasible solution λ^* is said to be efficient if there exists no other feasible solution which is better than λ^* at least on one objective and as good as λ^* on all other objectives. An efficient solution can be formally defined as follows.

Definition 1. In formulation (2), a feasible solution $\lambda^* \in \Omega$ is an efficient solution if and only if there does not exist any other feasible solution $\lambda \in \Omega$ such that $f_i(\lambda) \leq f_i(\lambda^*)$ for all $i=1,\ldots, m$ and $f_\tau(\lambda) < f_\tau(\lambda^*)$ for at least one $\tau \in \{1, \ldots, m\}$.

Any efficient solutions of a *MOO* problem can be generated using a minimax formulation [22,27]. Suppose λ is an efficient solution of model (2) and f_i^* is the minimum feasible value of objective i. There exists a weighting vector w satisfying $w_1 = 1$ and $w_i > 0$ for $i=2, \ldots, m$ and a reference point f^{ref} such that λ can be generated by solving the following weighted minimax reference point problem [27]:

$$\begin{aligned} \min_{\lambda} \max_{1 \leq i \leq m} \quad & \{w_i | (f_i(\lambda) - f_i^{ref})|\} \\ \text{s.t.} \quad & \lambda \in \Omega \end{aligned} \tag{3}$$

The weighted minimax reference point formulation will be called the ideal point model if the ideal point $f^* = [f_1^* \ldots f_m^*]^T$ is used as the reference point $f^{ref} = [f_1^{ref} \ldots f_m^{ref}]^T$. In the minimax reference point formulation, for a given weight vector, the *DM* is assumed to

be satisfied with an efficient solution $\lambda \in \Omega$ at which $f(\lambda)$ has the shortest weighted distance to f^{ref} measured in ∞-norm in the objective space. If f^{ref} is no worse than an ideal point, or $f^{ref} \le f^*$, then the weighted minimax formulation given in formulation (3) can be equivalently transformed into the following form by introducing an auxiliary variable θ [18,27]:

$$\begin{aligned} &\text{Min} \quad \theta \\ &\text{s.t.} \quad w_i(f_i(\lambda) - f_i^{ref}) \le \theta, \quad i = 1, \ldots, m; \ \lambda \in \Omega, \ \theta \ge 0 \end{aligned} \tag{4}$$

It will be shown in the following section that under certain conditions the minimax formulations can be used to generate the same *DEA efficiency scores*, composite inputs and outputs as the input-oriented dual *CCR* models produce.

3. Interactive minimax reference point approach for *HETA*

In this section, we investigate an interactive approach based on minimax reference point models to support integrated performance analysis and resource planning. In the approach, *DEA* is primarily employed for non-parametric performance modelling, and equivalence between *DEA* models and minimax models are investigated to enable equivalent performance analysis in a *MOO* process. The characteristics of efficiency frontier are explored and consistent efficiency measures are investigated for *HETA*. A gradient projection method based on the ideal point model is then investigated to facilitate interactive trade-off analysis based on the results of efficiency analysis.

3.1. Equivalence between minimax and DEA models

The input-orientated dual *CCR* model given in formulation (1) can be equivalently rewritten by

$$\begin{aligned} &\text{Min} \quad \theta_{j_0} \\ &\text{s.t.} \quad \theta_{j_0} x_{ij_0} - \sum_{j=1}^{n} \lambda_j x_{ij} \ge 0, \quad i = 1, \ldots, m, \ \theta_{j_0} \ge 0 \\ &\qquad \lambda \in \Omega_{j_0} = \left\{ \lambda \,\middle|\, \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rj_0}, \ r = 1, \ldots, s; \ \lambda_j \ge 0, \ j = 1, \ldots, n \right\} \end{aligned} \tag{5}$$

In this section, we prove that formulation (4) is equivalent to formulation (5) under certain conditions.

In formulation (5) the ith composite input can be represented by $f_i(\lambda)$ as follows:

$$f_i(\lambda) = \sum_{j=1}^{n} \lambda_j x_{ij} \ (i = 1, \ldots, m) \ \text{ and } \lambda = [\lambda_1, \ldots, \lambda_n]^T \tag{6}$$

In the following trade-off analysis, the ith composite input $f_i(\lambda)$ will be defined as an objective for minimisation, so there are m objectives in total. The minimum feasible value of the ith composite input for the observed DMU_0 is denoted by $\bar{f}_i^{j_0} = f_i(\lambda^*)$ where λ^* can be found by solving the following single objective optimisation problem:

$$\begin{aligned} &\text{Min} \quad f_i(\lambda) = \sum_{j=1}^{n} \lambda_j x_{ij} \\ &\text{s.t.} \quad \lambda \in \Omega_{j_0} \end{aligned} \tag{7}$$

Note that $\bar{f}^{j_0} = [\bar{f}_1^{j_0}, \ldots, \bar{f}_m^{j_0}]^T$ is the ideal point in the objective space spanned on Ω_{j_0} by the m objective functions defined in (6) for the observed DMU_0.

Suppose the feasible decision space Ω in formulation (4) is set to be the same as defined in formulation (5), or $\Omega = \Omega_{j_0}$. The equivalence relationship between the input-oriented dual *CCR* model (1) or (5) and the minimax formulation (4) can be established by the following theorem.

Theorem 1. *Suppose $x_{ij_0} > 0$ for any $i = 1, \ldots, m$ and $j_0 = 1, \ldots, n$. The input-oriented dual CCR model (5) can be equivalently transformed to the minimax formulation (4) using Eq. (6), formulation (7) and the following equations:*

$$w_i = 1/x_{ij_o} \tag{8}$$

$$f_i^{ref} = \frac{F^{\min}}{w_i} = x_{ij_0} F^{\min} \tag{9}$$

$$\theta = \theta_{j_0} - F^{\min} \tag{10}$$

$$F^{\min} = \min_{1 \le i \le m} \left\{ w_i \bar{f}_i^{j_0} \right\} = \min_{1 \le i \le m} \left\{ \frac{\bar{f}_i^{j_0}}{x_{ij_0}} \right\} \tag{11}$$

Proof. Using Eqs. (6) and (8), the input-orientated dual *CCR* model (5) can be equivalently rewritten as follows:

$$\begin{aligned} &\text{Min} \quad \theta_{j_0} \\ &\text{s.t.} \quad \theta_{j_0} \frac{1}{w_i} - f_i(\lambda) \ge 0, \quad i = 1, \ldots, m; \ \lambda \in \Omega_{j_0} \end{aligned} \tag{12}$$

The first m objective constraints in formulation (12) can be equivalently transformed as follows, where "$\Leftrightarrow$" means "is equivalent to". For any $i = 1, \ldots, m$, we have

$$\begin{aligned} \theta_{j_0} \frac{1}{w_i} - f_i(\lambda) \ge 0 &\Leftrightarrow w_i f_i(\lambda) \le \theta_{j_o} \\ &\Leftrightarrow w_i f_i(\lambda) - F^{\min} \le \theta_{j_0} - F^{\min} \\ &\Leftrightarrow w_i (f_i(\lambda) - f_i^{ref}) \le \theta \end{aligned} \tag{13}$$

Moreover, the objective function of formulation (12) becomes

$$\text{Min } \theta_{j_0} = \text{Min}(\theta_{j_0} - F^{\min}) = \text{Min } \theta \tag{14}$$

For any $\lambda \in \Omega_{j_0}$, since $\bar{f}_i^{j_0} = f_i(\lambda^*)$, from Eq. (11) we have

$$f_i^{ref} = \frac{F^{\min}}{w_i} \le \frac{w_i \bar{f}_i^{j_0}}{w_i} = \bar{f}_i^{j_0} = \min_{\lambda \in \Omega_{j_0}} f_i(\lambda) \quad \text{for any } i = 1, \ldots, m \tag{15}$$

So

$$f_i(\lambda) - f_i^{ref} \ge 0 \quad \text{for any } \lambda \in \Omega_{j_0} \text{ and } i = 1, \ldots, m \tag{16}$$

Also there is

$$\theta = \theta_{j_0} - F^{\min} \ge w_i f_i(\lambda) - F^{\min} \ge w_i f_i(\lambda) - w_i \bar{f}_i^{j_0} \ge 0 \quad \text{for any } i = 1, \ldots, m \tag{17}$$

E.O.D.

Theorem 1 shows that if the reference point in model (4) is set by $f^{ref} = [f_1^{ref}, \ldots, f_m^{ref}]^T = F^{\min}[x_{1j_0}, \ldots, x_{mj_0}]^T$ and w_i by Eq. (8), then the input-oriented dual *CCR* model will be identical to the following minimax reference point formulation:

$$\begin{aligned} &\text{Min} \quad \theta \\ &\text{s.t.} \quad w_i \left(\sum_{j=1}^{n} \lambda_j x_{ij} - f_i^{ref} \right) \le \theta, \quad i = 1, \ldots, m; \ \lambda \in \Omega_{j_0} \end{aligned} \tag{18}$$

By identical it is meant that they share the same decision and objective spaces and have the same optimal solution. Since $f^{ref} = F^{\min}[x_{1j_0}, \ldots, x_{mj_0}]^T \le \bar{f}^{j_0} = [\bar{f}_1^{j_0}, \ldots, \bar{f}_m^{j_0}]^T$, such a reference point f^{ref} is called super-ideal point and the minimax reference point formulation established using the super-ideal point is therefore referred to as the super-ideal point model in this paper. If $x_{ij_0} \ge 0$ for all $i = 1, \ldots, m$, $f^{ref} = F^{\min}[x_{1j_0}, \ldots, x_{mj_0}]^T \ge 0$. Note that the point $F^{\min}[x_{1j_0}, \ldots, x_{mj_0}]^T$ lies in the same line emitting from the origin to the point $[x_{1j_0}, \ldots, x_{mj_0}]^T$. So, the origin, or $f^{ref} = 0$, is a special super-ideal point. The super-ideal point model (18) can be used to generate the same efficiency score and efficient composite inputs and outputs of the observed DMU_0 as from the dual *CCR* model (1)

if in model (18) w_i is calculated by Eq. (8) and f_i^{ref} by Eqs. (9) and (11).

3.2. Equivalent efficiency analysis in an MOO process

In this section, we conduct analytical and graphical investigation into how the above equivalence relationship can be explored to support *HETA*. In models (5) and (18), $\lambda = [\lambda_1 \ldots \lambda_n]^T$ is a vector of decision variables and Ω_{j_0} defines the feasible decision space for the observed DMU_0. Any feasible solution $\lambda \in \Omega_{j_0}$ represents a composite unit for the observed DMU_0. The *DEA* efficient solution, denoted by λ^{DEA}, is an efficient solutions in Ω_{j_0}, or $\lambda^{DEA} \in \Omega_{j_0}$.

If $\lambda_j \neq 0$ and $\lambda_i = 0$ for all $i = 1, \ldots, m$ with $i \neq j$, then *DMU j* itself is a composite solution in Ω_{j_0}. In this case, the feasible value of λ_j can be calculated by requiring $\lambda_j y_{rj} \geq y_{rj_o}$ for all $r = 1, \ldots, s$, which leads to the following result:

$$\bar{\lambda}_j = \max_{1 \leq r \leq s} \left\{ \frac{y_{rj_0}}{y_{rj}} \right\} \tag{19}$$

The value $\bar{\lambda}_j$ calculated using Eq. (19) is a scaling factor for *DMU j*, which can be used to decide the location of *DMU j* in the objective space constructed for DMU_0, as shown later.

$f_i(\lambda) = \sum_{j=1}^{n} \lambda_j x_{ij}$ is the *i*th composite input. In model (18), $f_i(\lambda)$ $(i = 1, \ldots, m)$ together constitute an objective space. Therefore, the corresponding *MOO* problem, in which the super-ideal point model (18) is constructed, is the following m objective linear programming problem:

$$\begin{aligned} \text{Min} \quad & f = [f_1(\lambda), \ldots, f_m(\lambda)]^T \\ \text{s.t.} \quad & \lambda = [\lambda_1 \ldots \lambda_n]^T \in \Omega_{j_0} \end{aligned} \tag{20}$$

The m dimensional objective space is expanded by the m objective functions $f_i(\lambda)$ $(i = 1, \ldots, m)$. The feasible objective space, denoted by $f(\Omega_{j_0})$, is the projection of the feasible decision space Ω_{j_0} using the objective functions. All the efficient solutions of model (20) is represented by $E(f(\Omega_{j_0}))$, which includes all possible efficient composite solutions for the observed DMU_0 to benchmark against. In this section, we show that the *DEA* efficient solution is one of the efficient solutions in $E(f(\Omega_{j_0}))$, or $f(\lambda^{DEA}) \in E(f(\Omega_{j_0}))$.

We will show how $E(f(\Omega_{j_0}))$ can be graphically and analytically generated and represented for an observed DMU_0 if there are up to three inputs with any number of outputs and any number of *DMU*s. In general, if there are more than three inputs, any solution in $E(f(\Omega_{j_0}))$ can be explored using the minimax reference point model (18) with the reference point set as the ideal point, or $f^{ref} = \bar{f}_{j_0} = [\bar{f}_{1j_0}, \ldots, \bar{f}_{sj_0}]^T$, and w_r allowed to change in R^+.

Note that an original *DMU j* can be represented in $f(\Omega_{j_0})$ by

$$f_i(\bar{\lambda}_j) = \bar{\lambda}_j x_{ij} \quad \text{for } i = 1, \ldots, m \tag{21}$$

Since $\bar{\lambda}_{j_0} = 1$, the observed DMU_0 is represented in $f(\Omega_{j_0})$ using its true (un-scaled) input values. In general, all other original *DMU*s are represented in $f(\Omega_{j_0})$ using their scaled input values. The locations of the original *DMU*s using Eq. (21) provide a basis to characterise $E(f(\Omega_{j_0}))$ as shown later in this paper.

In the rest of this subsection, we generate $E(f(\Omega_{j_0}))$ for a simple *DEA* model with two inputs, two outputs and three *DMU*s in order to show analytically and graphically how efficiency analysis can be conducted in a *MOO* process on the basis of the reference point model (18). Consider a *DEA* model defined in Table 1, where x_1 and x_2 are the two inputs and y_1 and y_2 the two outputs.

Solving model (1) or model (18) for example 1, we can find that *DMU B* and *DMU C* are both efficient but *DMU A* is inefficient with an *DEA* efficiency score of 0.446. Furthermore, the values of the decision variables of the *DEA* efficient solution are $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T = [0, 0.179, 0.179]^T$ for *DMU A*, with the optimal composite inputs given by $x = [x_1 x_2]^T = [1.8,\ 0.9]^T$ and the optimal

Table 1
Example 1 with three *DMU*s.

DMU	x_1	x_2	y_1	y_2	*DEA* efficiency score
A	4	2	2.5	5	0.446
B	2	4	16	8	1
C	8	1	6	20	1

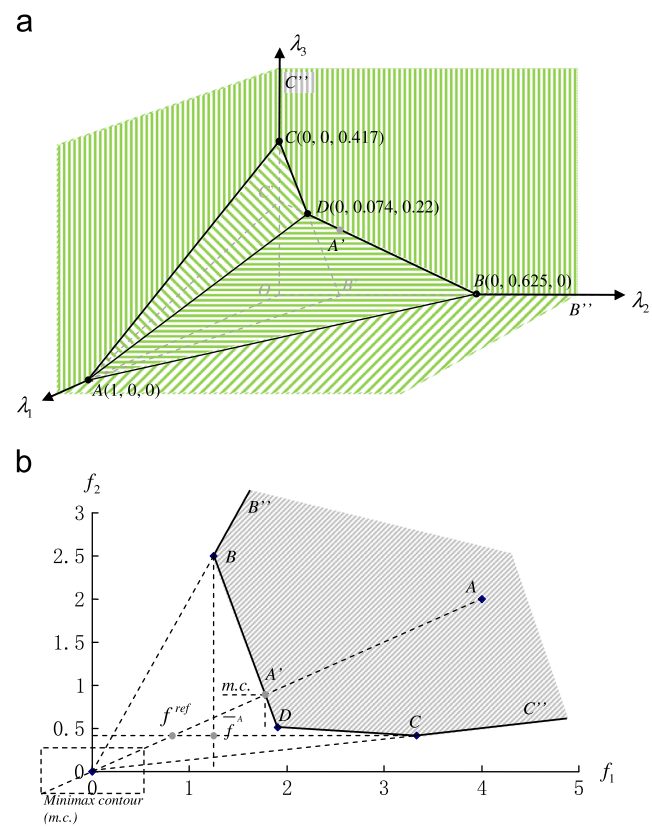

Fig. 1. (a) The decision space for *DMU A* in Example 1. (b) The objective space for *DMU A* in Example 1.

composite outputs by $y = [y_1 y_2]^T = [3.94,\ 5.01]^T$. In the following, we first show how to generate these results graphically using an equivalent *MOO* model defined by formulation (20).

The equivalent *MOO* problem for *DMU A* in example 1 is given as follows:

$$\begin{aligned} \text{Min} \quad & f = [f_1(\lambda) = 4\lambda_1 + 2\lambda_2 + 8\lambda_3, \quad f_2(\lambda) = 2\lambda_1 + 4\lambda_2 + \lambda_3] \\ \text{s.t.} \quad & \lambda = [\lambda_1, \lambda_2, \lambda_3]^T \in \Omega_A = \left\{ \lambda \,\middle|\, \begin{array}{c} 2.5\lambda_1 + 16\lambda_2 + 6\lambda_3 \geq 2.5 \\ 5\lambda_1 + 8\lambda_2 + 20\lambda_3 \geq 5 \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \lambda_3 \geq 0 \end{array} \right\} \end{aligned} \tag{22}$$

The feasible decision space Ω_A of problem (22) is the first quadrant above the planes $\overline{ACD}$ and $\overline{ABD}$, as illustrated in Fig. 1(a), which is an unbounded region.

The feasible objective space of problem (22) is the projection of the feasible decision space to the objective space using the objective functions $f_1(\lambda)$ and $f_2(\lambda)$, denoted by $f(\Omega_A)$. Since both $f_1(\lambda)$ and $f_2(\lambda)$ are linear functions, we can generate the feasible objective space $f(\Omega_A)$ by locating the extreme points *A*, *B*, *C* and *D* in the objective space. The coordinates of these points in the decision space are shown in Fig. 1(a). Their coordinates in the objectives space can be calculated using the objective functions $f_1(\lambda)$ and $f_2(\lambda)$, as shown in the second and third columns in Table 2.

Table 2
Extreme points in objective space for Example 1.

DMU	f_1	f_2	$\bar{\lambda}_j[x_1 \quad x_2]^T$
A	4	2	$1\times[4 \quad 2]^T$
B	1.25	2.5	$0.625\times[2 \quad 4]^T$
C	3.336	0.417	$0.417\times[8 \quad 1]^T$
D	1.907	0.514	$0.074\times[2 \quad 4]^T+0.22\times[8 \quad 1]^T$

In general, the objective coordinates can be generated using the scaling factors of the *DMU*s, as shown in the last column in Table 2. For instance, the scaling factor for *DMU B* can be found using Eq. (19) by $\bar{\lambda}_B=\max\{2.5/16, 5/8\}=0.625$. So, the coordinates of point *B* in $f(\Omega_A)$ can be generated by scaling the objective values of *DMU B* by $0.625\times[2 \quad 4]^T=[1.25 \quad 2.5]^T$. Point *D* in Fig. 1(a) is composed of *DMU B* scaled by a factor of 0.074 and *DMU C* scaled by a factor of 0.22. Its objective coordinates are thus given by $0.074\times[2 \quad 4]^T+0.22\times[8 \quad 1]^T=[1.907 \quad 0.514]^T$.

As shown by the shaded region in Fig. 1(b), the feasible objective space $f(\Omega_A)$ is an unbounded region. Note that point B'' is any point beyond point *B* on a line emitting from the origin through point *B*, referred to as the expansion point of point *B*, and point C'' is any point beyond point *C* on a line emitting from the origin through point *C*, referred to as the expansion point of point *C*. Therefore, the data envelope for *DMU A* in example 1 is composed of $\overline{B''B}\cup\overline{BD}\cup\overline{DC}\cup\overline{CC''}$. Its efficient frontier is given by $\overline{BD}\cup\overline{DC}$. In Fig. 1(b), the ideal point $\bar{f}^A$ is given by taking the minimum feasible objective values, so $\bar{f}^A=[\bar{f}_1^A, \bar{f}_2^A]=[1.25, 0.417]^T$. The super-ideal point f^{ref} is given by $f^{ref}=F^{\min}[4, 2]^T$ with $F^{\min}=\min\{1.25/4, 0.417/2\}=0.2085$. So $f^{ref}=[0.834, 0.417]^T$, which is on a line emitting from the origin to point *A*, represented by $2f_1-4f_2=0$, and is dominated by the origin point. The minimax contour at the origin is a rectangle with the origin as its centre and the above line passing through its southwest and northeast corners.

Solving the input-oriented *CCR* dual model (5) is equivalent to solving the super-ideal point model (18). In Fig. 1(b), solving the super-ideal point model for *DMU A* is to expand the minimax contour until it just touches the feasible objective space at point A', which is the intersection of the above line and the efficient line segment $\overline{BD}$ represented by $1.986f_1+0.657f_2=4.125$. So, the objective coordinates of point A' are given by $[1.782, 0.891]^T$. The *DEA* efficient score e_A for *DMU A* can then be calculated by

$$e_A=\frac{\overline{OA'}}{\overline{OA}}=\frac{\sqrt{1.782^2+0.981^2}}{\sqrt{4^2+2^2}}=0.446 \tag{23}$$

which is the same efficiency score as the one generated by solving the input-oriented dual *CCR* model.

The *DEA* composite inputs and outputs for *DMU A* can also be graphically generated as follows. Note that point A' is a convex combination of point *B* and point *D* given by

$$\text{Point } A'=\left(1-\frac{\overline{BA'}}{\overline{BD}}\right)\times\text{Point } B+\frac{\overline{BA'}}{\overline{BD}}\times\text{Point } D=0.19\times\text{Point } B+0.81\times\text{Point } D \tag{24}$$

Since point *B* represents the scaled *DMU B* with $\bar{\lambda}_B=0.625$ and point *D* represents a composite *DMU* composed of 0.074 of *DMU B* and 0.22 of *DMU C*, point A' also represents a composite *DMU* composed of *DMU B* scaled by 0.179 (or $0.19\times0.625+0.81\times0.074$) and *DMU C* scaled by 0.179 (or 0.81×0.22). So $\lambda=[\lambda_1 \quad \lambda_2 \quad \lambda_3]^T=[0, 0.179, 0.179]^T$, which leads to the same *DEA* composite inputs $x=[x_1, x_2]^T=[1.8, 0.9]^T$ and composite outputs $y=[y_1 y_2]^T=[3.94, 5.01]^T$.

Point A' is one of the efficient solutions that *DMU A* can benchmark against. Other efficient solutions in $\overline{BD}\cup\overline{DC}$ could also be potential benchmarks for *DMU A*, depending upon the *DM*'s preferences. In general, any efficient solutions for *DMU A* can be found by solving the following ideal point model with the weighting parameter w_i changed systematically.

$$\text{Min} \quad \theta$$
$$\text{s.t.} \quad w_i\left(\sum_{j=1}^{n}\lambda_j x_{ij}-\bar{f}_i^{j_0}\right)\le\theta, \quad i=1,\ldots,m; \ \theta\ge0, \ \lambda\in\Omega_{j_0} \tag{25}$$

3.3. Computational procedure for exploring efficient frontier for HETA

In the previous subsection, we used a simple *DEA* example with three *DMU*s to show both the decision and objective spaces graphically. In this subsection, an analytical procedure is investigated to generate the data envelopes and efficient frontiers of complicated *DEA* problems for exploring the characteristics of *HETA*, so that an appropriate interactive method can be employed to support *HETA*, as discussed later in this section. The analytical procedure for generating a data envelopes and its efficient frontier consists of the following steps.

Step 1: For a *DEA* problem, choose an observed *DMU* and define its objective functions as composite inputs, as shown in section 3.2.
Step 2: Calculate the scaling factors for all original *DMU*s using Eq. (19) and the coordinates of all scaled original *DMU*s in the objective space.
Step 3: Construct an initial data envelope by linking the identified points in the first quadrant, which includes all the scaled original *DMU*s, as shown in section 3.2.
Step 4: Expand the current data envelope by formulating a searching problem to check whether there is any feasible solution beyond a facet on the current envelope along the outwards normal direction of the facet.
Suppose there are two inputs and $I=[x_{11}, x_{12}]^T$ and $J=[x_{21}, x_{22}]^T$ are two adjacent extreme points on the current data envelope with $x_{12}\ge x_{22}$, so the line segment between *I* and *J* is a facet on the envelope. Then the searching function along the direction of the outwards normal vector of the facet is given by

$$g(f_1,f_2)=\begin{cases}-(x_{22}-x_{12})f_1(\lambda)+(x_{21}-x_{11})f_2(\lambda) & \text{if } \vec{N}^{IJ}\cdot\vec{f}_1\ge0\\ -(x_{12}-x_{22})f_1(\lambda)+(x_{11}-x_{21})f_2(\lambda) & \text{if } \vec{N}^{IJ}\cdot\vec{f}_1<0\end{cases} \tag{26}$$

where $\vec{N}^{IJ}$ is the outwards normal vector of the line $\overline{IJ}$ and $\vec{f}_1$ is the horizontal axis (f_1). $\vec{N}^{IJ}\cdot\vec{f}_1\ge0$ in (26) means that the angle between $\vec{N}^{IJ}$ and $\vec{f}_1$ is less than or equal to 90 degree whilst $\vec{N}^{IJ}\cdot\vec{f}_1<0$ means that the angle is larger than 90°. The searching problem is defined by

$$\begin{array}{ll}\text{Max} & g(f_1(\lambda), f_2(\lambda))\\ \text{s.t.} & \lambda\in\Omega_{j_0}\end{array} \tag{27}$$

where Ω_{j_0} is the decision space for the observed DMU_0. If the optimal solution is $f_1^*=x_{11}$ and $f_2^*=x_{12}$ or $f_1^*=x_{21}$ and $f_2^*=x_{22}$ with $g^*=|x_{21}x_{12}-x_{11}x_{22}|$, there will be no feasible solution outside $\overline{IJ}$, which is then recorded as the expanded facet of the data envelope. Otherwise, a new feasible solution is found and the data envelope is adjusted accordingly.
Step 5: If there is any unexpanded facet on the data envelope, go to Step 4.

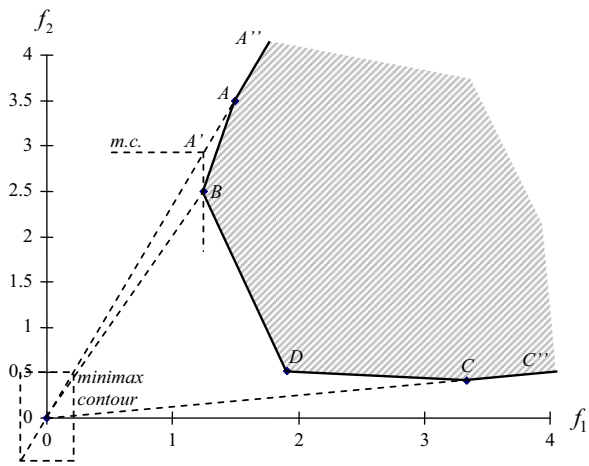

Fig. 2. The objective space for *DMU A* in the modified example.

3.4. Consistent measurement of technical and preferential efficiency

In Fig. 1(b), the line emitting from the origin to point A intersects with the efficient frontier at point A', leading to the measurement of the *DEA* efficiency score, or $e_A = \overline{OA'}/\overline{OA}$, which is consistent with the suggestion of the composite *DMU* A' as the benchmark for *DMU A*. However, such consistency may not always hold.

If the *DEA* model shown in Table 1 is slightly modified so that the input levels for *DMU A* are changed to $x_1 = 1.5$ and $x_2 = 3.5$ respectively, solving the dual *CCR* model for each *DMU* of this modified example shows that *DMU*s *B* and *C* are still efficient whilst *DMU A* is inefficient with a new *DEA* efficiency score of 0.833. The *DEA* composite unit for *DMU A* is *DMU B* scaled by a factor of 0.625 with the composite input levels given by 1.25 and 2.5 respectively. The objective space for *DMU A* in this modified example is drawn in Fig. 2. Its efficient frontier is still $E(f(\Omega_A)) = \overline{BD} \cup \overline{CD}$, but its data envelope becomes $\overline{A''A} \cup \overline{AB} \cup \overline{BD} \cup \overline{DC} \cup \overline{CC''}$. Point A becomes an extreme point on the data envelope but is inefficient.

As shown in Fig. 2, solving the super-ideal point model (18) with the origin as the reference point is to expand the minimax contour until its right hand side just touches the efficient frontier at point B. The corresponding upper-right corner of the minimax contour is given by $A' = [1.25, 2.917]^T$. From Fig. 2, two issues can be observed in the conventional *DEA*. The first issue is that the *DEA* efficiency score is calculated with point A' assumed as the benchmark for *DMU A*, or

$$e_A = \frac{\overline{OA'}}{\overline{OA}} = \frac{\sqrt{1.25^2 + 2.917^2}}{\sqrt{1.5^2 + 3.5^2}} = 0.833 \tag{28}$$

However, point A' is not feasible. This leads to the second issue. In the conventional *DEA* it is Point B that is actually suggested as the benchmark for *DMU A*, which is inconsistent with the efficiency measurement. That is, it is not a radial movement from A to A' that makes *DMU A* become efficient.

To address the above mentioned two issues, it seems appropriate to define a new technical efficiency score, or *TES* for short, for *DMU A* as follows:

$$TES_A = \frac{\overline{OB}}{\overline{OB} + \overline{AB}} = \frac{\sqrt{1.25^2 + 2.5^2}}{\sqrt{1.25^2 + 2.5^2} + \sqrt{(1.5 - 1.25)^2 + (3.5 - 2.5)^2}} = 0.731 \tag{29}$$

In example 1 the *DEA* efficiency score e_A and technical efficiency score TES_A for *DMU A* are the same. In the modified example, we have $TES_A < e_A$. In general, the technical efficiency score can be defined as follows.

Definition 2. Suppose a *DEA* problem has m inputs. In the objective space, suppose the observed *DMU A* is represented by $A = [x_{A1} \cdots x_{Am}]^T$ and its targeted input vector by $T = [x_{T1} \cdots x_{Tm}]^T$. Then the technical efficiency score for *DMU A* is defined by

$$TES_A = \frac{\overline{OT}}{\overline{OT} + \overline{AT}} = \sqrt{\sum_{i=1}^{m} x_{Ai}^2} \Big/ \left(\sqrt{\sum_{i=1}^{m} x_{Ai}^2} + \sqrt{\sum_{i=1}^{m} (x_{Ti} - x_{Ai})^2} \right) \tag{30}$$

The technical efficiency score of a *DMU* can be intuitively interpreted as a degree of minimum efforts that the *DMU* needs to make to reduce its inputs for achieving feasible efficiency. The closer the technical efficiency score is to one, the smaller the efforts needed to achieve the feasible efficiency. Analytically, a technical efficiency score measures the relative distance from a *DMU* to its nearest feasible efficient benchmark identified in the objective space through minimax optimisation. Regarding the relationship between the *DEA* efficiency score and the technical efficiency score, it can be shown that the following conclusion holds. Suppose $T = [x_{T1} \cdots y_{Tm}]^T$ is the suggested target for *DMU A* with $A = [x_{A1} \cdots x_{Am}]^T$. Then, the *DEA* efficiency score of *DMU A* will be equal to its technical efficiency score if the following conditions are met:

$$\frac{x_{Ai}}{x_{A1}} = \frac{x_{Ti}}{x_{T1}} \quad \forall i = 1, \ldots, m \tag{31}$$

If the suggested target is the most preferred solution of the decision maker maximising his implicit utility function, the technical efficiency score defined in Eq. (30) will be referred to as the preferential efficiency score, measuring the degree to which the observed *DMU* needs to reduce or balance its inputs to achieve the most preferred target, defined as follows.

Definition 3. Suppose a *DEA* problem has m inputs. In the objective space, suppose the observed *DMU A* is represented by $A = [x_{A1} \cdots x_{Am}]^T$ and its preferred input vector by $P = [x_{P1} \cdots x_{Pm}]^T$. Then a preferential efficiency score for *DMU A* is defined by

$$PES_A = \frac{\overline{OP}}{\overline{OP} + \overline{AP}} = \sqrt{\sum_{i=1}^{m} x_{Ai}^2} \Big/ \left(\sqrt{\sum_{i=1}^{m} x_{Ai}^2} + \sqrt{\sum_{i=1}^{m} (x_{Pi} - x_{Ai})^2} \right) \tag{32}$$

The preferential efficiency score of a *DMU* can be intuitively interpreted as a degree of minimum efforts that the *DMU* needs to make to balance its inputs for achieving the most preferred solution. The closer the preferential efficiency score is to one, the smaller the efforts needed to achieve the most preferred *DMU*. Analytically, a preferential efficiency score measures the relative distance from a *DMU* to its most preferred *DMU* in the context of minimax optimisation.

3.5. Interactive trade-off analysis and minimax optimisation

As a result of efficiency analysis, a benchmark is generated for a *DMU*, such as Point A' in Fig. 1(b) and Point B in Fig. 2 for *DMU A*. A benchmark can be justified for the consistent and objective comparison of efficiency if it is created on a common basis across all *DMU*s. This is because consistency and objectivity are paramount in efficiency assessment. However, a question arises as to whether the benchmark should also be used to plan resources for the *DMU*. If not, a further question is how to find an alternative efficient solution as a benchmark for resource planning, given that in resource planning it is the *DM*'s preferences that become paramount. To help address the questions raised above, an interactive approach is proposed, in which the *DM*'s indifference trade-offs are acquired to identify a new reference point and then a minimax optimisation problem is

solved to find a more preferred benchmark. This method is summarised as the following main steps.

Step 1: Check whether a current benchmark (e.g. generated from efficiency analysis) is what the *DM* prefers the most. While this question may be answered in different ways, in *MCDA* a popular criterion is to check whether the *DM*'s utility (or value) is maximised at this solution. Suppose the *DM*'s utility function is denoted by $u(\lambda)=u(f_1(\lambda)\cdots f_m(\lambda))$. If $u(\lambda)$ is given *a priori*, a benchmark will be the *DM*'s most preferred solution if $u(\lambda)$ is maximised at the benchmark within Ω_{j_0}. The problem is that $u(\lambda)$ is unknown in general, so this straightforward test is not feasible. However, local information about *DM*'s partial preferences may be acquired. For example, the *DM* may be capable of judging whether a given amount of the increased consumption of one resource (input) can be offset by a certain amount of the decreased consumption of another resource without changing his overall utility. In this paper, such an indifference trade-off or marginal rate of substitution is suggested as a means to acquire partial preference information from the *DM* as it can be used to estimate the local utility gradient of $u(\lambda)$, such as $\vec{u}(B)$ at Point B in Fig. 3 where the corresponding utility contour and its tangent line at Point B are also illustrated by the dotted curve and line.

Without loss of generality, suppose the first input $f_1(\lambda)$ is chosen as reference for increase by a given amount of Δf_1 from the current solution λ. Suppose Δf_1 can be exactly offset by decreasing other inputs $f_i(\lambda)$ by Δf_i for $i=2,\ldots,m$ without changing the *DM*'s overall utility. Then, the gradient of the utility function $u(\lambda)$ at λ can be estimated as follows [27]:

$$\nabla u(\lambda)=\left[\frac{\partial u}{\partial f_1}\cdots\frac{\partial u}{\partial f_i}\cdots\frac{\partial u}{\partial f_m}\right]^T=\frac{\partial u}{\partial f_1}\vec{u}(\lambda)$$
$$\text{with } \vec{u}(\lambda)\approx\left[1,\frac{\Delta f_1}{\Delta f_2}\cdots\frac{\Delta f_1}{\Delta f_i}\cdots\frac{\Delta f_1}{\Delta f_m}\right]^T \tag{33}$$

Suppose the normal vector of the facet of the efficient frontier at λ is given by $\vec{N}(\lambda)$. It is proven [27] that $\vec{u}(\lambda)$ will overlap $\vec{N}(\lambda)$ if λ is the most preferred solution maximising $u(\lambda)$, with $\vec{N}(\lambda)$ calculated by

$$\vec{N}(\lambda)=[w_1\beta_1\cdots w_i\beta_i\cdots w_m\beta_m]^T=w_1\beta_1\left[1,\frac{w_2\beta_2}{w_1\beta_1}\cdots\frac{w_i\beta_i}{w_1\beta_1}\cdots\frac{w_m\beta_m}{w_1\beta_1}\right]^T \tag{34}$$

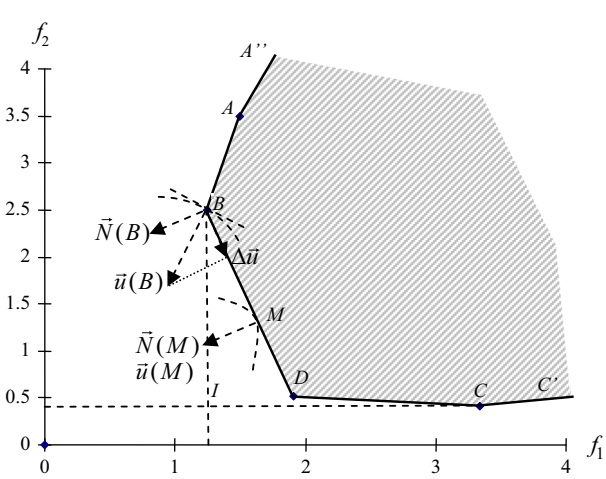

Fig. 3. Illustration of utility gradient projection.

where β_i is the dual variable of the ith objective constraint of the following ideal point formulation:

$$\text{Min} \quad \theta$$
$$\text{s.t.} \quad w_i\left(\sum_{j=1}^{n}\lambda_j x_{ij}-\bar{f}_i^{j_0}\right)\le\theta \quad \text{for } i=1,\ldots,m; \quad \lambda\in\Omega_{j_0} \tag{35}$$

whose optimal solution will be λ if w_i is given by

$$w_i=1/(f_i(\lambda)+\delta-\bar{f}_i^{j_0}) \quad \text{for } i=1,\ldots,m \tag{36}$$

δ is a perturbation parameter which is sufficiently small, with $\delta=0$ if $f_i(\lambda)>\bar{f}_i^{j_0}$.

Checking whether the current solution λ is the *DM*'s most preferred solution is therefore equivalent to checking whether $\vec{u}(\lambda)$ is proportional to $\vec{N}(\lambda)$ at λ, or

$$\left[1,\frac{\Delta f_1}{\Delta f_2}\cdots\frac{\Delta f_1}{\Delta f_i}\cdots\frac{\Delta f_1}{\Delta f_m}\right]^T\propto\left[1,\frac{w_2\beta_2}{w_1\beta_1}\cdots\frac{w_i\beta_i}{w_1\beta_1}\cdots\frac{w_m\beta_m}{w_1\beta_1}\right]^T$$
$$\text{or} \quad \Delta f_i=\frac{w_1\beta_1}{w_i\beta_i}\Delta f_1 \quad \text{for } i=2,\ldots,m \tag{37}$$

$\Delta f_i=(w_1\beta_1/w_i\beta_i)\Delta f_1$ is the optimal indifference trade-off between inputs $f_i(\lambda)$ and $f_1(\lambda)$, which can be used to guide the acquisition of the *DM*'s preferences, as discussed later. Condition (37) means that if λ is already efficient it will be infeasible to move in the direction of a normal vector at λ on the efficient frontier. So, it will be infeasible to improve $u(\lambda)$ if $\vec{u}(\lambda)$ is in the same direction as $\vec{N}(\lambda)$. In Fig. 3, the condition given by Eq. (37) is met at point M but not at point B.

Step 2: Find a new reference point with higher utility by utility gradient projection. If $\vec{u}(\lambda)$ is not proportional to $\vec{N}(\lambda)$, then a better efficient solution with higher utility than the current value of $u(\lambda)$ can always be found around λ on the efferent frontier. The projection of $\vec{u}(\lambda)$ onto the efferent frontier, as shown by $\Delta\vec{u}$ in Fig. 3, provides a direction along which the utility $u(\lambda)$ can be improved. The projection of $\vec{u}$ to a facet with $\vec{N}$ as its normal vector is given by

$$\Delta\vec{u}=[\Delta f_1\cdots\Delta f_i\cdots\Delta f_m]^T=-\vec{u}+\frac{(\vec{u})^T\cdot\vec{N}}{(\vec{N})^T\cdot\vec{N}}\vec{N} \tag{38}$$

Suppose α is a step size. A new reference point can then be given by

$$f^{ref}=[f_1^{ref}\cdots f_i^{ref}\cdots f_m^{ref}]^T \quad \text{with} f_i^{ref}=f_i(\lambda)+\alpha\Delta f_i \tag{39}$$

If α is sufficiently small, there will be $u(f^{ref})>u(\lambda)$ [26].

Step 3: Update the weighting parameters of the ideal point problem (35) and resolve it as follows:

$$w_i=1/(f_i^{ref}-\bar{f}_i^{j_0}) \quad \text{for } i=1,\ldots,m \tag{40}$$

Solving the ideal point problem (35) again with w_i updated by Eq. (40) results in a new efficient solution λ^{new}, which lies on the intersection of the efficient frontier and a line passing the ideal point and the reference point f^{ref}. For a sufficiently small step size α, there will be $u(\lambda^{new})>u(\lambda)$. At the new solution, we can go back to step 1 to check whether it is the *DM*'s most preferred solution. This interactive process can be repeated until condition (37) is met such as at point M in Fig. 3.

4. A case study for supplier assessment of fashion retailer

4.1. Problem description and conventional DEA results

This case study concerns a fashion accessory retailer that owns a network of chain stores specialised in high street fashion jewellery such as necklace, bracelets and earrings and other

accessories including belts and bags [25]. In the case study, two inputs, two outputs and twelve *DMU*s (the suppliers of the retailer) are taken into account. The two quantitative inputs chosen are average price per product and percentage of late deliveries. The two outputs are the quantitative attribute of total purchase and the qualitative attribute of product quality in terms of expected average utility. More measures were used by the retailer for performance analysis such as customer service. However, one widely adopted rule of thumb suggests that the number of *DMU*s should be at least two or three times larger than the total number of inputs and outputs; otherwise the discrimination power of *DEA* would be affected [9,20,30]. For this case study, we therefore only choose the above two input and two output measures. This is also because there is little correlation between the two inputs or between the two outputs as discussed later.

The quantitative data was directly obtained from the company, with the qualitative data being the average utility scores for the *DMU*s generated using the multiple criteria assessment, which reflect the value judgments of the management of the company [25]. The input and output values used for performance modelling are shown in Table 3. It should be noted that the correlation coefficient between the two outputs "Total purchase" and "Product quality" is 0.076, showing that the two outputs are almost uncorrelated. The correlation coefficient between the two inputs "Average price" and "Late deliveries" is -0.1, indicating a quite low degree of negative correlation between the two inputs. Also note that the number of *DMU*s is three times larger than the total number of inputs and outputs. As such, this is a practical case that can be studied using *DEA* for performance analysis.

The *DEA* efficiency scores and composite inputs and outputs can be found by running the input-oriented dual *CCR* model (1) or the identical super-ideal point model (18) with the origin as the reference point. The *DEA* efficiency scores of all the suppliers are shown in Table 4. It can be seen that suppliers 3, 4, 5, 6, 10, 11 and 12 are all inefficient and the other suppliers are efficient. Each inefficient supplier should benchmark against the efficient suppliers. For instance, supplier 12 has a *DEA* efficiency score of 0.498 and its *DEA* composite unit or benchmark is composed of supplier 1 by a degree of 0.374 and supplier 2 by a degree of 0.324.

The *DEA* composite inputs and outputs of the benchmark *DMU* for each supplier are shown in Table 5. For each efficient supplier, they are the same as its original inputs and outputs. For each inefficient supplier, its *DEA* composite inputs are at most as large as its original inputs whilst its *DEA* composite outputs are no smaller than its original outputs. For the inefficient supplier 12, condition (31) is not met. In fact, we have $9/1.94 \neq 2.74/0.97$. So, the *DEA* efficiency score for supplier 12 is not measured consistently and from Tables 4 and 5, its technical efficiency score is given by

$$TES_{12} = \frac{\overline{O, 12^T}}{\overline{O, 12^T} + \overline{12, 12^T}} = \frac{\sqrt{0.97^2 + 2.74^2}}{\sqrt{0.97^2 + 2.74^2} + \sqrt{(1.94-0.97)^2 + (9-2.74)^2}} = 0.315 \tag{41}$$

Table 3
Input–output data for supplier assessment.

DMU	Supplier	Inputs		Outputs	
		Average price (RM)	Late deliveries (%)	Total purchase (RM)	Product quality (utility)
1	Asia Jewellers Co.	1.34	3	5640.80	0.69
2	Beylai Jewelry Co. Ltd	1.44	5	38948.61	0.65
3	De Cartino Enterprise	2.13	6	2714.73	0.44
4	Fashion Show Beads Shop	2.01	9	13574.56	0.40
5	M & N Accessories	3.44	4	3249.00	0.36
6	Ondori Ngai Hin Sdn Bhd	2.98	4	2503.00	0.39
7	Sejong Trading	1.34	1	108124.50	0.13
8	Swarovski Singapore Trading Pte Ltd	5.10	2	100241.16	0.79
9	Sweet Girl Enterprise	1.40	1	21985.05	0.38
10	Shantou Tea Building	1.50	6	12091.10	0.37
11	Vivinual Enterprise	1.33	3	2245.00	0.31
12	Xiticomme Accessories Sdn Bhd	1.94	9	14723.00	0.47

where point 12^T represents the target for supplier 12, as shown for supplier 12 (*DMU* 12) in Table 5. It can be shown that condition (31) is not met for suppliers 3, 4 and 10 either, but it is met for the other inefficient suppliers. For example, there is $4/3.44 = 1.21/1.04$ for supplier 5. So, only for inefficient suppliers 5, 6 and 11 and for the efficient suppliers the technical efficiency score is the same as the *DEA* efficiency score. In the following, we use the concepts and techniques investigated in the previous sections to analyse the above results.

4.2. Equivalent efficiency analysis in a MOO process

In Table 5, the suppliers are divided into two types: efficient and inefficient. In this subsection, we generate and analyse the data envelopes and efficient frontiers for the efficient supplier 1 and the inefficient supplier 5 to illustrate the various features of efficiency analysis and provide a basis for resource allocation through interactive trade-off analysis to be discussed in the next section.

The *MOO* formulation (20) for the efficient supplier 1 is constructed as follows:

$$\begin{aligned}
&\text{Min} \quad f = \begin{bmatrix} f_1(\lambda) = 1.34\lambda_1 + 1.44\lambda_2 + 2.13\lambda_3 + 2.01\lambda_4 + 3.44\lambda_5 \\ +2.98\lambda_6 + 1.34\lambda_7 + 5.1\lambda_8 + 1.4\lambda_9 + 1.5\lambda_{10} + 1.33\lambda_{11} + 1.94\lambda_{12} \\ f_2(\lambda) = 3\lambda_1 + 5\lambda_2 + 6\lambda_3 + 9\lambda_4 + 4\lambda_5 + 4\lambda_6 + \lambda_7 + 2\lambda_8 + \lambda_9 + 6\lambda_{10} + 3\lambda_{11} + 9\lambda_{12} \end{bmatrix} \\
&\text{s.t.} \quad \lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}]^T \in \Omega_1 \\
&\Omega_1 = \left\{ \lambda \left| \begin{array}{l} 5640.8\lambda_1 + 38948.61\lambda_2 + 2714.73\lambda_3 + 13574.56\lambda_4 + 3249\lambda_5 + 2503\lambda_6 \\ +108124.5\lambda_7 + 100241.16\lambda_8 + 21985.05\lambda_9 \\ +12091.1\lambda_{10} + 2245\lambda_{11} + 14723\lambda_{12} \geq 5640.8 \\ 0.69\lambda_1 + 0.65\lambda_2 + 0.44\lambda_3 + 0.4\lambda_4 + 0.36\lambda_5 + 0.39\lambda_6 + 0.13\lambda_7 \\ +0.79\lambda_8 + 0.38\lambda_9 + 0.37\lambda_{10} + 0.31\lambda_{11} + 0.47\lambda_{12} \geq 0.69 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12} \geq 0 \end{array} \right. \right\}
\end{aligned} \tag{42}$$

Since there are twelve decision variables, it is not possible to draw the decision space of the above *MOO* problem. However, the procedure investigated in Section 3.3 can be used to generate the data envelope and the efficient frontier in the objective space for supplier 1. The scaling factors and coordinates for the twelve suppliers in the objective space constructed for supplier 1 are

Table 4
DEA efficiency scores and composite units for suppliers.

DMU	Supplier	Observed *DMU*'s COMPOSITE UNIT												
		DEA score	1	2	3	4	5	6	7	8	9	10	11	12
1	Asia Jewellers Co.	1.000	1.000											
2	Beylai Jewelry Co. Ltd	1.000		1.000										
3	De Cartino Enterprise	0.398	0.633											
4	Fashion Show Beads Shop	0.418	0.300	0.305										
5	M & N Accessories	0.303	0.229								0.525			
6	Ondori Ngai Hin Sdn Bhd	0.354	0.324								0.443			
7	Sejong Trading	1.000							1.000					
8	Swarovski Singapore Trading Pte Ltd	1.000								1.000				
9	Sweet Girl Enterprise	1.000									1.000			
10	Shantou Tea Building	0.515	0.287	0.269										
11	Vivinual Enterprise	0.457	0.454											
12	Xiticomme Accessories Sdn Bhd	0.498	0.374	0.324										

Table 5
DEA composite input–output levels for suppliers.

DMU	Supplier	Composite inputs		Composite outputs	
		Average price (RM)	Late deliveries (%)	Total purchase (RM)	Product quality (utility)
1	Asia Jewellers Co.	1.34	3	5640.80	0.69
2	Beylai Jewelry Co. Ltd	1.44	5	38948.61	0.65
3	De Cartino Enterprise	*0.85*	*1.90*	*3572.61*	*0.44*
4	Fashion Show Beads Shop	*0.84*	*2.43*	*13574.56*	*0.40*
5	M & N Accessories	*1.04*	*1.21*	*12836.63*	*0.36*
6	Ondori Ngai Hin Sdn Bhd	*1.05*	*1.42*	*11566.61*	*0.39*
7	Sejong Trading	1.34	1	108124.50	0.13
8	Swarovski Singapore Trading Pte Ltd	5.10	2	100241.16	0.79
9	Sweet Girl Enterprise	1.40	1	21985.05	0.38
10	Shantou Tea Building	*0.77*	*2.21*	*12091.10*	*0.37*
11	Vivinual Enterprise	*0.61*	*1.36*	*2560.10*	*0.31*
12	Xiticomme Accessories Sdn Bhd	*0.97*	*2.74*	*14723.00*	*0.47*

shown in Table 6. The twelve suppliers are drawn in the objective space as shown in Fig. 4, leading to the initial data envelope of $\overline{12'',12} \cup \overline{12,2} \cup \overline{2,1} \cup \overline{1,9} \cup \overline{9,8} \cup \overline{8,8''}$, the vertex points of which are composed of the original *DMU*s only, with point 12″ being the expansion point of point 12 and point 8″ the expansion point of point 8.

Following the procedure investigated in Section 3.3, we find that there is no more feasible solution outside this initial data envelope. The efficient frontier for supplier 1 is given by $\overline{1,9} \cup \overline{9,8}$, any of which could be a potential target for supplier 1, depending upon the *DM*'s preferences. Since point 1 is on the efficient frontier, supplier 1 is efficient. Solving the super-ideal point model (18) for supplier 1 with the origin as the reference point is to expand the minimax contour (m.c. in Fig. 4) along the line emitting from the origin to point 1. The northeast corner of the minimax contour touches the efficient frontier at point 1 with the *DEA* efficiency score of $e_1 = \overline{O,1}/\overline{O,1} = 1$. Note that points 2 and 7 are not on the efficient frontier constructed for supplier 1, though suppliers 2 and 7 are both efficient.

The *MOO* formulation (20) for the inefficient supplier 5 is constructed as follows:

$$\text{Min} \quad f = [f_1(\lambda), f_2(\lambda)]$$

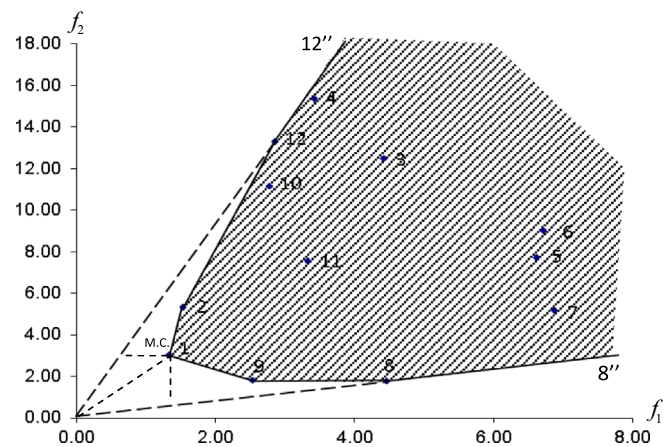

Fig. 4. Data envelope and efficient frontier for supplier 1.

$$\text{s.t.} \quad \lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}]^T \in \Omega_5 \tag{43}$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are the same as defined in problem (42) and Ω_5 is different from Ω_1 only in that the right hand side constants 5640.8 and 0.69 given in Ω_1 for supplier 1 are replaced by 3249 and 0.36 respectively in Ω_5 for supplier 5.

The feasible objective space for supplier 5 is drawn in Fig. 5, and the vertex points of the data envelope are composed of the original *DMU*s and a new composite *DMU* (point 13 in Fig. 5), given by $\overline{12'',12} \cup \overline{12,2} \cup \overline{2,13} \cup \overline{13,9} \cup \overline{9,8} \cup \overline{8,8''}$. In the objective space of Fig. 5, the coordinates of point 5 are given by (3.44, 4), point 8 by (2.31, 0.91), point 9 by (1.32, 0.94) and point 13 by (0.71, 1.54). The efficient frontier is $\overline{13,9} \cup \overline{9,8}$, any of which could be a potential target for supplier 5. Note that point 13 is a composite *DMU* that overtakes *DMU* 1 (point 1) as a vertex point in the efficient frontier, which is not an unusual case as shown for supplier 12 in the next section.

Note that points 1, 2 and 7 are not on the efficient frontier for supplier 5, although suppliers 1, 2 and 7 are all efficient. It is important to observe that in Table 4 the *DEA* benchmark for supplier 5 is composed of supplier 1 by a degree of 0.229 and supplier 9 by 0.525, whist suppliers 1 is not on the efficient frontier in Fig. 5. How can this happen? The answer is that point 13 is on the frontier, which is expanded from and composed of suppliers 1 and 9. This observation is important to help design an appropriate interactive process to search for a benchmark most preferred by the decision maker, as shown for supplier 12 in next section.

In Fig. 5, solving the super-ideal point model (18) for supplier 5 with the origin as the reference point is to expand the minimax

Table 6
Typical points in objective space for supplier 1.

DMU	$\bar{\lambda}_1$	$\bar{\lambda}_2$	$\bar{\lambda}_3$	$\bar{\lambda}_4$	$\bar{\lambda}_5$	$\bar{\lambda}_6$	$\bar{\lambda}_7$	$\bar{\lambda}_8$	$\bar{\lambda}_9$	$\bar{\lambda}_{10}$	$\bar{\lambda}_{11}$	$\bar{\lambda}_{12}$	f_1	f_2
1	1.00												1.34	3.00
2		1.07											1.54	5.33
3			2.08										4.43	12.47
4				1.71									3.43	15.36
5					1.93								6.63	7.71
6						2.25							6.72	9.01
7							5.14						6.89	5.14
8								0.87					4.46	1.75
9									1.81				2.54	1.81
10										1.85			2.78	11.13
11											2.51		3.34	7.54
12												1.48	2.86	13.29

contour along the line emitting from the origin to point 5. The northeast corner of the minimax contour (m.c. in Fig. 5) touches the efficient frontier at point $5'$ with the *DEA* efficiency score calculated by $e_5 = \overline{O,5'}/\overline{O,5}$. Point $5'$ is the intersection of this line, whose function is given by $4f_1 - 3.44f_2 = 0$, and the line $\overline{13,9}$, whose function is given by $0.6f_1 + 0.61f_2 = 1.3654$. The coordinates of point $5'$ in the objective space are thus given by $5' = [1.0428, 1.2126]^T$. So,

$$e_5 = \frac{\overline{O,5'}}{\overline{O,5}} = \frac{\sqrt{1.0428^2 + 1.2126^2}}{\sqrt{3.44^2 + 4^2}} = 0.303 \tag{44}$$

This is the same result as shown in Table 4 for supplier 5.

4.3. Interactive trade-off analysis and resource planning

As shown in Section 4.1, supplier 12 is given a benchmark that is inconsistent with the efficiency measurement. In this subsection, supplier 12 is investigated in detail to show the features of efficiency analysis and the need for trade-off analysis for achieving balanced resource allocation with the *DM*'s preferences taken into account interactively. The *MOO* formulation (20) for supplier 12 is constructed as follows:

$$\begin{aligned} \text{Min} \quad & f = [f_1(\lambda), f_2(\lambda)] \\ \text{s.t.} \quad & \lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}]^T \in \Omega_{12} \end{aligned} \tag{45}$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are the same as defined in problem (42) and Ω_{12} is different from Ω_1 only in that the right hand side constants 5640.8 and 0.69 in Ω_1 for supplier 1 are replaced by 14723 and 0.47 respectively in Ω_{12} for supplier 12. The objective space constructed for supplier 12 is shown in Fig. 6. The initial data envelope is composed of $\overline{12'',12} \cup \overline{12,2} \cup \overline{2,9} \cup \overline{9,8} \cup \overline{8,8''}$.

Using the procedure investigated in Section 3.3, the above initial data envelope can be expanded to the final data envelope: $\overline{12'',12} \cup \overline{12,2} \cup \overline{2,14} \cup \overline{14,13} \cup \overline{13,15} \cup \overline{15,9} \cup \overline{9,8} \cup \overline{8,8''}$ as shown in Fig. 6, and the efficient frontier is composed of $\overline{14,13} \cup \overline{13,15} \cup \overline{15,9} \cup \overline{9,8}$ as shown in Fig. 7 in detail where the local feasible region round the efficient frontier for supplier 12 is amplified. The coordinates for point 12 is given by (1.94, 9), point 13 by (1.02, 2.07), point 14 by (0.97, 2.74) and point 15 by (1.29, 1.65). It is important to observe that the efficient *DMU*s 1, 2 and 7 are not on the efficient frontier constructed for supplier 12, whilst in Table 4 the *DEA* benchmark for supplier 12 is composed of supplier 1 by a degree of 0.374 and supplier 2 by 0.324. The explanation for such a phenomenon is that the benchmark for supplier 12 is actually the composite *DMU* 14 (point 14), as shown below, which is composed of suppliers 1 and 2.

As shown in Fig. 7, to solve the super-ideal point model (18) for supplier 12 with the origin as the reference point is to expand the minimax contour along the line emitting from the origin to point 12.

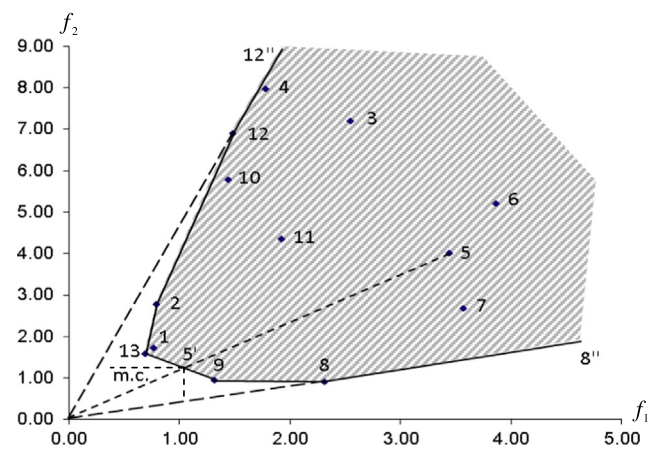

Fig. 5. Efficiency analysis for *DMU* 5.

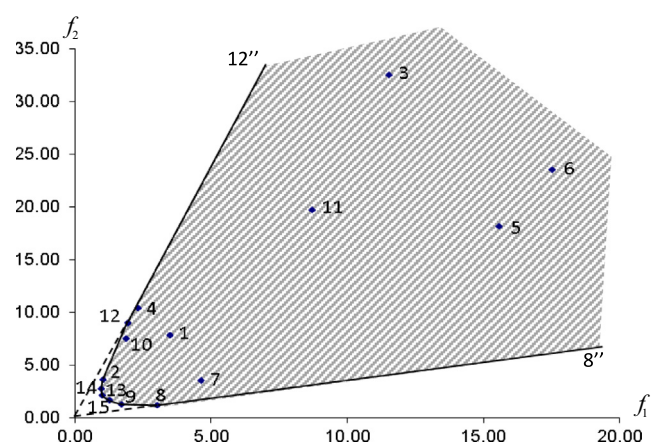

Fig. 6. Data envelop and efficient frontier for *DMU* 12.

The right side of the minimax contour touches the efficient frontier at point 14, with $f_1(\lambda^{14}) = 0.97$ and $f_2(\lambda^{14}) = 2.74$, and the *DEA* efficiency score is calculated by $e_4 = \overline{O,12'}/\overline{O,12}$ where point $12'$ is the northeast corner of the minimax contour with the coordinates given by $[0.97, 4.5]^T$, which is the intersection of the line emitting from the origin to point 12, whose function is given by $9f_1 - 1.94f_2 = 0$, and the line of the right side of the minimax contour, whose function is given by $f_1 = 0.97$. So, the *DEA* efficiency score is given by

$$e_{12} = \frac{\overline{O,12'}}{\overline{O,12}} = \frac{\sqrt{0.97^2 + 4.5^2}}{\sqrt{1.94^2 + 9^2}} = 0.5 \tag{46}$$

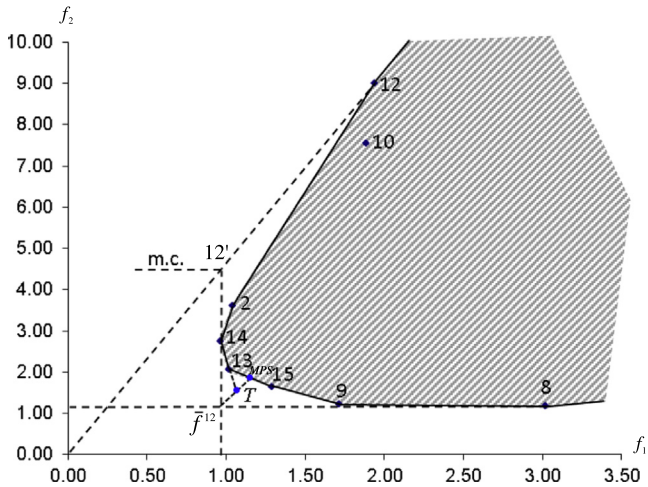

Fig. 7. Trade-off analysis for *DMU* 12.

This is the same *DEA* efficiency score as shown in Table 4 for supplier 12. Since point 12′ is not feasible, however, point 14 rather than point 12′ is actually suggested as the benchmark for supplier 12 in the conventional *DEA* as shown in Table 5, which is inconsistent with the measurement of the *DEA* efficiency score. The following technical efficiency score should be used to measure the degree to which supplier 12 needs to make the better use of its resources to achieve full efficiency:

$$TES_{12} = \frac{\overline{O,14}}{\overline{O,14}+\overline{12,14}} = \frac{\sqrt{0.97^2+2.74^2}}{\sqrt{0.97^2+2.74^2}+\sqrt{(1.94-0.97)^2+(9-2.74)^2}} = 0.315 \tag{47}$$

Note that the benchmark for supplier 12 generated from the conventional *DEA* process, as given in Table 5 (or point 12^T in Eq. (41)), is the same as point 14 in Fig. 7 and Eq. (47). This explains why the result of Eq. (47) is the same as that of Eq. (41).

In the enlarged efficient frontier shown in Fig. 7, the ideal point $\bar{f}^{12}$ takes the minimum feasible values of 0.97 for the first composite input ($f_1(\lambda)$) and 1.19 for the second composite input ($f_2(\lambda)$). The *DEA* benchmark for supplier 12 is given by point 14, which is an extreme efficient solution at the upper end of the efficient frontier, where the lowest feasible *Average Price* ($f_1(\lambda)$) is achieved but the percentage of *Late Deliveries* ($f_2(\lambda)$) is the highest among all solutions on the efficient frontier.

Point 14 was suggested as the target for supplier 12 to benchmark against in the conventional *DEA* simply because of the fixed radial movement strategy along the line emitting from the origin to point 12, technically embedded in the input-oriented dual *CCR* model (1) without necessarily taking into account the *DM*′s preferences. In general, this strategy may not be what the *DM* prefers the most and trade-off analysis between $f_1(\lambda)$ and $f_2(\lambda)$ may need to be conducted to take into account the *DM*′s preferences for planning resources.

As shown in Fig. 7, improving $f_2(\lambda)$ at the expense of $f_1(\lambda)$ along the efficient line $\overline{14,13}$ seems attractive, as indicated by its normal vector $\overrightarrow{N}^{14,13} = [-0.6661, -0.0504]^T = -0.6661[1, 1/13.2]^T$, which shows that a small sacrifice (increase) of $f_1(\lambda)$ can lead to a large improvement (decrease) of $f_2(\lambda)$. More precisely, the sacrifice of one unit of $f_1(\lambda)$ along the efficient line $\overline{14,13}$ can lead to the improvement of 13.2 units of $f_2(\lambda)$. On the other hand, trade-offs along the efficient line $\overline{13,15}$ lead to 1.55 units of improvement for $f_2(\lambda)$ at the expense of each unit of $f_1(\lambda)$, because its normal vector $\overrightarrow{N}^{13,15} = [-0.4217, -0.2714]^T = -\ 0.4217[1, 1/1.55]^T$. Along the efficient line $\overline{15,9}$, however, the sacrifice of $f_1(\lambda)$ becomes even less attractive to improve $f_2(\lambda)$, as $\overrightarrow{N}^{15,9} = [-0.426, -0.4283]^T = -0.426[1, 1/0.995]^T$. In other words, one unit of sacrifice from $f_1(\lambda)$ can only lead to 0.995 unit of improvement for $f_2(\lambda)$ along $\overline{15,9}$. Furthermore, improving $f_2(\lambda)$ along the efficient line $\overline{9,8}$ does not make sense because one unit of improvement for $f_2(\lambda)$ would need 31.6 units of sacrifice from $f_1(\lambda)$, because $\overrightarrow{N}^{9,8} = [-0.0413, -1.3052]^T = -1.3052[1/31.6, 1]^T$. The above trade-off analysis is discussed in association with the normal vector of an efficient facet (line). This association is useful because the normal vectors of an efficient frontier for a *MOO* problem can be generated using the minimax formulation without extra costs as shown in Section 3.5.

The interactive trade-off analysis process investigated in Section 3.5 is applied to analyse supplier 12 and demonstrated as follows. For example, suppose the *DM* of supplier 12 decides to investigate whether or not his overall utility could be improved from point 14 that was initially suggested as the benchmark for *DMU* 12 in the conventional *DEA*. The normal vector of the efficient frontier at Point 14 can be found by constructing the ideal point problem (35) and using Eq. (36) to set the weighting parameter, which is the same as $\overrightarrow{N}^{14,13}$, or $\overrightarrow{N}(\lambda^{12}) = \overrightarrow{N}^{14,13} = -0.6661[1, 1/13.2]^T$. An optimal indifference trade-off question can then be presented to the *DM* by asking "whether his overall utility would remain unchanged if from the current level of the benchmark for '*Average Price*' and '*Late Deliveries*', or $f_1(\lambda^{14}) = 0.97$ and $f_2(\lambda^{14}) = 2.74$, '*Average Price*' is increased by one unit and the '*Late Deliveries*' is decreased by 13.2 units". If the answer is "yes", Point 14 will be the *DM*′s most preferred solution and the process terminated. Otherwise, the process continues.

Suppose the *DM*′s answer to the above optimal indifference trade-off question is "no". Note that since Point 14 is an extreme point the *DM* can only improve his overall utility by improving "*Late deliveries*" ($f_2(\lambda)$) at the expense of "*Average price*" ($f_1(\lambda)$). Indeed, he should do so along a line overlapping the efficient facet $\overline{14,13}$. Since $\overline{14,13}$ has been drawn already and the improvement direction is known at Point 14, there is no need to estimate the *DM*′s local utility gradient. To search for a reference point along this line, suppose the *DM* is prepared to sacrifice 0.09 units of $f_1(\lambda)$ to improve $f_2(\lambda)$ by an estimated amount of 1.188 (or 0.09×13.2) units along the line $\overline{14,13}$. A new reference point is then given by $T = [0.97+0.09, 2.74-1.188] = [1.06, 1.552]^T$, as shown in Fig. 7, which is not feasible.

Note that the ideal point for *DMU* 12 is given by $\bar{f}^{12} = [0.97, 1.19]^T$. Use Eq. (36) to update the weights by $w_1 = 1/(1.06-0.97) = 11.1111$ and $w_2 = 1/(1.552-1.19) = 2.7624$. To find the efficient solution closest to the new target *T*, formulate the following minimax ideal point problem:

$$\begin{aligned} &\text{Min} \quad \theta \\ &\text{s.t.} \quad w_1(f_1(\lambda)-0.97) \le \theta, \quad w_2(f_2(\lambda)-1.19) \le \theta, \quad \lambda \in \Omega_{12} \end{aligned} \tag{48}$$

The optimal solution of the above problem can be generated by using Solver in Excel and given by $MPS = [1.146, 1.898]^T$, which is the intersection point of the efficient line $\overline{13,15}$ and the line emitting from the ideal point $\bar{f}^{12}$ to the efficient frontier through the reference point *T* as shown in Fig. 7.

The Lagrange (simplex) multipliers or the dual variables of the first two objective constraints in formulation (48) are given by $\beta = [\beta_1, \beta_2]^T = [-0.278, -0.722]^T$, which can be obtained from the Sensitivity Report generated by Solver in Excel. The normal vector of the efficient frontier at point *MPS* is then given from Eq. (34) by $\overrightarrow{N}^{MPS} = [w_1\beta_1, w_2\beta_2]^T = [-11.1111 \times 0.278, -2.7624 \times 0.722]^T = [-3.0889, -1.9945]^T = -3.0889[1, 1/1.55]^T$, which is in parallel

with $\overrightarrow{N}^{13,15} = -0.4217[1, 1/1.55]^T$, the normal vector of the efficient facet $\overline{13,15}$.

The *DM* is then presented with a new optimal indifference trade-off question by asking "whether he agrees that one unit of $f_1(\lambda)$ can be exactly offset by 1.55 units of $f_2(\lambda)$ from this current target of $f_1(\lambda^{MPS}) = 1.146$ and $f_2(\lambda^{MPS}) = 1.898$". If the answer is "no", the trade-off process will continue. Otherwise, *MPS* will be the most preferred solution that maximises the *DM*'s implicit utility function around this current target. The preferential efficiency score for supplier 12 is then given by

$$PES_{12} = \frac{\overline{O,MPS}}{\overline{O,MPS} + \overline{12,MPS}} = \frac{\sqrt{1.146^2 + 1.898^2}}{\sqrt{1.146^2 + 1.898^2} + \sqrt{(1.94 - 1.146)^2 + (9 - 1.898)^2}} = 0.2368 \tag{49}$$

The very low PES_{12} value indicates that supplier 12 has to make a lot of efforts for improving its performance to achieve this most preferred efficiency target. In Fig. 7, this means graphically that point 12 is far away from point *MPS* compared with the distance between point *MPS* and the origin.

5. Conclusion

This paper reported an investigation in exploring equivalence relationships between *DEA* and *MOO* models. It was shown that minimax reference point models are equivalent to input-oriented dual *CCR* models under certain conditions. This leads to the development of an interactive minimax reference point approach for *Hybrid Efficiency and Trade-off Analyses* (*HETA*), with the decision maker's preferences taken into account interactively. The proposed *HETA* approach can be used to support past performance assessment and future resource planning in a consistent and interactive manner. The graphical and analytical procedures explored in this paper for generating data envelopes and efficient frontiers helped to deepen the understanding of *HETA* as a *MOO* process and led to the construction of the technical and preferential efficiency measures for *HETA*. The case study showed the implementation process of the interactive *HETA* approach and its potential applications. Both the graphical and analytical procedures can be applied to analyse performance analysis problems having many *DMU*s. The graphical procedure can be used to analyse higher dimensional *DEA* problems with three or more inputs by exploring two inputs at a time. If all inputs need to be explored simultaneously, the analytical procedure can be applied.

It is worth emphasising that both input-oriented dual *CCR* models and minimax reference point models share the same Pareto frontier, although the former is formulated to find a particular solution on the Pareto frontier, defined as the *DEA* efficient solution in this paper. This property of sharing the same Pareto frontier ensures that the performance of a *DMU* is assessed objectively in conjunction with other *DMU*s, on the same basis of which the performance target is set for future improvement with the *DM* preferences taken into account interactively. This feature makes the proposed *HETA* approach practical and distinctive from other methods by preserving the consistency and objectivity in performance assessment as well as the flexibility of taking into account the *DM*'s preferences in resource planning for performance improvement.

Performance analysis and resource planning is a complex process. While the proposed *HETA* approach provides a useful analytical means to support this process, more factors need to be considered which could be both quantitative and qualitative and involve uncertainty such as incomplete data and imprecise preference information. Also, this investigation is focused on using input-oriented dual *DEA* models for non-parametric performance modelling. Other modelling techniques may provide alternative means for similar purposes, which needs to be further investigated. Finally, it should be noted that the computational procedure proposed to explore efficient frontier is rather generic and is mainly used to illustrate the concepts and computational process of *HETA*. To deal with large scale *DEA* problems with hundreds of *DMU*s and more inputs/outputs, more powerful algorithms and user interfaces need to be developed to explore efficient frontiers to support informative trade-off analysis.

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