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Integrated efficiency and trade-off analyses using a *DEA*-oriented interactive minimax reference point approach

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ABSTRACT

A *DEA*-oriented Interactive Minimax Reference Point (*DEA-IMRP*) approach was recently developed to support integrated performance assessment and target setting for consistent management control and planning. To conduct the Integrated Efficiency and Trade-off (*IET*) analyses using the *DEA-IMRP* approach, it is important to understand the characteristics of the efficiency frontier and interactive trade-off analysis process. In this paper, the features of the *IET* analyses are investigated in detail. Graphical and analytical methods and procedures are explored for generating and analysing data envelopes and efficient frontiers for multiple input and multiple output *DEA* models using the *DEA-IMRP* approach. This computational investigation generates useful insights into the *IET* analyses and leads to the definition of new efficiency measures, which are instrumental to help conduct trade-off analysis for setting realistic performance targets. A numerical example is studied to illustrate the findings graphically. A case study for UK retail banks is conducted using the new methods and procedures investigated in this paper.

1. Introduction

Data envelopment analysis (*DEA*) is a tool for assessing past performances as part of management control function and has been used to measure and analyse the relative efficiencies of various types of decision making units (*DMU*s), such as banks and university departments, which possess shared functional goals with incommensurate inputs and outputs [1–3]. In the classical *DEA* models proposed by Charnes et al. [4] and Banker et al. [5], however, the decision maker (*DM*)'s preferences are not taken into account in the generation of efficiency scores and target levels. Since then, a number of techniques have been developed to incorporate the *DM*'s preferences in *DEA*, including the goal and target setting models proposed by Golany [6], Thanassoulis and Dyson [7] and Athanassopoulos [8,9], and weight restriction models by imposing bounds on individual weights [10] or by specifying a proper set of "preference weights" [11], by creating assurance region [12], by restricting weight ratios and proportions [13], and using the cone ratio concept to adjust weights [14,15]. However, these techniques require *prior* preferences from the *DM*s as well as knowledge about what are achievable.

Multiple criteria decision analysis (MCDA) in general and multiple objective linear programming (MOLP) in particular can be used for planning future performances. The model structures of *DEA* and MOLP have much in common and research on integrating *DEA* and MOLP has attracted increasing attentions in order to support both past performance assessment and future target setting in an integrated and consistent manner [16]. For instance, Golany [6] developed an interactive model to allocate a set of input levels as resources and to select the most preferred output levels from a set of alternative points on the efficient frontier. Post and Spronk [17] combined the use of *DEA* and interactive goal programming to adjust the upper and lower feasible boundaries of the input and output levels. Li and Reeves [18] used a MOLP framework to evaluate several efficiency measures without requiring prior weights from decision makers. Joro et al. [19,20] showed the structural similarity between *DEA* and MOLP. Halme et al. [21] and Korhonen et al. [22] investigated value efficiency for incorporating decision makers' preferences. Estellita Lins et al. [23] proposed a multiple objective approach for posterior preference incorporation to obtain a target at every extreme-efficient point on the frontier. Chen et al. [24] proposed a linear programming model for generating the efficiency of a DMU using the difference of input and output rather than their ratio, and they further formulated a MOLP problem to generate common weights for all DMUs.

In recognition of the importance and popularity of interactive procedures based on reference point approaches for supporting

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multiple objective decision making, a *DEA*-oriented *Interactive* Minimax Reference Point (*DEA-IMRP*) approach method was recently developed to incorporate the *DM*'s preference information into performance assessment and target setting without necessarily requiring *prior* judgments [25] using the minimax formulations and the gradient projection method [26–28]. In *DEA-IMRP*, three minimax models are explored, all equivalent to the output-oriented *CCR* dual model in *DEA* and different from each other in their reference points and weighting schemas. Based on the equivalence analysis, a *MOLP* formulation is constructed where the features of data envelopes, efficient frontiers, efficiency measures and interactive trade-off analysis can be explored, which is the theme of this paper.

To help conduct the Integrated Efficiency and Trade-off (*IET*) analysis along an efficient frontier using the *DEA-IMRP* approach, it is fundamental to understand the features of the efficient frontier. This paper is aimed at investigating graphical and analytical methods and procedures for generating and analysing data envelopes and efficient frontiers for multiple input and multiple output *DEA* problems using *DEA-IMRP*. This computational investigation is intended to generate new insights into the *IET* analyses, leading to the definition of efficiency measures including a new technical efficiency score (*TES*) and preferred efficiency score (*PES*). *TES* provides a revised performance measure and *PES* provides a logical measure between an observed *DMU* and its most preferred efficient solution, showing the extent to which the observed *DMU* needs to improve its performances to achieve the most preferred solution. A numerical example is studied to illustrate the findings graphically and to generate data envelopes and efficient frontiers analytically for *DEA* problems of practical size. A case study for UK retail banks is conducted using the proposed methods and procedures.

The rest of the paper is organised as follows. In Section 2, the principle and main features of the *DEA-IMRP* approach is briefly introduced with the initial exploration of the dual decision and objective spaces. Section 3 reports the graphical and analytical investigation into the integrated efficiency and trade-off analyses, illustrated by a simple *DEA* problem, leading to the definition of the new technical efficiency score and preferred efficiency score and the development of a new analytical procedure for analysing data envelopes and efficient frontiers. In Section 4, the performance analysis for UK retail banks is conducted using the proposed methods and procedures. The paper is concluded in Section 5.

2. The *DEA-IMRP* approach for integrated performance assessment and target setting

2.1. DEA and super-ideal point model for efficiency and trade-off analysis

The *DEA*-oriented Interactive Minimax Reference Point (*DEA-IMRP*) approach was developed by Yang et al. [25] and its main features are described in this subsection. Suppose a *DEA* problem includes n *DMU*s, s outputs denoted by y_{rj} (the rth output of *DMU* j) and m inputs denoted by x_{ij} (the ith input of *DMU* j). The formulation of the output-oriented *CCR* primal model is then given by [29].

$$\begin{aligned} \text{Min} \quad & h_0 = \sum_{i=1}^{m} v_i x_{ij_0} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} > 0 \quad j = 1,\ldots,n \\ & \sum_{r=1}^{s} u_r y_{rj_0} = 1, \quad u_r, v_i > 0 \text{ for all } r,i \end{aligned} \tag{1}$$

where u_r is the weight parameter for output r and v_i the weight parameter for input i. $e_0 = 1/h_0$ denotes the optimal efficiency score with a possible range of $0 \le e_0 \le 1$ for the observed DMU_0. The score of $e_0 = 1$ represents full efficiency if all slacks are also zero [29] and $0 < e_0 < 1$ shows the presence of inefficiency. Each DMU can be evaluated as the observed DMU_0 and is allowed to assign its own set of u_r and v_i, which renders the observed DMU_0 as efficient as possible. In other words, the efficiency measure e_0 is optimised within the constraints constructed for each *DMU* by the set of n *DMUs*. In the output-orientated *CCR* primal model (1), the weighted outputs are fixed to unity and the weighted inputs minimised. The output weights u_r and input weights v_i are adjusted accordingly to generate an efficiency score.

While the *CCR* primal model can generate an efficiency score, the optimal output weight u_r and input weight v_i, the *CCR* dual model can be used to generate not only the efficiency score but also an imaginary composite *DMU* with the efficient composite inputs and outputs that the observed DMU_0 should benchmark against. The output-oriented *CCR* dual model is given as follows [29]:

$$\begin{aligned} \text{Max} \quad & h_0 = \theta_{j_0} \\ \text{s.t.} \quad & \theta_{j_0} y_{rj_0} - \sum_{j=1}^{n} \lambda_j y_{rj} \le 0, \quad r = 1,\ldots,s \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ij_0} \quad i = 1,\ldots,m \quad \lambda_j \ge 0 \text{ for all } j \end{aligned} \tag{2}$$

In the output-oriented *CCR* dual model (2), for each observed DMU_0 an imaginary composite unit is constructed as the optimal solution of model (2) that outperforms DMU_0. This optimal solution is referred to as the *DEA* efficient solution or *DEA* composite unit for DMU_0. λ_j is a dual variable representing the reference weight for *DMU* j ($j = 1,\ldots, n$) and $\lambda_j > 0$ means that *DMU* j is used to construct the *DEA* composite unit for DMU_0. The *DEA* composite unit consumes at most the same amount of inputs as DMU_0 and produces outputs that are at least equal to a proportion θ_{j_0} of the outputs of DMU_0 with $\theta_{j_0} \ge 1$. The inverse of θ_{j_0} is the efficiency score of DMU_0. The parameter θ_{j_0} indicates the amount by which DMU_0 has to proportionally increase its outputs in order to become efficient. The increase is employed concurrently to all outputs and results in a radial movement towards the envelopment surface [15].

In Yang et al. [25], it was suggested that the following reference point model (3) can be used to conduct efficiency analysis in the same way as the output-oriented *CCR* dual model (2) and also to provide a basis for trade-off analysis in planning future performances or setting targets.

$$\begin{aligned} & \text{Min}\,\theta \\ & \text{s.t.} \quad w_r(f_r^* - f_r(\lambda)) \le \theta \quad r = 1,\ldots,s \\ & \lambda = [\lambda_1 \cdots \lambda_n]^T \in \Omega_{j_0} = \left\{ \lambda \,\middle|\, \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ij_0}, \right. \\ & \left. i = 1,\ldots,m,\ \lambda_j \ge 0,\ j = 1,\ldots,n \right\} \end{aligned} \tag{3}$$

where $f_r(\lambda) = \sum_{j=1}^{n} \lambda_j y_{rj}$ is the rth composite output for maximisation and λ_j is as defined in (2).

Model (3) is proven identical to the output-oriented *CCR* dual model (2) under the following conditions [25] with $e_0 = 1/(F^{\max} - \theta)$ being the optimal efficiency score.

$$\theta = F^{\max} - \theta_{j_0} \tag{4}$$

$$w_r = 1/y_{rj_0} \tag{5}$$

$$f^* = [f_1^*,\ldots,f_s^*]^T \text{ with } f_r^* = y_{rj_0}F^{\max} \text{ and } F^{\max} = \max_{1 \le r \le s}\left\{\frac{\bar{f}_{rj_0}}{y_{rj_0}}\right\} \tag{6}$$

$$\bar{f}_{rj_0} = f_r(\lambda^*) = \max_{\lambda \in \Omega_{j_0}} f_r(\lambda) = \sum_{j=1}^{n} \lambda_j y_{rj} \tag{7}$$

Note that $\bar{f} = [\bar{f}_{r1},\ldots,\bar{f}_{rs}]^T$ is an ideal point, taking the maximum value of $f_r(\lambda)$ within Ω_{j_0}, and $f^* \ge \bar{f}$. In other words, the reference point f^* used in model (3) is superior to the ideal point $\bar{f}$. The reference point model (3) is thus referred to as the super-ideal point model. The equivalence between the super-ideal point model and the output-oriented *CCR* Dual model (2) shows that the *DEA* efficient solution is a specific solution of the multiple objective linear programming problem (9) as defined and analysed in the next section.

2.2. Dual decision and objective spaces for efficiency and trade-off analyses

In models (2) and (3), $\lambda = [\lambda_1 \cdots \lambda_n]^T$ is a vector of decision variables and Ω_{j_0} defines the feasible dual decision space for the observed DMU_0. Any feasible solution $\lambda \in \Omega_{j_0}$ represents a composite unit for the observed DMU_0. The *DEA* efficient solution, denoted by λ^{DEA}, is one of the efficient solutions in Ω_{j_0}.

If $\lambda_j \ne 0$ and $\lambda_k = 0$ for all $k = 1,\ldots,n$ with $k \ne j$, then *DMU j* itself is a composite solution in Ω_{j_0}. In this case, the feasible value of λ_j can be calculated so that $\lambda_j x_{ij} \le x_{ij_0}$ for all $i = 1,\ldots,m$, which leads to the following result:

$$\underline{\lambda}_j = \min_{1 \le i \le m}\left\{\frac{x_{ij_0}}{x_{ij}}\right\} \tag{8}$$

The value $\underline{\lambda}_j$ calculated using Eq. (8) is a scaling factor for *DMU j*. As shown in next sections, this scaling factor can be used to determine the location of *DMU j* in the dual objective space. It is obvious that the scaling factor for the observed DMU_0 is always one, or $\underline{\lambda}_{j_0} = 1$. If a *DMU* consumes more resources in any of the m inputs than the observed DMU_0, then the scaling factor of the *DMU* will be less than one; if a *DMU* consumes fewer resources in all the m inputs than the observed DMU_0, then the scaling factor of the *DMU* will be larger than one. These observations are useful for the graphical interpretations of *DMU*s in the dual objective space as discussed later.

Note that if there is more than one non-zero element in a dual decision vector with all other elements being zero, for example only $\lambda_{j_1} \ne 0$ and $\lambda_{j_2} \ne 0$, then such a composite solution (imaginary unit) represents a linear combination of *DMU* j_1 and *DMU* j_2, or more precisely λ_{j_1} of *DMU* j_1 and λ_{j_2} of *DMU* j_2. The vast majority of feasible solutions in the dual decision space Ω_{j_0} are of this nature.

$f_r(\lambda) = \sum_{j=1}^{n} \lambda_j y_{rj}$ is the rth composite output. In model (3), $f_r(\lambda)$ $(r = 1,\ldots,s)$ was defined as a dual objective function. Therefore, the corresponding *MOLP* problem, on which the super-ideal point model (3) is constructed, is the following s-objective linear programming problem.

$$\begin{aligned} \text{Max} \quad & f = [f_1(\lambda),\ldots,f_s(\lambda)]^T \\ \text{s.t.} \quad & \lambda = [\lambda_1,\ldots,\lambda_n]^T \in \Omega_{j_0} \end{aligned} \tag{9}$$

The dual objective space is expanded by the s dual objective functions and has s dimensions. The feasible dual objective space is the projection of the feasible dual decision space using the functions $f_r(\lambda)$ $(r = 1,\ldots,s)$, denoted by $f(\Omega_{j_0})$. The set of the efficient solutions of problem (9) is represented by $E(f(\Omega_{j_0}))$, which includes all possible efficient composite solutions for the observed DMU_0 to benchmark against. In this paper, we will show that the *DEA* efficient solution is one solution in $E(f(\Omega_{j_0}))$, or $f(\lambda^{DEA}) \in E(f(\Omega_{j_0}))$. We will show how $E(f(\Omega_{j_0}))$ can be graphically and analytically generated and represented for an observed DMU_0 if there are up to three outputs with any number of inputs and any number of *DMU*s. We will also discuss how $E(f(\Omega_{j_0}))$ can be searched using the *DEA-IMRP* approach [25] if there are more than three outputs.

From (8), to represent an original *DMU j* in the decision space Ω_{j_0}, the decision variables are given by $\lambda = [0,\ldots,0,\underline{\lambda}_j,0,\ldots,0]^T$. So an original *DMU j* can be represented in the objective space $f(\Omega_{j_0})$ by

$$f_r(\lambda) = \sum_{i=1}^{n} \lambda_i y_{ri} = \underline{\lambda}_j y_{rj} \quad \text{for } r = 1,\ldots,s \tag{10}$$

Since $\underline{\lambda}_{j_0} = 1$, the observed DMU_0 is represented in $f(\Omega_{j_0})$ using its true (un-scaled) output values. In general, all other original *DMU*s are represented in $f(\Omega_{j_0})$ using their scaled output values. The locations of the original *DMU*s calculated using Eq. (10) provide a basis to characterise $E(f(\Omega_{j_0}))$ as shown later.

2.3. Ideal point and shortest distance models for setting targets

$E(f(\Omega_{j_0}))$ is always non-empty, may only have one efficient solution in special cases but in general include many efficient solutions. As shown later, the *DEA* efficient solution is designed to be the intersection of $E(f(\Omega_{j_0}))$ with a line emitting from the origin through the point of the observed DMU_0 in the dual objective space, used as a benchmark for the observed DMU_0, and its location is technically embedded in the formation of the *CCR* dual model. It is therefore referred to as a technical benchmark, which, however, does not necessarily take into account the preferences of the decision makers (*DM*s) and may not be the most preferred solution of the decision maker. As such, it is necessary to search $E(f(\Omega_{j_0}))$ for finding the most preferred solution (*MPS*) that can maximise the *DM*'s implicit utility function.

Many methods could be employed for locating *MPS* on the basis of formulation (9). However, the focus of this paper is not about how to find *MPS*, which is discussed in Yang et al. [25], but to investigate how $E(f(\Omega_{j_0}))$ can be characterised both graphically and analytically. This computational investigation will lead to the modification of the technical efficiency measure in relation to the *DEA* efficient solution and the definition of a new preferred efficiency measure in relation to *MPS*. As such, we only briefly introduce the ideal point model, which can be used to help find *MPS*, and the shortest distance model to support the location of a group most preferred solution.

The ideal point model is defined as follows:

$$\begin{aligned} &\text{Min}\,\theta \\ &\text{s.t.} \quad w_r(\bar{f}_{rj_0} - f_r(\lambda)) \le \theta \quad r = 1,\ldots,s, \quad \lambda = [\lambda_1,\ldots,\lambda_n]^T \in \Omega_{j_0} \end{aligned} \tag{11}$$

where $w_r \ge 0$ (r=1, …, s) are weighting parameters subject to systematic adjustment. It has been proven that all efficient solutions in $E(f(\Omega_{j_0}))$ can be found by adjusting w_r (r=1, …, s) in R^+ [28,30,31]. For example, the gradient projection and local region search method based on problem (11) can be used to locate efficient solutions preferred by the *DM* of each original *DMU* via an interactive trade-off analysis process [25].

Suppose a group *MPS* (*GMPS*) is assigned either by a single leading *DM* having the overall responsibility for an organisation or group, or by choosing a convex combination of the individual *MPS*s, or by simply picking up an existing efficient *DMU*. However, a *GMPS* generated using such methods may lie within, on or outside $E(f(\Omega_{j_0}))$ and thus may not be achievable by the observed DMU_0. As such, a *GMPS* needs to be mapped back to $E(f(\Omega_{j_0}))$ to achieve a locally efficient *MPS* (*LMPS*) for the observed DMU_0.

Suppose a *GMPS* is represented by *m GMPS* inputs x_i^{GMPS} ($i=1, \ldots, m$) and *s GMPS* outputs y_r^{GMPS} ($r=1, \ldots, s$). A *LMPS* for the observed DMU_0 could be generated as a solution closest to the *GMPS* in the dual objective space using the following shortest distance model, also derived from the *MOLP* formulation (9).

$$\begin{aligned} &\text{Min}\, d \\ &\text{s.t.} \quad w_r(f_r^{GMPS} - f_r(\lambda)) \le d \\ &\qquad -w_r(f_r^{GMPS} - f_r(\lambda)) \le d, \quad r=1,\ldots,s; \;\; \lambda \in \Omega_{j_0} \end{aligned} \tag{12}$$

where $f_r^{GMPS} = \underline{\lambda} y_r^{GMPS}$ with $\underline{\lambda} = \min_{1 \le i \le m} \{x_{ij_0}/x_i^{GMPS}\}$. Note that the above model shares the same dual decision and objective spaces as the *MOLP* model (9). Also, w_r is the relative weight of the objective $f_r(\lambda)$, which can be assigned for each *DMU* individually.

3. Analytical investigation into the *IET* analyses using the *DEA-IMRP* approach

The dual decision and objective spaces investigated in the previous section provide a basis for analysing the characteristics of the efficient frontier $E(f(\Omega_{j_0}))$. In this section, we first explore $E(f(\Omega_{j_0}))$ graphically, leading to the development of an analytical process to generate data envelope and efficiency frontier for facilitating the integrated efficiency and trade-off analyses. Then, we investigate a technical efficiency measure and a preferred efficiency measure to modify the *DEA* score and support trade-off analysis.

3.1. Efficiency analysis using the DEA-IMRP approach

As shown in the previous section, the dimensions of the dual decision and objective spaces are decided by the number of *DMU*s and the number of outputs, respectively. In this section, we examine a *DEA* problem having three or more *DMU*s to show how the *CCR* dual model (2) and the super-ideal model (3) can be solved graphically. This investigation will lead to the modification of the conventional *DEA* efficiency score and the definition of a preferred efficiency score.

An example *DEA* model is shown in Table 1. The solution of the example by model (2) or (3) shows that *DMU B* and *DMU C* are both efficient but *DMU A* is inefficient with a *DEA* score of 0.552. Furthermore, the values of the dual decision variables for *DMU A* are given by $\lambda = [\lambda_1 \;\; \lambda_2 \;\; \lambda_3]^T = [0, 0.133, 0.4]^T$, its *DEA* composite inputs by $x = [x_1 \; x_2]^T = [4.0, 1.7]^T$ and its *DEA* composite outputs by $y = [y_1 \; y_2]^T = [4.53, 9.06]^T$. In the following, we first show how to generate these results graphically using the *MOLP* model (9).

The *MOLP* problem (9) for *DMU A* in example 1 is given as follows:

$$\text{Max } f = \begin{bmatrix} f_1(\lambda) = 2.5\lambda_1 + 16\lambda_2 + 6\lambda_3 \\ f_2(\lambda) = 5\lambda_1 + 8\lambda_2 + 20\lambda_3 \end{bmatrix}$$

$$\text{s.t.} \quad \lambda = [\lambda_1, \lambda_2, \lambda_3]^T \in \Omega_A, \;\; \Omega_A = \left\{ \lambda \left| \begin{array}{l} 4\lambda_1 + 6\lambda_2 + 8\lambda_3 \le 4 \\ 2\lambda_1 + 4\lambda_2 + 3\lambda_3 \le 2 \\ \lambda_1 \ge 0, \lambda_2 \ge 0, \lambda_3 \ge 0 \end{array} \right. \right\} \tag{13}$$

The feasible dual decision space for problem (13), or Ω_A, is spanned by the three dual variables λ_1, λ_2 and λ_3 and is the shaded area enclosed by the five planes: $\overline{ABD}$, $\overline{ACD}$, $\overline{AOB}$, $\overline{AOC}$ and $\overline{BOCD}$, as shown in Fig. 1(a). It is clear from Fig. 1(a) that $\underline{\lambda}_1 = 1$, $\underline{\lambda}_2 = 0.5$ and $\underline{\lambda}_3 = 0.5$. So, point *A* stands for 100% of *DMU A*, point *B* for 50% of *DMU B*, and point *C* for 50% of *DMU C*. Note that the line $\overline{AD}$ is the intersection of the two planes: $4\lambda_1 + 6\lambda_2 + 8\lambda_3 = 4$ and $2\lambda_1 + 4\lambda_2 + 3\lambda_3 = 2$, and point *D* the intersection of these two planes plus the plane $\lambda_1 = 0$. Also note that point *D* is not an original *DMU* but a composite DMU composed of 0.286 of *DMU B* and 0.286 of *DMU C* because $\lambda_1 = 0$, $\lambda_2 = 0.286$ and $\lambda_3 = 0.286$ at point *D*. It will be shown later that point *D* is instrumental to generate the *DEA* score for *DMU A*.

The dual objective space for *DMU A* in Example 1 is spanned by f_1 and f_2, and is the shaded area as shown in Fig. 1(b). The coordinates of points *A*, *B*, *C* and *D* in the objective space can be found by calculating the values of f_1 and f_2 at these points in the decision space. For example, at point *D*, we have

$$f_1(D) = 2.5 \times 0 + 16 \times 0.286 + 6 \times 0.286 = 6.28,$$
$$f_2(D) = 5 \times 0 + 8 \times 0.286 + 20 \times 0.286 = 8.$$

Alternatively, the scaling factors for the *DMU*s corresponding to points *A*, *B* and *C* can be calculated. From the definition of Ω_A in Eqs. (13) and (8), we have $\underline{\lambda}_A = 1$, $\underline{\lambda}_B = 0.5$ and $\underline{\lambda}_C = 0.5$. Therefore, the coordinates (2.5, 5) at point *A* represents the true output levels of 2.5 and 5 for *DMU A*, whilst the coordinates (8, 4) at point *B* are 0.5 times the true output levels of 16 and 8 for *DMU B* and the coordinates (3, 10) at point *C* are 0.5 times the true output levels of 6 and 20 for *DMU C*, as shown in Fig. 1(b).

In Fig. 1(b), the data envelope for *DMU A* is $\overline{OC} \cup \overline{CD} \cup \overline{DB} \cup \overline{BO}$ and its efficient frontier is composed of the line segments $\overline{DB}$ and $\overline{CD}$, or $E(f(\Omega_A)) = \overline{BD} \cup \overline{CD}$. *DMU A* is not an efficient *DMU* or on the data envelope. It is important to note that point *D* is an extreme

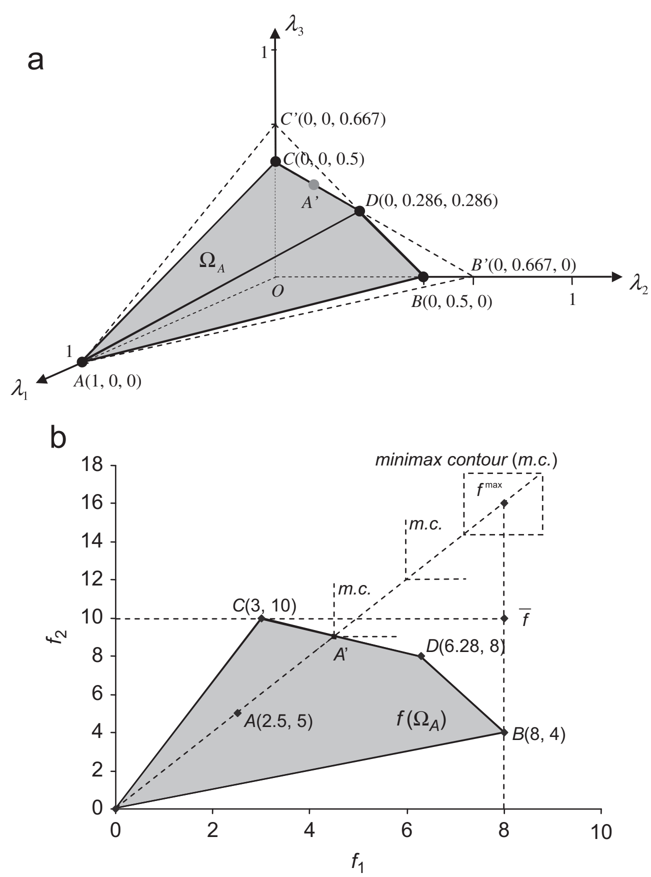

Fig. 1. (a) Dual decision space for *DMU A* in example 1 and (b) dual objective space for *DMU A* in example 1.

Table 1
Example 1 with three *DMU*s.

DMU	x_1	x_2	y_1	y_2	*DEA* score
A	4	2	2.5	5	0.552
B	6	4	16	8	1
C	8	3	6	20	1

point in the efficient frontier but it represents a composite *DMU*, or 0.286 of *DMU B* and 0.286 of *DMU C*. This observation can help to explain why a composite efficient solution could be a linear combination of more than two *DMU*s, as shown in the next section. The fact that a composite *DMU* can also become an extreme point in the efficient frontier, contrary to what have been mentioned in some *DEA* text books and papers, is a surprising yet inspiring discovery as shown by point *D* in Fig. 1(a) and (b). Also note that the line segment $\overline{BC}$ connecting the two efficient *DMU*s *B* and *C* is not part of the efficient frontier for *DMU A*.

Fig. 1(b) shows that *DMU A* is inefficient and its *DEA* score is graphically defined by $\overline{OA}/\overline{OA'}$, or $e_A=\overline{OA}/\overline{OA'}$ where A' is the intersection point of $\overline{CD}$ and the line emitting from the origin *O* through point *A*. The coordinates of point A' can be calculated as follows. The equation of the line passing the two points *C* and *D* is given by $2f_1+3.28f_2=38.8$ and the equation of the line passing the two points *O* and *A* is given by $5f_1+2.5f_2=0$. The coordinates of point A' is then given by $f_1=4.53$ and $f_2=9.06$, which are the same as the *DEA* composite outputs for *DMU A*. Then, we have

$$e_A=\frac{\overline{OA}}{\overline{OA'}}=\frac{\sqrt{2.5^2+5^2}}{\sqrt{4.53^2+9.06^2}}=0.552$$

which is the same as the *DEA* score generated by solving the *CCR* dual model (2) for *DMU A*, as shown in Table 1. Point A' in the decision space is shown in Fig. 1(a).

Note that points *B* and *C* are on the efficient frontier constructed for *DMU A*. Therefore, *DMU*s *B* and *C* corresponding to points *B* and *C* are efficient. This can also be confirmed by constructing the dual objective spaces for *DMU B* and *DMU C* in the same way as for *DMU A*. So, the efficiency scores for *DMU*s *B* and *C* are both one.

The composite input levels of point A' can be generated graphically as follows. Note that point A' is a convex combination of points *C* and *D*, or $A'=\alpha C+(1-\alpha)D$ where $0\le\alpha\le 1$ with $\alpha=\overline{A'D}/\overline{CD}=2.046/3.842=0.5325$. So, $A'=0.5325C+0.4675D$. Since point *C* stands for 0.5 of *DMU C* and point *D* for 0.286 of *DMU B* and 0.286 of *DMU C*, point A' therefore stands for 0.4 $(=0.5325\times 0.5+0.4675\times 0.286)$ of *DMU C* and 0.134 $(=0.4675\times 0.286)$ of *DMU B*, which means that point A' is a linear combination of *DMU B* and *DMU C* and the values of the dual variables for point A' are given by $\lambda_1=0$, $\lambda_2=0.134$ and $\lambda_3=0.4$. Therefore, the composite inputs for point A' can be generated by

$$x=[x_1x_2]^T=[0.134\times 6+0.4\times 8, 0.134\times 4+0.4\times 3]^T=[4.0,1.7]^T$$

We now demonstrate how the super-ideal point model (3) for *DMU A* is graphically solved to generate the same solution point A'. From Eq. (7), the ideal point takes the maximum feasible composite outputs, so $\overline{f}=[8,10]^T$. From Eq. (6), we have $F^{\max}=\max\{(8/2.5),(10/5)\}=3.2$

Therefore, the super-ideal point is given by $f^{\max}=3.2\times[2.5,5]^T=[8,16]^T$, proportional to point *A*. In other words, point $f^{\max}$ and point *A* are on the same line passing point *O*. Point $\overline{f}$ and point $f^{\max}$ are as shown in Fig. 1(b). The minimax contour (*m.c.* for short) at $f^{\max}$ is defined as a rectangle with $f^{\max}$ as its centre and the above line passing its southwest and northeast corners. Solving the super-ideal point model (3) is to expand the rectangle until it just touches the efficient frontier. As shown in Fig. 1(b), the southwest corner of the minimax contour touches the efficient frontier first with the point A' as the touching point.

In Fig. 1(b), it is fortunate that the line emitting from point *O* through point *A* intersects with the efficient frontier at point A'. In this case, the measurement of the *DEA* score e_A is justifiable and consistent with the suggested target represented by point A'. Now, let's consider a slightly modified example as shown in Table 2 where the output levels for *DMU A* are revised to 1 and 6 respectively.

Table 2
Example 2 with three *DMU*s.

DMU	x_1	x_2	y_1	y_2	*DEA* score
A	4	2	1	6	0.6
B	6	4	16	8	1
C	8	3	6	20	1

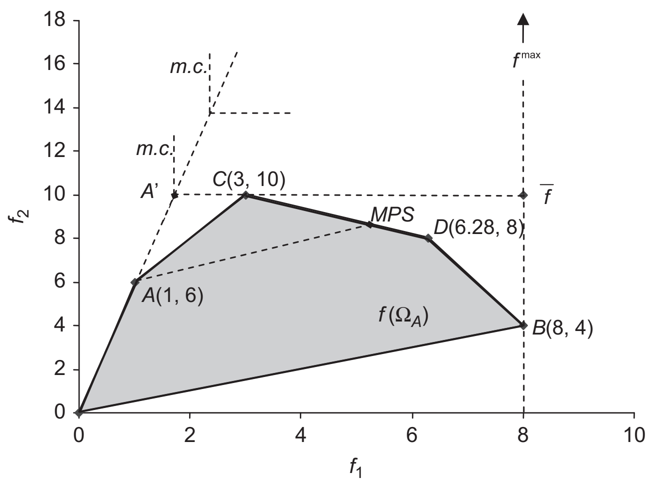

Fig. 2. Dual objective space for *DMU A* in example 2.

Solving the output-oriented *CCR* dual model (2) for each *DMU* in example 2 shows that *DMU*s *B* and *C* are still efficient whilst *DMU A* is inefficient with a *DEA* score of 0.6. The *DEA* composite unit for *DMU A* is 0.5 of *DMU C* with the composite output levels of 3 and 10, respectively. Now, we use the above graphical method to generate and analyse these results. In the same way as for example 1, the dual objective space for *DMU A* in example 2 can be generated, as shown in Fig. 2. It is clear from Fig. 2 that the data envelope for *DMU A* becomes $\overline{OA}\cup\overline{AC}\cup\overline{CD}\cup\overline{DB}\cup\overline{BO}$ and the efficient frontier for *DMU A* in example 2 is still $E(f(\Omega_A))=\overline{BD}\cup\overline{CD}$. So, point *A* becomes an extreme point on the data envelope but it is still inefficient.

From Eq. (7), the new ideal point takes the maximum feasible composite outputs, so $\overline{f}=[8,10]^T$, which is the same as in Fig. 1(b). From Eq. (6), we have $F^{\max}=\max\{(8/1),(10/6)\}=8$. So, the super-ideal point is given by $f^{\max}=8\times[1,6]^T=[8,48]^T$, well above the ideal point $\overline{f}$ and on the line emitting from point *O* through point *A*, as indicated in Fig. 2.

Note that the super-ideal point model (3) is equivalent to the output-oriented *CCR* dual model (2). Solving model (3) for *DMU A* in example 2 graphically is to expand the minimax contour (*m.c.*) from $f^{\max}$ until it touches the feasible dual objective space. In Fig. 2, the lower side of the minimax contour first touches the efficient frontier at point *C* with the coordinates of its southwest corner of the minimax contour given by $A'=[1.67,10]^T$. The *DEA* score for *DMU A* generated by solving the output-oriented *CCR* dual model is thus given by

$$e_A=\frac{\overline{OA}}{\overline{OA'}}=\frac{\overline{OA}}{\overline{OA}+\overline{AA'}}=\frac{\sqrt{1^2+6^2}}{\sqrt{1.67^2+10^2}}=0.6$$

3.2. *New indices for measurement of technical and preferred efficiency scores*

The analysis conducted for problem 2 in Section 3.1 shows that the conventional *DEA* used a composite *DMU* corresponding to point A' as a reference to measure the efficiency score for *DMU A*. Two problems can be raised from this analysis. The first problem is that point A' is infeasible, which means that the measurement does not make sense. The second problem is that the *CCR* dual model for *DMU A* actually suggested that the benchmark unit for *DMU A* should be point *C* (or 0.5 of *DMU C*) rather than point A'. So, the efficiency measurement is not consistent with the target setting. In other words, such a *DEA* score does not measure the amount by which *DMU A* should improve itself to achieve the suggested target.

To maintain the consistency between the suggested target and the defined efficiency score, we define the following technical efficiency score for *DMU A*, or *TES* for short.

$$TES_A = \frac{\overline{OA}}{\overline{OA}+\overline{AC}} = \frac{\sqrt{1^2+6^2}}{\sqrt{1^2+6^2}+\sqrt{(3-1)^2+(10-6)^2}} = 0.576$$

It is easy to show that in example 1 the *DEA* score e_A and the technical efficiency score TES_A for *DMU A* are the same. In example 2, we have $TES_A < e_A$.

In general, a technical efficiency score is defined as follows:

Definition 1. Suppose a *DEA* problem has *s* outputs. In the dual objective space, suppose the observed *DMU A* is represented by $A=[y_{A1},\ldots,y_{As}]^T$ and its suggested target by $T=[y_{T1},\ldots,y_{Ts}]^T$. Then the technical efficiency score for *DMU A* is defined by

$$TES_A = \frac{\overline{OA}}{\overline{OA}+\overline{AT}} = \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \left(\sqrt{\sum_{r=1}^{s} y_{Ar}^2} + \sqrt{\sum_{r=1}^{s} (y_{Tr}-y_{Ar})^2} \right) \tag{14}$$

Regarding the relationship between the *DEA* score and the technical efficiency score, we have the following results:

Theorem 1. *If a suggested target for a DMU is an efficient solution and lies on the line emitting from the origin through the DMU in its dual objective space, then the DEA score of the DMU must be equal to its technical efficiency score.*

Proof. Suppose $T=[y_{T1}\cdots y_{Ts}]^T$ is the suggested target for *DMU A* with $A=[y_{A1},\ldots,y_{As}]^T$. Then, the condition of Theorem 1 means that there must be $[y_{A1},\ldots,y_{As}]^T = M[y_{T1},\ldots,y_{Ts}]^T$ with *M* being a non-zero real number and $M \le 1$, or $y_{Ar}=M\times y_{Tr}$ for all $r=1,\cdots,s$. Therefore, we have

$$TES_A = \frac{\overline{OA}}{\overline{OA}+\overline{AT}} = \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \left(\sqrt{\sum_{r=1}^{s} y_{Ar}^2} + \sqrt{\sum_{r=1}^{s} (y_{Tr}-y_{Ar})^2} \right)$$

$$= \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \left(\sqrt{\sum_{r=1}^{s} (My_{Tr})^2} + \sqrt{\sum_{r=1}^{s} (y_{Tr}-My_{Tr})^2} \right)$$

$$= \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \left(M\sqrt{\sum_{r=1}^{s} (y_{Tr})^2} + (1-M)\sqrt{\sum_{r=1}^{s} (y_{Tr})^2} \right)$$

$$= \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \sqrt{\sum_{r=1}^{s} (y_{Tr})^2} = \frac{\overline{OA}}{\overline{OT}} = e_A \quad \square$$

Corollary 1. *Suppose $T=[y_{T1},\ldots,y_{Ts}]^T$ is the suggested target for DMU A with $A=[y_{A1},\ldots,y_{As}]^T$. The DEA score of DMU A must be equal to its technical efficiency score if the following conditions are met.*

$$\frac{y_{Ar}}{y_{Ar_0}} = \frac{y_{Tr}}{y_{Tr_0}} \quad \forall r=1,\ldots,s \text{ and for any given } r_0=1,\ldots,s \tag{15}$$

Proof. If condition (15) is met, we will have

$$y_{Ar} = \frac{y_{Ar_0}}{y_{Tr_0}} y_{Tr} \quad \forall r=1,\ldots,s$$

Let $M=(y_{Ar_0}/y_{Tr_0})$. We then have $[y_{A1},\ldots,y_{As}]^T = M[y_{A1},\ldots,y_{As}]^T$ $\square$

Theorem 1 and Corollary 1 can be used to check whether the generated *DEA* score and suggested composite outputs are consistent. If condition (15) is not met, the inconsistence will exist and it is suggested that the technical efficiency score calculated using Eq. (14) be used to replace the *DEA* score. Intuitively, the technical efficiency score of a *DMU* can be interpreted as a degree of minimum efforts that the *DMU* needs to make to expand its outputs for achieving feasible efficiency. The closer the technical efficiency score is to one, the smaller the efforts needed to achieve the feasible efficiency. Theoretically, a technical efficiency score measures the relative distance from a *DMU* to its nearest feasible efficient benchmark *DMU* as identified in the objective space through the minimax or Chebychev optimisation.

If the suggested target is the most preferred solution maximising the implicit utility function of the decision maker, we have the following definition:

Definition 2. Suppose a *DEA* problem has *s* outputs and the observed *DMU A* is denoted by $A=[y_{A1},\ldots,y_{As}]^T$ in the dual objective space. Suppose its suggested target is the most preferred solution maximising the implicit utility function of the decision maker, denoted by $P=[y_{P1},\ldots,y_{Ts}]^T$. Then the preferred efficiency score for *DMU A* is defined by

$$PES_A = \frac{\overline{OA}}{\overline{OA}+\overline{AP}} = \sqrt{\sum_{r=1}^{s} y_{Ar}^2} \Bigg/ \left(\sqrt{\sum_{r=1}^{s} y_{Ar}^2} + \sqrt{\sum_{r=1}^{s} (y_{\mathrm{Pr}}-y_{Ar})^2} \right) \tag{16}$$

In Fig. 2, for example, if the coordinates of the most preferred solution for *DMU A*, denoted by point *MPS*, are given by $MPS=[5.6, 8.4]^T$, then

$$PES_A = \frac{\overline{OA}}{\overline{OA}+\overline{A,MPS}} = \frac{\sqrt{1^2+6^2}}{\sqrt{1^2+6^2}+\sqrt{(5.6-1)^2+(8.4-6)^2}} = 0.569$$

Note that the preferred efficiency score for the observed *DMU* is always less than one except that it is an efficient *DMU* and its most preferred solution is itself.

Similar to technical efficiency score, intuitively the preferred efficiency score of a *DMU* can be interpreted as a degree of minimum efforts that the *DMU* needs to make to balance its outputs for achieving the most preferred solution (or target *DMU*) along its efficient frontier. The closer the preferred efficiency score is to one, the smaller the efforts needed to achieve the most preferred *DMU*. Theoretically, a preferred efficiency score measures the relative distance from a *DMU* to its most preferred *DMU* as identified along its efficient frontier through a dedicated trade-off analysis with the *DM*'s preferences taken into account.

3.3. Generation of data envelope and efficient frontier for integrated efficiency and trade-off analysis

In the previous sections, we restricted our discussions to up to three *DMU*s in order to show both decision and objective spaces. In this section, we relax the restriction to any number of *DMU*s and any number of inputs with up to three outputs in order to analytically generate and graphically show data envelope and efficient frontier for facilitating integrated efficiency and trade-off analyses. If there are more than three outputs, interactive *MOLP* methods can be used to search efficient frontier, such as the gradient projection method based on the ideal-point model [25,26,28].

In this section, we investigate a procedure for generating data envelope and efficient frontier. Due to limited space and without loss of generality, we consider two outputs only. However, the procedure can be extended to cases with three and more outputs. We use example 2 to illustrate the procedure that constitutes the following steps:

***Step* 1:** For an output-oriented *DEA* problem, choose an observed *DMU* and define its dual objective functions. In example 2, we choose *DMU A*.

***Step* 2:** Calculate the scaling factors for all original *DMU*s using Eq. (8). Then, calculate and draw the coordinates of all scaled original *DMU*s in the dual objective space. For example 2, the scaling factor for *DMU*s *A, B* and *C* are given by $\underline{\lambda}_1=1$, $\underline{\lambda}_2=0.5$ and $\underline{\lambda}_3=0.5$, respectively. So the coordinates for the three *DMU*s are given by $A=[1,\ 6]^T$, $B=[8,\ 4]^T$ and $C=[3,\ 10]^T$, denoted by points *A, B* and *C* in Fig. 3.

***Step* 3:** Draw the initial data envelope of the identified points. Note that the origin is another point because it is feasible that all decision variables take zero. For example 2, at this stage the data envelope is composed of the lines $\overline{OA}$, $\overline{AC}$, $\overline{CB}$ and $\overline{BO}$, as shown in Fig. 3.

***Step* 4:** Choose two adjacent extreme points on the current data envelope in any order and formulate a searching problem to check if there is any other feasible solution beyond the line passing the two points. The search should be in the direction of the outwards normal vector of the line. Suppose $I=[y_{11},\ y_{12}]^T$ and $J=[y_{21},\ y_{22}]^T$ are two adjacent points on the current data envelope with $y_{12}\geq y_{22}$. Then the searching function along the outwards normal vector direction is given by

$$g(f_1,f_2)=\begin{cases}-(y_{22}-y_{12})f_1(\lambda)+(y_{21}-y_{11})f_2(\lambda) \text{ if } \overrightarrow{N}^{IJ}\bullet\overrightarrow{f}_1\geq 0\\ -(y_{12}-y_{22})f_1(\lambda)+(y_{11}-y_{21})f_2(\lambda) \text{ if } \overrightarrow{N}^{IJ}\bullet\overrightarrow{f}_1<0\end{cases}\qquad(17)$$

where $\overrightarrow{N}^{IJ}$ is the outwards normal vector of the line $\overline{IJ}$ and $\overrightarrow{f}_1$ is the horizontal (f_1) axis. So, $\overrightarrow{N}^{IJ}\bullet\overrightarrow{f}_1\geq 0$ in (17) means that the angle between $\overrightarrow{N}^{IJ}$ and $\overrightarrow{f}_1$ is less than or equal to 90°whilst $\overrightarrow{N}^{IJ}\bullet\overrightarrow{f}_1<0$ means that the angle is larger than 90°. The search problem is defined by

$$\begin{aligned}&\text{Max}\quad g(f_1(\lambda),f_2(\lambda))\\&\text{s.t.}\quad \lambda\in\Omega_{j_0}\end{aligned}\qquad(18)$$

where Ω_{j_0} is the decision space constructed for the observed DMU_0. If the optimal solution is $f_1^*=y_{11}$ and $f_2^*=y_{12}$ or $f_1^*=y_{21}$ and $f_2^*=y_{22}$ with $g^*=|y_{21}y_{12}-y_{11}y_{22}|$, there will be no feasible solution outside $\overline{IJ}$, which is then recorded as the expanded line segment on the data envelope. Otherwise, a new feasible solution is found and the data envelope is adjusted.

***Step* 5:** If there is any unexpanded line segment on the data envelope, go to **Step 4**.

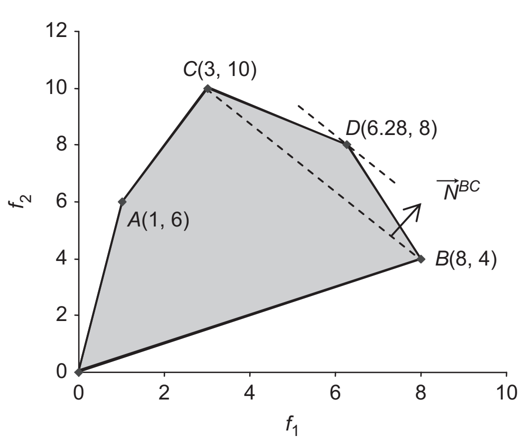

Fig. 3. Generating data envelope for *DMU A* in example 2.

In Fig. 3, in the initial data envelope generated in Step 3, suppose $\overline{CB}$ is chosen as the line segment for possible expansion of the initial data envelope. Note that the outwards normal vector of $\overline{CB}$ is $\overrightarrow{N}^{BC}=[6,5]^T$. Then we have the following search problem.

$$\begin{aligned}&\text{Max}\quad g(f_1,f_2)=6f_1(\lambda)+5f_2(\lambda)\\&\text{s.t.}\quad \lambda=[\lambda_1,\lambda_2,\lambda_3]^T\in\Omega_A\end{aligned}\qquad(19)$$

where Ω_A is defined in (13), $f_1(\lambda)=2.5\lambda_1+16\lambda_2+6\lambda_3$ and $f_2(\lambda)=5\lambda_1+8\lambda_2+20\lambda_3$. The optimal solution of the problem is given by $f_1^*=6.28$ and $f_2^*=8$ with $g^*=77.71>68=(8\times 10-3\times 4)=|y_{21}y_{12}-y_{11}y_{22}|$, leading to point *D* in Fig. 3. The updated data envelope is given by $\overline{OA}\cup\overline{AC}\cup\overline{CD}\cup\overline{DB}\cup\overline{BO}$. In the same way, it is easy to show that if any of the five line segments is chosen for expansion no more new feasible solution can be found. So, this updated data envelope is the true data envelope for *DMU A* in example 2 and its efficient frontier is therefore given by $\overline{CD}\cup\overline{DB}$.

4. A case study for UK retail bank performance management

In this section, we apply the results investigated in the previous sections to conduct a case study. Its purpose is not only to show how the analytical methods and procedures investigated in this paper can be applied to deal with problems of practical size but also to reveal the features of the data envelopes and efficient frontiers of practical *DEA* problems to gain valuable insight into efficiency and trade-off analyses.

4.1. DEA for retail banks and calculation of technical efficiency score

The case study is to examine the efficiencies of major UK retail banks using the data collected [32]. The reference set consists of seven *DMU*s (retail banks), and four inputs and two outputs are considered. The data set is as shown in Table 3.

The output-oriented *CCR* dual model (2) and the super-ideal point model are run to find the *DEA* efficiency scores [25]. As shown

Table 3
Original data set for retail banks.

DMU	Bank	Inputs				Outputs	
		No. of branches ('000)	No. of ATMs ('000)	No. of staff ('0,000)	Asset size (£bn)	Customer satisfaction*	Total revenue (£m)
1	Abbey Nat.	0.77	2.18	2.35	2.96	6.79	10.57
2	Barclays	1.95	3.19	8.43	3.53	2.55	13.35
3	Halifax	0.80	2.10	3.21	2.41	9.17	8.14
4	HSBC	1.75	4.00	13.30	4.85	5.82	23.67
5	Lloyds TSB	2.50	4.30	9.27	2.40	6.57	14.01
6	NatWest	1.73	3.30	7.70	3.09	4.86	12.04
7	RBS	0.65	1.73	2.67	1.34	7.28	7.36

* Customer satisfaction values are converted scores based on the average expected utilities of the respondents of a local survey conducted in Manchester.

Table 4
Efficiency analysis of retail banks.

DMU	*DEA* dual model		Minimax model								$\theta = F^{\max} - \theta_{j_0}$
			Composite inputs and outputs								
	DEA score	θ_{j0}	$F^{\max}$	θ	x_1	x_2	x_3	x_4	y_1	y_2	
1	1.000	1.000	1.000	0.000	0.77	2.18	2.35	2.96	6.79	10.57	0.000
2	0.778	1.285	5.449	4.164	1.32	3.19	8.43	3.53	7.45	17.15	4.164
3	1.000	1.000	1.238	0.238	0.80	2.10	3.21	2.41	9.17	8.14	0.238
4	1.000	1.000	3.001	2.001	1.75	4.00	13.30	4.85	5.82	23.67	2.001
5	1.000	1.000	1.989	0.989	2.50	4.30	9.27	2.40	6.57	14.01	0.989
6	0.747	1.338	2.905	1.567	1.32	3.30	7.30	3.09	10.48	16.11	1.567
7	1.000	1.000	1.000	0.000	0.65	1.73	2.67	1.34	7.28	7.36	0.000

Table 5
Typical points in dual objective space for NatWest.

Composite DMU	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	f_1	f_2
1	1.05							7.10	11.06
2		0.88						2.23	11.69
3			1.29					11.80	10.47
4				0.58				3.37	13.71
5					0.69			4.55	9.7
6						1.00		4.86	12.04
7							1.91	13.89	14.05
8				0.31			1.19	10.48	16.11
9	0.14			0.56				4.16	14.58
10				0.42			0.79	8.20	15.77
11				0.34	0.06		0.96	9.39	16.06
12			0.7				1.07	14.12	13.49
13		0.62		0.18				2.66	12.67

Table 6
Typical points in dual objective space for Barclays.

Composite DMU	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	f_1	f_2
1	1.19							8.11	12.63
2		1.00						2.55	13.35
3			1.47					13.47	11.96
4				0.63				3.69	15.01
5					0.74			4.87	10.38
6						0.97		4.69	11.63
7							1.84	13.41	13.57
8	0.14			0.51			0.49	7.45	17.15
9				0.49			0.7	7.98	16.84

in Table 4, only Barclays and NatWest are found to be inefficient within the reference set of the seven banks. NatWest has a *DEA* efficiency score of 0.747 and its *DEA* composite unit on the efficient frontier is represented as a linear combination of 0.310 of HSBC and 1.192 of RBS. Barclays has a *DEA* efficiency score of 0.778 and its *DEA* composite unit is represented as a linear combination of 0.135 of Abbey National, 0.512 of HSBC and 0.489 of RBS.

First, it can be shown that neither NatWest nor Barclays meets condition (15). In fact, for NatWest in Tables 5 and 6, we have $[y_{61}, y_{62}]^T = [4.86, 12.04]^T$ and $[y_{T1}, y_{T2}]^T = [10.48, 16.11]^T$, so $(y_{62}/y_{61}) = (12.04/4.86) = 2.477 \neq (y_{T2}/y_{T1}) = (16.11/10.48) = 1.537$. In Table 4, the *DEA* efficiency score of NatWest is thus not calculated consistently and needs to be revised to its following technical efficiency score.

$$TES_6 = \frac{\sqrt{4.86^2 + 12.04^2}}{\sqrt{4.86^2 + 12.04^2} + \sqrt{(10.48 - 4.86)^2 + (16.11 - 12.04)^2}} = 0.652 \tag{20}$$

For Barclays, we have $[y_{21}, y_{22}]^T = [2.55, 13.35]^T$ and $[y_{T1}, y_{T2}]^T = [7.45, 17.15]^T$, so $(y_{22}/y_{21}) = (13.35/2.55) = 5.235 \neq (y_{T2}/y_{T1}) = (17.15/7.45) = 2.3$. In Table 4, the efficiency score of Barclays is not calculated consistently either and needs to be revised to its following technical efficiency score.

$$TES_2 = \frac{\sqrt{2.55^2 + 13.35^2}}{\sqrt{2.55^2 + 13.35^2} + \sqrt{(7.45 - 2.55)^2 + (17.15 - 13.35)^2}} = 0.687 \tag{21}$$

4.2. Data envelope and efficient frontier for inefficient NatWest

Now we apply the graphical and analytical methods and procedures investigated in the previous sections to explore the characteristics of the data envelope and efficient frontier for each of the *DMU*s and graphically explain the results shown in Table 4. The first attention is paid to the inefficient NatWest. The *MOLP* problem (9) for NatWest is constructed by

$$\begin{aligned} &\text{Max} \quad f = \begin{bmatrix} f_1(\lambda) = 6.79\lambda_1 + 2.55\lambda_2 + 9.17\lambda_3 + 5.82\lambda_4 + 6.57\lambda_5 + 4.86\lambda_6 + 7.28\lambda_7 \\ f_2(\lambda) = 10.57\lambda_1 + 13.35\lambda_2 + 8.14\lambda_3 + 23.67\lambda_4 + 14.01\lambda_5 + 12.04\lambda_6 + 7.36\lambda_7 \end{bmatrix} \\ &\text{s.t.} \quad \lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7]^T \in \Omega_6 \\ &\Omega_6 = \left\{ \lambda \left| \begin{array}{l} 0.77\lambda_1 + 1.95\lambda_2 + 0.8\lambda_3 + 1.75\lambda_4 + 2.5\lambda_5 + 1.73\lambda_6 + 0.65\lambda_7 \le 1.73 \\ 2.18\lambda_1 + 3.19\lambda_2 + 2.1\lambda_3 + 4\lambda_4 + 4.3\lambda_5 + 3.3\lambda_6 + 1.73\lambda_7 \le 3.3 \\ 2.35\lambda_1 + 8.43\lambda_2 + 3.21\lambda_3 + 13.3\lambda_4 + 9.27\lambda_5 + 7.7\lambda_6 + 2.67\lambda_7 \le 7.7 \\ 2.96\lambda_1 + 3.53\lambda_2 + 2.41\lambda_3 + 4.85\lambda_4 + 2.4\lambda_5 + 3.09\lambda_6 + 1.34\lambda_7 \le 3.09 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \ge 0 \end{array} \right. \right\} \end{aligned} \tag{22}$$

Obviously, it is not possible to graph the decision space Ω_6 for NatWest, as there are seven decision variables. It is not straightforward to draw the feasible objective space either. However, the procedure developed in Section 3.3 can be used to generate the data envelope and efficient frontier for NatWest. First, using Eqs. (8) and (10) we can calculate the scaling factors for the seven individual *DMU*s, as shown in the first seven data rows in Table 5, and draw the first seven points for the original *DMU*s in the dual objective space constructed for NatWest, leading to the initial data envelope of $\overline{0,2} \cup \overline{2,4} \cup \overline{4,7} \cup \overline{7,3} \cup \overline{3,0}$, as shown in Fig. 4.

Now we show how to expand the initial data envelope to find the true data envelope for NatWest. We start the process by arbitrarily choosing $\overline{4,7}$ to construct problem (18) as follows:

$$\begin{aligned} &\text{Max} \quad g(f_1(\lambda), f_2(\lambda)) = -0.34 f_1(\lambda) + 10.52 f_2(\lambda) \\ &\text{s.t.} \quad \lambda \in \Omega_6 \end{aligned} \tag{23}$$

where $f_1(\lambda), f_2(\lambda)$ and Ω_6 are given in problem (22). Solving problem (23) leads to composite *DMU* 8 shown in Table 5 and Fig. 4, and $\overline{4,7}$ is thus expanded to $\overline{4,8} \cup \overline{8,7}$ in the updated data envelop.

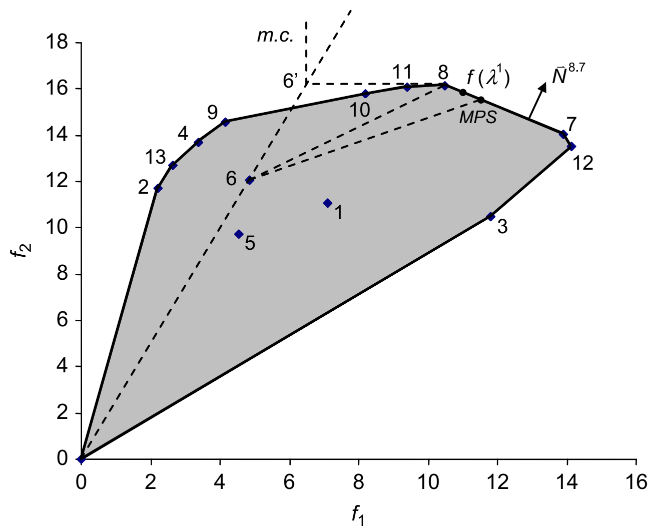

Fig. 4. Data envelope and efficient frontier for NatWest.

In a similar way, we can expand the data envelope in the following sequence from $\overline{4,8}$ to $\overline{4,9}\cup\overline{9,8}$; from $\overline{9,8}$ to $\overline{9,10}\cup\overline{10,8}$; from $\overline{10,8}$ to $\overline{10,11}\cup\overline{11,8}$; from $\overline{7,3}$ to $\overline{7,12}\cup\overline{12,3}$; and from $\overline{2,4}$ to $\overline{2,13}\cup\overline{13,4}$. The values of $f_1(\lambda)$ and $f_2(\lambda)$ at points 9, 10, 11, 12 and 13 are given in Table 5. Then, the newly updated data envelope is given by $\overline{0,2}\cup\overline{2,13}\cup\overline{13,4}\cup\overline{4,9}\cup\overline{9,10}\cup\overline{10,11}\cup\overline{11,8}\cup\overline{8,7}\cup\overline{7,12}\cup\overline{12,3}\cup\overline{3,0}$. From any of these 11 line segments, it is impossible to further expand the data envelope. So this is the true data envelope for NatWest and its feasible objective space is enclosed by these 11 line segments, as shown in the shaded area of Fig. 4. It is clear from Fig. 4 that the efficient frontier for NatWest is given by $\overline{8,7}\cup\overline{7,12}$. Hence, only the scaled original *DMU* 7 is on the efficient frontier constructed for NatWest and all the other efficient solutions for NatWest are composite *DMU*s.

Solving the super-ideal point model (3) or the *CCR* dual model (2) for NatWest is to expand the minimax contour down from the super-ideal point (not shown in Fig. 4) along the line emitting from the origin through point 6 to touch the data envelop at point 8, which is exactly the *DEA* composite unit suggested for NatWest as shown in Tables 5 and 6. Note that the southwest corner of the optimal minimax contour is point 6′ which is infeasible with $f_1(\lambda^{6'})=6.5$ and $f_2(\lambda^{6'})=16.11$. However, the *DEA* efficiency score of 0.747 for NatWest shown in Table 4 is given by $e_6=(\overline{O,6}/\overline{O,6'})=(12.98/17.37)=0.747$. Because point 6′ is infeasible, such efficiency measurement does not make any sense to the managers of NatWest and this efficiency score is not measured consistently, because point 6′ is not feasible and point 8 is actually suggested as the *DEA* composite unit for NatWest. Thus, this *DEA* score should not be used to measure NatWest's efficiency. Instead, the technical efficiency score of $TES_6=(\overline{O,6}/(\overline{O,6}+\overline{6,8}))=0.652$ calculated using Eq. (20) is suggested as the consistent efficiency score for NatWest.

The interactive trade-off analysis method investigated by Yang et al. [25] is a generic process for searching the efficient frontier and can be implemented in objective spaces of any order. In this case study with four inputs, seven *DMU*s and only two outputs, we can graphically illustrate the interactive process for NatWest conducted by Yang et al. [25]. As shown in Fig. 4, the process was actually started from point 8, the *DEA* suggested solution, moved down to point $f(\lambda^1)=[11.00,15.79]^T$ and terminated at point $MPS=[11.50,15.49]^T$, all on the same facet $\overline{8,7}$ of the efficient frontier with the normal vector $\overrightarrow{N}^{8,7}=[2.06,3.41]^T$. If the *MPS* is used as the benchmark for NatWest, its preferential efficiency score is given by $PES_6=(\overline{O,6}/(\overline{O,6}+\overline{6,MPS}))=12.98/(12.98+7.48)=0.634$.

The above procedure can be used to support the management of NatWest for improving performance in line with its long term policy and strategy. The graphical representation of the efficient frontier clearly shows the range of its feasible improvement targets. For example, NatWest could choose to move to point 8 to become efficient by maximising its total revenue without increasing any input, just as suggested by the conventional efficiency analysis using the CCR model. Such a strategy would require NetWest to increase its total revenue from the current 12.04 units to 16.11 units, an increase of 34%, and its customer satisfaction from the current 4.86 units to 10.48 units, an increase of 116%. Alternatively, they can choose to move to point 12 to become efficient by maximising customer satisfaction without increasing any input, leading to the total revenue of 13.49 units, an increase of 12%, and the customer satisfaction of 14.12 units, an increase of 191%. If neither of these two extreme strategies is preferred by the NatWest management, they can consider a compromised benchmark along the efficient frontier between point 8 and point 12. A visual aid such as Fig. 4 can be used to support the trade-off analysis for searching the most preferred benchmark with clarity and confidence. The *MPS* ($[11.50,15.49]^T$) shown in Fig. 4 represents a desired target for NatWest to benchmark against, at which the total revenue and customer satisfaction are increased by 29% and 137%, respectively, representing a balanced or compromised business strategy to improve its customer satisfaction substantially as its top priority, whilst also improving its business results. If more outputs need to be dealt with, similar trade-off analysis can be conducted using the analytical procedure as explored by Yang et al. [25], although it is still helpful to explore graphically the characteristics of a partial efficient frontier between any pair of outputs for supporting an informative trade-off analysis. A dedicated decision support system is needed to implement such an integrated efficiency and trade-off analysis for large scale problems.

4.3. Data envelope and efficient frontier for inefficient Barclays

For the other inefficient *DMU* Barclays, we can construct its *MOLP* problem (9) as follows:

$$\begin{aligned}
&\max \quad f=[f_1(\lambda),f_2(\lambda)]\\
&\text{s.t.} \quad \lambda=[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7]^T\in\Omega_2\\
&\Omega_2=\left\{\lambda\left|\begin{array}{l}0.77\lambda_1+1.95\lambda_2+0.8\lambda_3+1.75\lambda_4+2.5\lambda_5+1.73\lambda_6+0.65\lambda_7\le 1.95\\ 2.18\lambda_1+3.19\lambda_2+2.1\lambda_3+4\lambda_4+4.3\lambda_5+3.3\lambda_6+1.73\lambda_7\le 3.19\\ 2.35\lambda_1+8.43\lambda_2+3.21\lambda_3+13.3\lambda_4+9.27\lambda_5+7.7\lambda_6+2.67\lambda_7\le 8.43\\ 2.96\lambda_1+3.53\lambda_2+2.41\lambda_3+4.85\lambda_4+2.4\lambda_5+3.09\lambda_6+1.34\lambda_7\le 3.53\\ \lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7\ge 0\end{array}\right.\right\}
\end{aligned} \tag{24}$$

where $f_1(\lambda)$ and $f_2(\lambda)$ are the same as defined in problem (22) for NatWest and Ω_2 is the decision space for Barclays. Note that the only deference between Ω_2 and Ω_6 is that in Ω_2 the constants on the right hand sides of the constraints are the input values for Barclays rather than for NatWest as in Ω_6.

Using Eqs. (8) and (10), we can calculate the scaling factors for the seven individual *DMU*s in the dual objective space for Barclays, as shown in the first seven data rows in Table 6. Using the procedure described in Section 3.3, we can identify two composite *DMU*s 8 and 9 on the data envelope, as shown in the last two data rows in Table 6. It is important to note that point 8 is composed of three efficient original *DMU*s, or 0.135 of Abbey National (*DMU* 1), 0.512 of HSBC (*DMU* 4) and 0.489 of RBS (*DMU* 7).

The objective space for Barclays is shown in Fig. 5. The data envelope for Barclays is given as $\overline{0,2}\cup\overline{2,4}\cup\overline{4,8}\cup\overline{8,9}\cup\overline{9,7}\cup\overline{7,3}\cup\overline{3,0}$ and its efficient frontier as $\overline{8,9}\cup\overline{9,7}\cup\overline{7,3}$. Note that point 2

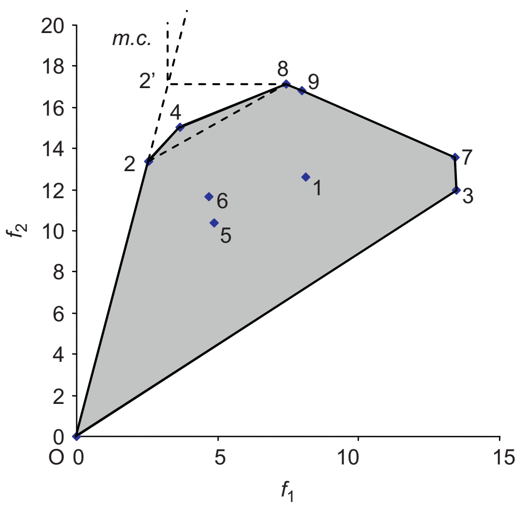

Fig. 5. Objective space for Barclays.

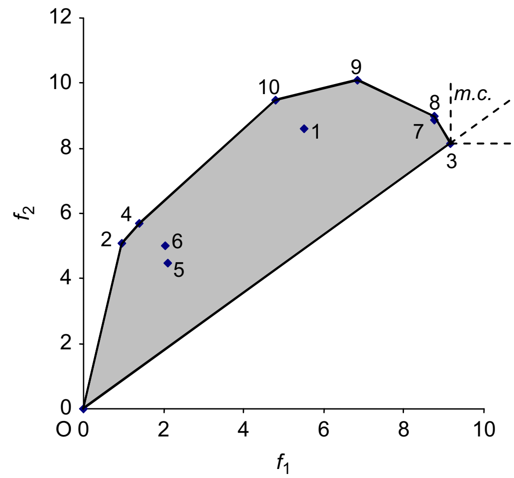

Fig. 6. Objective space for Halifax.

(Barclays) is an extreme point on its data envelope but it is inefficient. Solving the super-ideal point model (3) for Barclays leads to point 8, which is the same result as shown in Table 4. The standard *DEA* score is measured with reference to point $2'=[3.28, 17.15]^T$, or $e_2=(\overline{O,2}/\overline{O,2'})=13.59/17.48=0.778$, also as shown in Table 4. As investigated in the previous sub-section, this *DEA* efficiency score is inconsistently calculated because point $2'$ is not a feasible solution. As such, the technical efficiency score of 0.687, calculated using Eq. (21), is suggested as the appropriate efficiency score for Barclays.

It is vital to note that both points 8 and 9 represent composite *DMU*s and are the extreme efficient solutions of the efficient frontier constructed for Barclays. Indeed, point 8 is composed of the three original *DMU*s: Abbey National, HSBC and RBS. Similar to the explanations for point D in Fig. 1(a) and (b), the reason for point 8 becoming an extreme point of the efficient frontier for Barclays is that the second, third and forth input constrains in Ω_2 are all binding at point 8. Although it is not possible to draw this point in the decision space as there are 7 decision variables (λ_1 to λ_7), point 8 is clearly shown in the objective space for Barclays in Fig. 5 using the method developed in this paper. Similarly, at point 9 the second and third constrains in Ω_2 are both binding, leading to point 9 being an extreme point as well.

Fig. 5 graphically explains that the *DEA* efficient solution for Barclays is point 8 and thus can be the linear combination of the three original *DMU*s, though neither Abbey National nor HSBC is on the efficient frontier for Barclays. Also note that Barclays could benchmark against any convex combinations of the composite *DMU*s 8 and 9, or linear combinations of Abbey National, HSBC and RBS. Furthermore, Barclays can benchmark against any convex combinations of the composite *DMU* 9 and the scaled *DMU* 7, or any convex combinations of the scaled *DMU* 7 and the scaled *DMU* 3.

From the above discussions, it can be noted that if an efficient *DMU* is used to construct the *DEA* composite unit of another inefficient *DMU* it does not necessarily mean that the efficient *DMU* would be on the efficient frontier constructed for the inefficient *DMU*. Also, it is not unusual that extreme efficient solutions are composite *DMU*s. In fact, if all extreme efficient solutions of an efficient frontier were original *DMU*s but not any composite *DMU*, it would not be possible to generate a *DEA* composite unit as a linear combination of more than two *DMU*s in the output-oriented *CCR* model if there were only two outputs. This is because any efficient solution of the efficient frontier is a convex combination of two adjacent extreme efficient solutions in the dual objective space.

4.4. Characteristics of other DMUs' objective spaces

As shown in Table 4, Halifax, HSBC and Lloyds are all efficient *DMU*s and their efficient frontiers should include a range of efficient solutions, as indicated by $F^{\max}>1$ for these three *DMU*s. The dual objective space for Halifax is shown in Fig. 6. The first seven points stand for the seven original *DMU*s and are generated using the following scaling factors: 0.81, 0.38, 1.00, 0.24, 0.32, 0.42 and 1.2, respectively. Point 8 stands for 0.033 of *DMU* 1 and 1.173 of *DMU* 7 with $f^8_{Halifax}=[8.76,8.99]^T$; point 9 for 0.518 of *DMU* 1, 0.069 of *DMU* 4 and 0.403 of *DMU* 7 with $f^9_{Halifax}=[6.85,10.08]^T$; point 10 for 0.588 of *DMU* 1 and 0.137 of *DMU* 4 with $f^{10}_{Halifax}=[4.79,9.47]^T$. So, the data envelope for Halifax is given by $\overline{0,2}\cup\overline{2,4}\cup\overline{4,10}\cup\overline{10,9}\cup\overline{9,8}\cup\overline{8,3}\cup\overline{3,0}$ and its efficient frontier by $\overline{9,8}\cup\overline{8,3}$. It is interesting to note that the minimax contour of the super-ideal point model (9) for Halifax first touches the efficient frontier at point 3, which represents Halifax, by the southwest corner of the minimax contour, as shown in Fig. 6. This means that both the *DEA* and technical efficiency scores for Halifax are equal to one. Whilst Halifax is already efficient by maximising customer satisfaction at point 3, it can also conduct trade-off analysis to plan its improvement targets along $\overline{9,8}$ and $\overline{8,3}$ by shifting its priority to improve its business outputs (total revenue).

The data envelopes and efficient frontiers for HSBC and Lloyds TSB are shown in Fig. 7 and 8. It is interesting to note that the extreme points on the data envelopes for these two *DMU*s are both composed of the original *DMU*s only. For HSBC, the scaling factors for the seven original *DMU*s are given by 1.64, 0.9, 1.9, 1.0, 0.7, 1.01 and 2.31, respectively; for Lloyds TSB they are 0.81, 0.68, 1.0, 0.5, 1.0, 0.78 and 1.8, respectively. The efficient frontier for HSBC is given by $\overline{4,7}\cup\overline{7,3}$ as in Fig. 7 and that for Lloyds TSB by $\overline{5,7}$ as in Fig. 8. In Fig. 7, the minimax contour touches point 4 (HSBC) with its southwest corner and in Fig. 8 the minimax contour touches point 5 (Lloyds TSB) also with its southwest corner, which mean that both HSBC and Lloyds TSB are efficient and their *DEA* and technical efficiency scores are both equal to one. Depending upon the *DM*'s preferences, trade-off analysis can be conducted so that HSBC can benchmark against any convex combination of itself and the scaled RBS (i.e. any point on $\overline{4,7}$ in Fig. 7) or any convex combination of the scaled RBS and Halifax (i.e. any point on $\overline{7,3}$ in Fig. 7); similarly, Lloyds TSB can benchmark against any convex combination of itself and the scaled RBS (i.e. any point on $\overline{5,7}$ in Fig. 8).

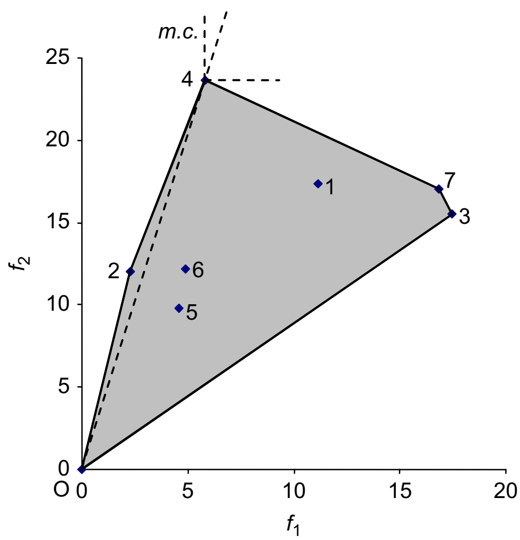

Fig. 7. Objective space for HSBC.

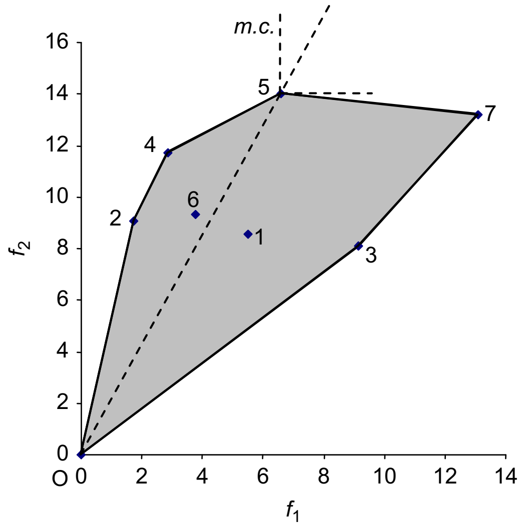

Fig. 8. Objective space for Lloyds TSB.

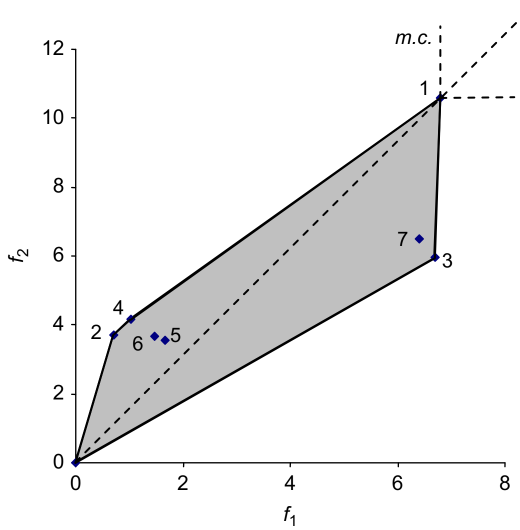

Fig. 9. Objective space for Abbey Nat.

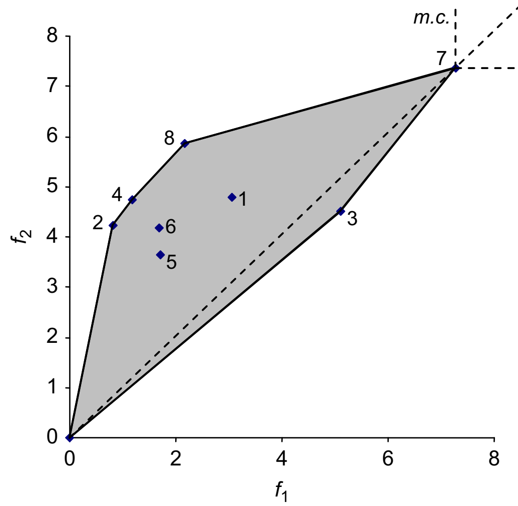

Fig. 10. Objective space for RBS.

The data envelopes and efficient frontiers for Abbey National and RBS are as shown in Fig. 9 and Fig. 10. For Abbey National, the scaling factors for the seven original *DMU*s are given by 1.0, 0.28, 0.73, 0.18, 0.25, 0.31 and 0.88, respectively; for RBS they are 0.45, 0.32, 0.56, 0.2, 0.26, 0.35 and 1.0, respectively. On the data envelope for RBS, there is a composite *DMU* (point 8) composed of 0.173 of Abbey National and 0.17 of HSBC with $f_{RBS}^{8} = [2.17,\ 5.86]^T$. Note that the reason for the composite *DMU* 8 (point 8) becoming an extreme point in the data envelope for RBS is that the third and fourth input constraints of the *MOLP* model constructed for RBS (not listed in this paper) are both binding at point 8, although point 8 is not projected to the efficient frontier of RBS. In Fig. 9, the minimax contour touches point 1 (Abbey National) with its southwest corner and in Fig. 10 the minimax contour touches point 7 (RBS) also with its southwest corner, which mean that both HSBC and Lloyds TSB are efficient and their *DEA* and technical efficiency scores are both equal to one. It is interesting to note that there is only one efficient solution in either Fig. 9 or 10, which is predicted by $F^{\max} = 1$ in Table 4 for Abbey National and RBS. This means that given the current reference set of the seven retail banks neither Abbey National nor RBS has alternative benchmark other than themselves.

From the above analyses, the following observations about efficient frontiers can be made. First, composite *DMU*s composed of original efficient *DMU*s can be extreme solutions on efficient frontiers such as point 8 in Figs. 4 and 5. It is because of this fact that inefficient *DMU*s could benchmark against the linear combinations of multiple original *DMU*s as for Barclays. Secondly, an efficient *DMU* is always on its own efficient frontier but may not be on the efficient frontier constructed for another *DMU*, as shown in Figs. 4–10. Thirdly, an inefficient *DMU* is not on its own or any other *DMU*'s efficient frontier, though it can be on the data envelope like *DMU* 2. Finally, it is possible that an efficient *DMU* cannot find a benchmark other than itself in the given reference set such as *DMU* 1 in Fig. 9 and *DMU* 7 in Fig. 10.

The above findings should be taken into account in designing any trade-off analysis processes to explore efficient frontiers for setting targets and/or for embedding the preferences of the decision maker in trade-off analysis. For example, the *DEA* composite *DMU* for Barclays was found to be 0.14 of *DMU* 1, 0.51 of *DMU* 4 and 0.49 of *DMU* 7 in standard *DEA*. From such a result, one might conclude that *DMU*s 1, 4 and 7 would all be on the efficient frontier constructed for Barclays. As shown in Fig. 5, however, neither *DMU* 1 nor *DMU* 4 is on the efficient frontier and the *DEA* composite unit is actually a single extreme efficient point itself, as shown by point 8 in Fig. 5 and Table 6. Therefore, it would be wrong to design a trade-off analysis process to explore the efficient frontier for Barclays if one assumed that *DMU*s 1 and 4 might be efficient solutions on the efficient frontier constructed for Barclays, though they do lie on their own efficient frontiers, as shown in Figs. 9 and 7, respectively.

Finally but not least importantly, it needs to be emphasised that trade-off analysis for setting targets with the DM's preferences taking into account interactively can be conducted not only for inefficient DMUs as explored in Sections 4.2 and 4.3 but also for an efficient DMU as investigated in Section 4, as long as its own efficient frontier consists of other (composite) DMUs as shown in Figs. 6–8, rather than itself only as shown in Figs. 9 and 10.

5. Concluding remarks

In this paper, the graphical and analytical methods and procedures were investigated on the basis of the *DEA*-oriented Interactive Minimax Reference Point approach. The investigation generated new insights into the integrated efficiency and trade-off analyses and revealed some interesting features of data envelopes and efficient frontiers. This is useful to support the design of pragmatic trade-off analysis processes and performance measures for setting future targets with decision makers' preferences taken into account in an interactive fashion. As a result of the investigation, we proposed the new technical efficiency score (*TES*), which can be intuitively interpreted as a degree of minimum efforts that a DMU needs to make to expand its outputs for achieving feasible efficiency, and the new preferred efficiency score (*PES*), which can be intuitively interpreted as a degree of minimum efforts that a DMU needs to make to balance its outputs for achieving the most preferred solution (or target DMU) along its efficient frontier. The numerical example and case study demonstrated these findings both graphically and analytically, which can help understand the integrated efficiency and trade-off analyses for performance assessment and planning.

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